BAHA ALZALG

COMBINATORIAL AND ALGORITHMIC MATHEMATICS

FROM FOUNDATION TO OPTIMIZATION

WILEY

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COMBINATORIAL AND ALGORITHMIC MATHEMATICS

From Foundation to Optimization

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WILEY

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In Fond Memory Of
Ala M. Alzaalig
1983–2021

Also Dedicated To
My Father Mahmoud,
My Mother Aysheh,
And My Wife Ayat

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PART IV OPTIMIZATION

PREFACE

While there has been a proliferation of high-quality books on the areas of combinatorics, algorithms, and optimization in recent years, especially now, there has remained a notable absence of a comprehensive reference work accessible to students that covers all essential aspects of these three fields, starting from their foundations. Our motivation for writing this book was to supply a clear and easily digestible resource designed specifically for traditional courses in combinatorial and algorithmic mathematics with optimization. These courses are typically pursued by students in mathematics, computer science, and engineering following their introductory calculus course. The book's origins trace back to a series of lecture notes developed for courses offered at the University of Jordan and the Ohio State University. Its primary target audience is students studying discrete structures, combinatorics, algorithms, and optimization, but it also caters to scientists across diverse disciplines that incorporate algorithms. The book is crowned with modern optimization methodologies. Without the optimization part, the book can be used as a textbook in a one- or two-term undergraduate course in combinatorial and algorithmic mathematics. The optimization part can be used in a one-term high-level undergraduate course, or a low- to medium-level graduate course.

The book is divided into four major parts, and each part is divided into three chapters. Part I is devoted to studying mathematical foundations. This includes mathematical logic and basic structures. This part is divided into three chapters (Chapters 1 – 3). In Chapter 1, we study the propositional logic and the predicate logic. In Chapter 2, we study basic set-theoretic structures such as sets, relations, and functions. In Chapter 3, we study basic analytic and algebraic structures such as sequences, series, subspaces, convex structures, and polyhedra.

Part II is devoted to studying combinatorial structures. Discrete mathematics is the study of countable structures. Combinatorics is that area of discrete mathematics that studies how to count these objects using various representations. Part II, which is divided into three chapters (Chapters 4 – 6), studies recursion techniques, counting methods, permutations, combinations, arrangements of objects and sets, and graphs. Specifically, Chapter 4 introduces graph basics and properties, Chapter 5 presents some recurrence-solving techniques, and Chapter 6 introduces some counting principles, permutations, and combinations.

Part III is devoted to studying algorithmic mathematics, which is a branch of mathematics that deals with the design and analysis of algorithms. Analyzing algorithms allows us to determine and express their efficiency. This part is divided into three chapters (Chapters 7 – 9). In Chapter 7, we discuss the asymptotic notations, one of the important tools in algorithmic analysis. Then we dive more into determining the computational complexity of various algorithms. In Chapter 8, we present and analyze standard integer, array and numeric algorithms. In Chapter 9, we present elementary combinatorial algorithms.

Part IV is devoted to studying linear and conic optimization problems, and is divided into three chapters (Chapters 10 – 12). Chapter 10 introduces linear optimization and studies its geometry and duality. This chapter also studies simplex and non-simplex algorithms for linear optimization. Second-order cone programming is linear programming over vectors belonging to second-order cones. Semidefinite programming is linear programming over positive semidefinite matrices. One of the chief attractions of these conic optimization problems is their diverse applications, many in engineering. In Chapter 11, we introduce second-order cone optimization applications and algorithms. In Chapter 12, we introduce semidefinite optimization applications (especially combinatorial applications) and algorithms.

Each chapter commences with a chapter overview, a selection of keywords, and a mini table of contents. This provides an initial overview of the chapter's content, its relevance, and what readers can expect to find in the chapter. At the end of every chapter, readers will find a set of exercises aimed at applying their knowledge of the chapter's concepts and enriching their understanding. Solutions to all of these exercises can be found in Appendix A. Additionally, to further aid readers, we have placed a list of references at the end of each chapter.

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I hold a deep sense of gratitude and appreciation towards the authors of some of the early books in this discipline, as their works have played a pivotal role in shaping the content of this book. Throughout the writing process, I have made every effort to diligently cite all references, ensuring the accuracy and integrity of the content. However, I must humbly acknowledge the potential for unintentional oversights. If, in any instance, you encounter an unattributed or inadequately cited statement, table, figure, example, or exercise within this book, I sincerely apologize. It was never my intent to omit proper attribution, and I am committed to rectifying any such omissions in future editions. Your understanding and support are greatly valued.

I am grateful to the Department of Mathematics at my home institution, the University of Jordan, for giving me sabbatical and unpaid leaves I have used to work on this book. I also thank the Department of Computer Science and Engineering at the Ohio State University for hosting me during the academic years of 2019/20 to 2021/22. The author is pleased to thank his host at OSU, Rephael Wenger. Despite being busy as an associate chair of the department, Rephael also found time to read certain chapters of the book and offered valuable feedback, for which the author is also appreciative.

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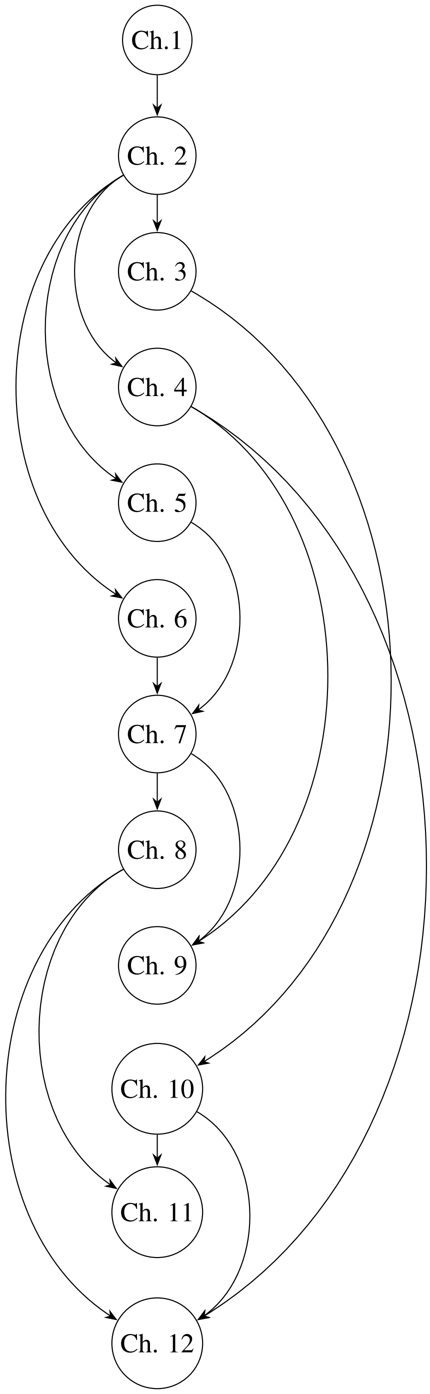
A preliminary version of this book was fully written and independently published on September 20, 2022 (ISBN-13: 979-8353826934). This self-publishing was done before launching ChatGPT on November 30, 2022. Nevertheless, it is important to acknowledge that certain insights and content in this book version may have been influenced by ChatGPT. It is worth noting that we have meticulously verified any information or data obtained from ChatGPT that has been integrated into this book edition in order to maintain its quality and reliability.

Finally, I offer my sincere gratitude and praise to Allah (Alhamdulillah) for granting me the strength, resilience, and ability to undertake the work involved in this book. His divine guidance and blessings have been an unshakable foundation upon which this project has been built.

B. M. A.

CORE OUTLINE



Part I

FOUNDATIONS

CHAPTER 1

MATHEMATICAL LOGIC

Chapter overview: This chapter provides a comprehensive introduction to fundamental branches of mathematical logic that form the basis for reasoning in mathematics, computer science, and philosophy. More precisely, it serves as a valuable resource for anyone seeking a firm grasp of propositional and predicate logic, from their foundational concepts to practical applications. Before bridging the propositional logic with the predicate logic, we shift focus to a central problem in computer science, the satisfiability problem. The chapter concludes with a set of exercises to encourage readers to apply their knowledge and enrich their understanding.

Keywords: Propositional logic, Satisfiability problem, Predicate logic

Contents

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The term "logic" has a broad definition, and countless logics have been explored by philosophers, mathematicians, and computer scientists. For most individuals, "logic" typically refers to either propositional logic or predicate logic. Propositional logic, a classical form, involves two truth values (true and false). Predicate logic, on the other hand, expands upon propositional logic by allowing explicit discussion of objects and their properties.

In this chapter, we first study the propositional logic which is the simplest, and most abstract logic we can study. After that, we study the predicate logic. We start by introducing the notion of a proposition.

Throughout this chapter (and the entire book), $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of all natural numbers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ denotes the set of all integers, and $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}$ and $b \neq 0\}$ denotes the set of all rational numbers. For example, 2 and $3/4$ are rational numbers, but $\sqrt{2}$ and π are irrational numbers. We also let $\mathbb{R}$ denote the set of all real (rational and irrational) numbers.

1.1 Propositions

In this section, we define a proposition and introduce two different types of propositions with examples. To begin with, we have the following definition.

Definition 1.1 *A proposition is a statement that can be either true or false.*

It is essential to emphasize that the proposition must hold either a true or false value, and it cannot simultaneously possess both. We give the following example.

Example 1.1 Identify each of the following statements as a proposition or not, and provide any required clarifications.

(a) 1+2 =3.

(b) 2+4 =4.

(c) Hillary Clinton is a former president of the United States.

(d) Bill Clinton is a former president of the United States.

(e) Be careful.

(f) Is Abraham Lincoln the greatest president that the United States has ever had?

(g) x+y =3.

Solution (a) "1+2 =3" is a proposition (a true proposition).

(b) "2+4 =4" is a proposition (a false proposition).

(c) "Hillary Clinton is a former president of the United States" is a proposition (a false proposition).

(d) "Bill Clinton is a former president of the United States" is a proposition (a true proposition).

(*e*) "Be careful" is not a proposition.

(*f*) "Is Abraham Lincoln the greatest president that the United States has ever had?" is not a proposition.

(*g*) "x+y =3" is not a proposition. In fact, the values of the variables have not been assigned. So, we do not know the values of x and y, and hence it neither true or false.

■

Sometimes a sentence does not provide enough information to determine whether it is true or false, so it is not a proposition. An example is, "Your answer to Question 13 is incorrect". The sentence does not tell us who we are talking about. If we identify the person, say "Adam's answer to Question 13 is incorrect", then the sentence becomes a proposition.

Note that, for a given statement, "*being not able to decide whether it is true or false (due to the lack of information)*" is different from "*being not able to know how to verify whether it is true or false*". Consider Goldbach's conjecture[1]: "Every even integer greater than 2 can be written as the sum of two primes"[2]. Despite its origin dating back to 1742 and computational data suggesting its validity, this conjecture remains an open problem in number theory. It is impossible for this sentence to be true sometimes, and false at other times. As of now, no one has proved or disproved this claim, classifying it as a proposition with an undetermined truth value because it cannot simultaneously be both true and false, and its resolution may await future mathematical breakthroughs.

There are some sentences that cannot be determined to be either true or false. For example, the sentence: "This statement is false". This is not a proposition because we cannot decide whether it is true or false. In fact, if the sentence: "This statement is false" is true, then by its meaning it is false. On the other hand, if the sentence: "This statement is false" is false, then by its meaning it is true. Therefore, the sentence: "This statement is false" can have neither true nor false for its truth value. This type of sentence is referred to as paradoxes.[3] The study of a paradox has played a fundamental role in the development of modern mathematical logic.

Note that the sentence "This statement is false", which is self contradicting, is different from the sentence "This statement is true", which is self consistent on either choice. Nevertheless, neither is a proposition. The later type of sentence is referred to as an anti-paradox.[4] The question that arises now is: Why the sentence "This statement is true" is not a proposition? This question is left as an exercise for the reader.

We now define atomic (or primitive) propositions. Intuitively, these are the set of smallest propositions.

Definition 1.2 *An atomic proposition is one whose truth or falsity does not depend on truth or falsity of any other proposition.*

According to Definition 1.2, we find that all the propositions in items $(a)-(d)$ of Example 1.1 are atomic.

[1]Christian Goldbach (1690–1764) was a German mathematician who also studied law. He is remembered today for Goldbach's conjecture, which is one of the oldest and best-known unsolved problems in number theory and all of mathematics.
[2]Prime numbers are those that are only divisible by one and themselves. For example, 7 is a prime number.
[3]A paradox is a logically self-contradictory statement that is not a proposition.
[4]An anti-paradox is a self-supporting, self-validating statement that is not a proposition.

Definition 1.3 *A compound proposition is a proposition that involves the assembly of multiple atomic propositions.*

The following connectives allow us to build up compound propositions:

$$\text{AND } (\wedge), \quad \text{OR } (\vee), \quad \text{NOT } (\neg), \quad \text{IF-THEN } (\rightarrow), \quad \text{IFF } (\leftrightarrow).$$

Example 1.2 The following proposition is compound.

$$\underbrace{\text{If } \underbrace{\underbrace{\text{it is raining}}_{\text{atomic prop. 1}} \ \underbrace{\text{and}}_{\wedge} \ \underbrace{\text{you are outside,}}_{\text{atomic prop. 2}}}_{\text{compound prop. 1}} \text{ then } \underbrace{\text{you get wet.}}_{\text{atomic prop. 3}}}_{\text{compound prop. 2}}$$

Note that

- compound proposition 1 = (atomic proposition 1) $\wedge$ (atomic proposition 2),
- compound proposition 2 = (compound proposition 1) $\rightarrow$ (atomic proposition 3).

In the realm of natural language, such as English, ambiguity is a recurring challenge that writers and readers encounter. The following remark says that slight variations in context could drastically alter the intended message.

Remark 1.1 *Sentences in natural languages can often be ambiguous and words can have different meanings in the context in which they are used.*

To illustrate Remark 1.1, we consider the following sentences:

- The sentence "*You can download Whats App or Skype to ring friends*" could mean that you can only download one of the applications, or it could mean that you can download just one or download both.
- The sentence "*I smelled a chlorine-like odor and felt ill*" implies that the odor of chlorine made you sick, but the sentence "*I am majoring in CS and minoring Math*" does not imply that majoring in CS caused you to minor in Math.
- The sentence "*I went to Chicago and took a plane*" could mean that you took a plane to travel to Chicago, or it could mean that you went to Chicago and then took a plane from Chicago to another destination, such as Las Vegas.

In mathematics and computer science, it is important to avoid ambiguity and for sentences to have a precise meaning. This is why people have invented artificial languages such as Java.

Notations Rather than writing out propositions in full, we will abbreviate them by using propositional variables: P, Q, R, . . ., etc.

Example 1.3 (Example 1.2 revisited) In Example 1.2, let P be the proposition "it is raining". Q be the proposition "you are outside", and R be the proposition "you get wet". Then we can abbreviate the compound proposition

$$\text{“If }\underbrace{\text{it is raining}}_{\text{P}}\ \underbrace{\text{and}}_{\wedge}\ \underbrace{\text{you are outside}}_{\text{Q}}\text{, then }\underbrace{\text{you get wet}}_{\text{R}}\text{”} \quad \text{as} \quad P \wedge Q \to R.$$

In the next section, we will learn more about the connectives AND, OR, NOT, IF-THEN, and IFF. This will enable us to gain a comprehensive understanding of how these connectives function within the context of formal logic and the role they play in constructing logical statements and arguments.

1.2 Logical operators

In this section, we study the following logical operators or connectives: Negation, conjunction, disjunction, exclusive disjunction, implication, and double implication.

Before starting with the logical operators, we introduce the truth tables which help us understand how such operators work by calculating all of the possible return values of atomic propositions.

Definition 1.4 *A truth table is a mathematical table used to show the truth or falsity of a compound proposition depending on the truth or falsity of the atomic propositions from which it is constructed.*

Examples of truth tables will be seen very frequently throughout this chapter.

Negation, conjunction and disjunction

This part is devoted to introducing the logical operators: Negation (NOT), conjunction (AND), disjunction (OR), and exclusive disjunction (XOR).

Negation "NOT" transforms a proposition into its opposite truth value via a negation.

Definition 1.5 *If P is an arbitrary proposition, then the negation of P is written ¬P (NOT P) which is true when P is false and is false when P is true.*

The truth table for "¬" is shown below.

P	¬P
T	F
F	T

Example 1.4 If P is the proposition "it is raining", then ¬P is the proposition "it is not raining".

Remark 1.2 *Negation does not mean "opposite". For instance, if x is a real number and ¬P is the proposition "x is not positive", then you cannot conclude that ¬P is the proposition "x is negative" because x could be 0, which is neither positive nor negative.*

Double negation In real numbers, two negative signs cancel each other out. Similarly, in propositional logic, two negations also cancel each other out. The double negation of P is ¬(¬P) or ¬¬P. The truth table for "¬¬" is shown below.

P	¬P	¬¬P
T	F	T
F	T	F

Logical equivalence Logical equivalence is a type of relationship between two propositional formulas in propositional logic.

Definition 1.6 *When the truth values for two propositional formulas P and Q are the same, the propositional formulas are called logically equivalent, and this is denoted as* $P \equiv Q$ *or* $P \Longleftrightarrow Q$.

For instance, from the truth table for "¬¬", we have P ≡ ¬¬P. This equivalence is called the double negation law.

The negation connective is unary as it only takes one argument. The upcoming connectives are binary as they take two arguments.

Conjunction "AND" connects two or more propositions via a conjunction.

Definition 1.7 *If P and Q are arbitrary propositions, then the conjunction of P and Q is written P∧Q (P AND Q) which is true when both P and Q are true.*

The truth table for "∧" is shown below.

P	Q	P ∧ Q
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.5 If P is the proposition "it is raining" and Q is the proposition "I have an umbrella", then P∧Q is the proposition "it is raining and I have an umbrella".

Disjunction "OR" connects two or more propositions via a disjunction.

Definition 1.8 *If P and Q are arbitrary propositions, then the disjunction of P and Q is written P∨Q (P OR Q) which is true when either P is true, or Q is true, or both P and Q are true.*

The truth table for "∨" is shown below.

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.6 If P is the proposition "I get married" and Q is the proposition "I live alone", then P∨Q is the proposition "I get married or I live alone".

Exclusive disjunction In English, the word "or" can be used in two different ways: inclusively or exclusively. For example, let us say that: a computer science course might require that a student be able to program in either Java or C++ before enrolling in the course. In this case, "or" is used inclusively: a student who can program in both Java and C++ would be eligible to take the course. On the other hand, for another example, let us say that: I get married or stay single. In this case, "or" is used exclusively: you can either get married or stay single, but not both.

Definition 1.9 *If P and Q are arbitrary propositions, then the exclusive-disjunction of P and Q is written P ⊕ Q (P XOR Q) which is true when exactly one of P and Q is true and false otherwise.*

The truth table for "⊕" is shown below.

P	Q	P ⊕ Q
T	T	F
T	F	T
F	T	T
F	F	F

Implication and double implication

In this part, we study the following logical operators: implication (IF-THEN) and double implication (IFF).

Implication "IF-THEN" connects two or more propositions via an implication, thus forming a conditional proposition.

Definition 1.10 *Let P and Q be arbitrary propositions. The implication P implies Q (also called the conditional of P and Q) is written P→Q (IF P THEN Q), which is false when P is true and Q is false, and true otherwise. Here, P is called the hypothesis and Q is called the conclusion.*

Example 1.7 If P is the proposition "you live in Russia" and Q is the proposition "you live in the coldest country in the world", then P→Q is the proposition "If you live in Russia, then you live in the coldest country in the world". ■

We have five different ways to read P → Q, as is seen in the following remark.

Remark 1.3 *Let P and Q be two atomic propositions. The following statements have the same meaning as the conditional statement "If P then Q".*

(*a*) *Q if P.*

(*b*) *P is sufficient for Q.*

(*c*) *Q is necessary for P.*

(*d*) *P only if Q.*

Example 1.8 Consider the implication:

- *If a person is president of the United States, then s/he is at least 35 years old.*

According to Remark 1.3, the above implication can be restated in the following equivalent ways:

- *A person is at least 35 years old if s/he is president of the United States.*
- *Being president of the United States is sufficient for being at least 35 years old.*
- *Being at least 35 years old is necessary for being president of the United States.*
- *A person is president of the United States only if s/he is at least 35 years old.*

■

The truth table for "→" is shown below.

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

Therefore, the implication is true if we are on the lines (tt), (ft), or (ff) only, and it is false if we are on the line (tf). We now justify the truth table for "→". Suppose that Sara is a math teacher and that she told her students in a class before the first midterm exam the statement that "If everyone in the class gets an A on the midterm, then I will make cookies for the class". Note that this statement says nothing about what will happen if not everyone gets an A. So, if a student did not get an A, Sara is free to make cookies or not and she will have told the truth in either case. Thus, the only case where Sara did not tell the truth (i.e., the implication is false) is the case that if everyone got A's in the midterm (i.e., the hypothesis is true) but Sara did not make cookies for the class (i.e., the conclusion is false). This is the case that made the (tf)-line in the implication truth table.

Example 1.9 Let $\mathbb{R}$ denote the set of real numbers and $x \in \mathbb{R}$[5]. Decide whether each of the following implications is true or false. Justify your answer.

(*a*) If $x \geq 10$, then $x \geq 0$.

(*b*) If $x \geq 0$, then $x \geq 10$.

(*c*) If $x^2 \geq 0$, then $x = 42$.

(*d*) If $x^2 < 0$, then $x = 42$.

(*e*) If the Riemann hypothesis is true, then $x^4 \geq 0$. [6]

Solution The very first look tells us that the proposition in item (*a*) is true, and those in items (*b*) and (*c*) are false, but we might be uncertain about the propositions in items (*d*) and (*e*). Since all these propositions have the P → Q form, we shall analyze each item according to the truth table for "→". As mentioned earlier, the conditional statement P → Q is true if we are on the lines (tt), (ft), or (ff) only, and it is false if we are on the line (tf).

(*a*) Since the hypothesis is $x \geq 10$ and the conclusion is $x \geq 0$, we are

$$\text{on the line: } \begin{cases} \text{(ff)}, & \text{if } x < 0; \\ \text{(ft)}, & \text{if } 0 \leq x < 10; \\ \text{(tt)}, & \text{if } 10 \leq x. \end{cases}$$

So, we cannot be on the (tf)-line for any $x \in \mathbb{R}$. Thus, the proposition is true.

(*b*) Since the hypothesis is $x \geq 0$ and the conclusion is $x \geq 10$, we are

$$\text{on the line: } \begin{cases} \text{(ff)}, & \text{if } x < 0; \\ \text{(tf)}, & \text{if } 0 \leq x < 10; \\ \text{(tt)}, & \text{if } 10 \leq x. \end{cases}$$

So, there is $x \in \mathbb{R}$ that makes us on the (tf)-line. Thus, the proposition is false.

(*c*) Take $x = 2$, then $x^2 = 4 \geq 0$, but $x \neq 42$. So, we are on the (tf)-line. Hence, the proposition is false.

(*d*) P is always false, so we cannot be on the (tf)-line. Hence, the proposition is true.

(*e*) The Riemann hypothesis is an unsolved conjecture in mathematics. That is, no one knows if it is true or false at this time. But this does not prevent us from answering the question for this item. In fact, as Q is always true, we cannot be on the (tf)-line. Hence, the proposition is true.

■

Example 1.10 Use a truth table to show that $P \to Q \equiv \neg P \vee Q$. This equivalence is called the implication law.

[5]To express the fact that an object x is a member of a set A, we write $x \in A$ (see Section 2.2 for definitions).
[6]The Riemann hypothesis states that all non trivial zeros of a mathematical function, the Riemann zeta function, have a real part equal to 0.5. This hypothesis is still a mysterious unsolved problem of mathematics.

Solution We prove that two propositional formulas are logically equivalent by showing that their truth values are the same. A truth table that contains the truth values for $P \to Q$ and $\neg P \lor Q$ is shown below and ends up the proof.

P	Q	$P \to Q$	$\neg P$	$\neg P \lor Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

■

Why implications are important in mathematics? Implications are important in mathematics because many mathematical theorems can be restated in the IF-THEN form, which enables us to prove them. See, for instance, the theorem that is stated and proved in the following example.

Example 1.11 Prove the result of the following theorem.

Theorem: The sum of two even integers is even.

Solution To prove the theorem, we restate it in IF-THEN form:

Theorem Restatement: If x and y are both even integers, then $x + y$ is even.

Proof If x and y are even, then each of x and y is the product of 2 and an integer. That is,

$$x = 2n \text{ and } y = 2m, \text{ for some } n, m \in \mathbb{Z},$$

where $\mathbb{Z}$ denotes the set of integers. Then $x + y = 2n + 2m = 2(n + m)$. Since the sum of any two integers is an integer, this means that $x + y$ is the product of 2 and an integer, and hence is even. The proof is complete. ■

Contrapositive of an implication The contrapositive of the implication $P \to Q$ is the implication $\neg Q \to \neg P$.

Example 1.12 The contrapositive of the proposition "If today is Monday, then Adam has a class today" is "If Adam do not have a class today, then today is not Monday". ■

The truth table for the contrapositive is given below.

P	Q	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

From the truth table for contrapositive and that for the implication, we conclude that $P \to Q \equiv \neg Q \to \neg P$. This is called the law of contrapositive.

The contrapositive is important and useful in mathematics for two reasons:

- Once a theorem in IF-THEN form is proven, there is no need to prove the contrapositive of the theorem. A proof by contrapositive, also known as contraposition, is a fundamental inference technique employed in proofs. It entails deriving a conditional statement from its contrapositive counterpart.
- If we are trying to prove a theorem in IF-THEN form, sometimes it is easier to prove its contrapositive.

Example 1.13 supports the first reason and Example 1.14 supports the second reason.

Example 1.13 The Pythagorean theorem states that in any right triangle, the square of the length of the hypotenuse (the side opposite the right angle) is to equal the sum of the squares of the lengths of the legs of the right triangle. The proof of this theorem is beyond the scope of our discussion.

In IF-THEN form, the Pythagorean theorem can be rewritten as: "If a triangle is right with hypotenuse c and legs a and b, then $a^2 = b^2+c^2$", or equivalently: "If a triangle is right-angled, then its three sides satisfy a Pythagorean triple", where a Pythagorean triple is a set of positive integers, a, b and c, that fits the rule $a^2 = b^2 + c^2$.

Since Pythagorean theorem was proven by Pythagoras, its contrapositive: "If triangle sides do not satisfy a Pythagorean triple, then the triangle is not right-angled" is obtained for free. ■

Example 1.14 Let x be a positive integer. Prove the result of the following theorem.

Theorem: If x^2 is odd, then x is odd.

Solution The implication in the theorem statement is equivalent to its contrapositive:

If x is even, then x^2 is even,

which is easier to prove. Let x be an even integer, then x can be written in the form $x = 2k$, for some positive integer k. It follows that $x^2 = (2k)^2 = 2(2k^2)$. Thus, x^2 is even. ■

Note that a proof by contraposition is different than the so-called direct proof. In a direct proof, we write a sequence of statements which are either evident or evidently follow from previous statements, and whose last statement is the desired conclusion (the one to be proved). For instance, in Example 1.14, when we proved the statement that "*If x is even, then x^2 is even*" we used a direct proof.

Converse of an implication The converse of the implication P → Q is the implication Q → P. The truth table for the converse is given below.

P	Q	P → Q	Q → P
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

From the above truth table, it is clear that Q → P ≢ P → Q.

Example 1.15 The converse of the proposition "If today is Monday, then Adam has a class today" is "If Adam has a class today, then today is Monday". Suppose that all Adam's classes are on Mondays, Wednesdays and Fridays though, then the original implication is true while its converse is false. This illustrates why we found that $P \to Q$ and $Q \to P$ are not logically equivalent. ■

Inverse of an implication The inverse of the implication $P \to Q$ is the implication $\neg P \to \neg Q$. The truth table for the inverse is given below.

P	Q	$\neg P$	$\neg Q$	$\neg P \to \neg Q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

It is clear that a conditional statement is not logically equivalent its inverse. That is, $P \to Q \not\equiv \neg P \to \neg Q$. It is also clear, from the truth table for the inverse and that for converse, that the inverse and converse of any conditional statement are logically equivalent. That is, $\neg P \to \neg Q \equiv Q \to P$.

Example 1.16 The inverse of the proposition "If today is Monday, then Adam has a class today" is "If today is not Monday, then Adam does not have a class today". ■

We now give the last logical operator, which is the double implication (IFF, which is read "if and only if").

Double implication The following definition defines a double implication as the combination of an implication and its converse.

Definition 1.11 *Let P and Q be arbitrary propositions. The double implication (also-called the biconditional) of P and Q is written $P \leftrightarrow Q$ (P IFF Q), which is true precisely when either P and Q are both true or P and Q are both false.*

Example 1.17 If P is the proposition "you live in Russia" and Q is the proposition "you live in the coldest country in the world", then $P \leftrightarrow Q$ is the proposition "You live in Russia iff you live in the coldest country in the world". ■

The truth table for "$\leftrightarrow$" is given below.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Note that $P \leftrightarrow Q$ means that P is both necessary and sufficient for Q. One of the chapter exercises asks to prove that $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$.

P	Q	R	$\neg R$	$P \wedge Q$	$(P \wedge Q) \leftrightarrow \neg R$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	T	F	F
F	T	T	F	F	T
F	T	F	T	F	F
F	F	T	F	F	T
F	F	F	T	F	F

Table 1.1: A truth table for $(P \wedge Q) \leftrightarrow \neg R$.

To prove an "if and only if" statement, you have to prove two directions. To disprove an "if and only if" statement, you only have to disprove one of the two directions. For instance, letting x be an integer, to prove that $x^2 = 0$ iff $x = 0$, we have to prove two the directions: First, we prove that if $x^2 = 0$ then $x = 0$. And second, we prove that if $x = 0$ then $x^2 = 0$ (the proofs are trivial). To disprove $x^2 = 4$ iff $x = 2$, we only disprove the direction: If $x^2 = 4$ then $x = 2$ (the disproof is trivial: take $x = -2$).

Example 1.18 Construct a truth table for the compound proposition

$$(P \wedge Q) \leftrightarrow \neg R.$$

Solution Since we have three propositional variables, P, Q and R, the number of rows in our truth table is eight rows (see Remark 1.4). The truth table for $(P \wedge Q) \leftrightarrow \neg R$ is given in Table 1.1. ■

We end this section with the following remark.

Remark 1.4 *The number of rows in a truth table for a propositional formula with n variables is 2^n rows.*

For instance, the numbers of rows in the truth tables for the propositional formulas $\neg P$, $P \wedge Q$ and $(P \wedge Q) \leftrightarrow \neg R$ are $2^1, 2^2$ and 2^3 rows, respectively.

1.3 Propositional formulas

In this section, we study special propositional formulas, which are tautologies, contradictions and contingencies. Then we study how to negate compound propositions and show how to derive propositional formulas. Before we dive into these, we illustrate the order in which the logical operators will be applied.

Order of logical operations In arithmetic, multiplication has higher precedence than addition. Hence, the value of the expression $4 + 5 \times 2$ is not 18, but 14. Similarly, in propositional logic, logical operators have operator precedence the same as arithmetic operators. We preserve the precedence order specified in Table 1.2.

Operation(s)	Operator(s)	Precedence
Parentheses	( )	1
Negation (NOT)	$\neg$	2
Conjunction (AND)	$\wedge$	3
Disjunction (OR, XOR)	$\vee, \oplus$	4
Implication (IF-THEN)	$\to$	5
Double implication (IFF)	$\leftrightarrow$	6

Table 1.2: The precedence order of logical operations.

Example 1.19 Determine the value of the propositional formula $P \vee Q \wedge R$ the propositional variables P, Q and R have values of false, true and false, respectively.

Solution From Table 1.2, the operator $\wedge$ has higher precedence than the operator $\vee$. So $P \vee Q \wedge R = P \vee (Q \wedge R)$. Plugging in the values of the variables, we have $F \vee (T \wedge F)$, which has the value of false. ■

Note that, for longer expressions, we recommend using parentheses to group expressions and control which ones are evaluated first. For example, the propositional formula $P \vee Q \wedge R \to P \vee R$ is written as $(P \vee (Q \wedge R)) \to (P \vee R)$, and the propositional formula $\neg P \vee Q \leftrightarrow R \to S$ is written as $((\neg P) \vee Q) \leftrightarrow (R \to S)$. Note also that if we have two logical operators with the same precedence, then their associativity is from left to right. For example, $P \oplus Q \vee R$ is written as $(P \oplus Q) \vee R$, and $P \to Q \to R$ is written as $(P \to Q) \to R$.

Tautologies, contradictions and contingencies

Tautologies, contradictions and contingencies are propositional formulas of particular interest in propositional logic. We have the following definition.

Definition 1.12 *A propositional formula is called*

(*a*) *a tautology if it is always true,*

(*b*) *a contradiction if it is always false,*

(*c*) *a contingency if it is neither a tautology nor a contradiction.*

Let P be an arbitrary proposition. From Table 1.3, it is seen that P $\vee\neg$P is a tautology, P $\wedge\neg$P is a contradiction, and P $\to$ (P $\wedge\neg$P) is a contingency.

P	$\neg$P	P $\vee\neg$P	P $\wedge\neg$P	P $\to$ (P $\wedge\neg$P)
T	F	T	F	F
F	T	T	F	T

Table 1.3: A tautology, contradiction and contingency.

P	Q	P ∧ Q	P ∨ Q	(P ∧ Q) → P	(P ∨ Q) → P
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	T	F
F	F	F	F	T	T

Table 1.4: A truth table for the propositional formulas of Example 1.20.

Example 1.20 Use a truth table to determine if each of the following propositional formulas is a tautology, contradiction or contingency.

(*a*) (P ∧ Q) → P.

(*b*) (P ∨ Q) → P.

(*c*) (P ∨ Q) → (¬P ∨ Q).

Solution From Table 1.4, it is seen that (P ∧ Q) → P is a tautology and (P ∨ Q) → P is a contingency. This answers items (*a*) and (*b*). Item (*c*) is left as an exercise for the reader. ■

Last, but not least, it is worth mentioning that Definition 1.12 suggests an alternative definition to the notion of logical equivalence: Two propositions P and Q are logically equivalent if P ↔ Q is a tautology.

A proof by contradiction is a common technique of proving mathematical statements. In a proof by contradiction, to prove that a statement P is true, we begin by assuming that P false, and show that this assumption leads to a contradiction, then this contradiction tells us that our original assumption is false, and hence the statement P is true. The following example illustrates how this proof technique is used.

Example 1.21 Prove that $\sqrt{2}$ is an irrational number.

Solution Suppose, in the contrary, that $\sqrt{2}$ is rational. It follows that there are two integers, say a and b, such that $\sqrt{2} = a/b$. Without loss of generality, we can assume that a and b have no common factors (otherwise, we can reduce the fraction and write it in its simplest form). Multiplying both sides of the equation $\sqrt{2} = a/b$ by b and squaring, we get $a^2 = 2b^2$. This means that a^2 is even. From the theorem stated in Example 1.14, a itself must be even. Thus, $a = 2m$ for some integer m. It follows that

$$2b^2 = a^2 = (2m)^2 = 4m^2,$$

and hence, after dividing by 2, we have $b^2 = 2m^2$. This means that b^2. By the same theorem stated in Example 1.14, b itself must be even. We have shown that both a and b are even, and hence they are both multiples of 2. This contradicts the fact that a and b have no common factors. This contradiction tells us that our original assumption is false, and hence $\sqrt{2}$ must not be rational. Thus, $\sqrt{2}$ is irrational. The proof is complete. ■

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Table 1.5: A truth table verifying the first DeMorgan's law.

Negating compound propositions

In this part, we study how to negate negations, conjunctions, disjunctions and implications.

Negating negations When we say "Results of the numerical experiment are not inconclusive" means "Results of the numerical experiment are conclusive". So, if you negate a negation they effectively cancel each other out. This leads us to the double negation law $\neg\neg P \equiv P$, which was previously stated in this chapter.

Negating conjunctions Suppose your roommate made the statement "I will get an A in Software I and an A in Foundations I". If your roommate's prediction is not correct, then s/he did not get an A in Software I or did not get an A in Foundations I. This leads us to the first DeMorgan's law:

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q.$$

Table 1.5 verifies the first DeMorgan's law.

Negating disjunctions Suppose your roommate made the statement "I will get an A in Software I or an A in Foundations I". If your roommate's prediction is not correct, then s/he did not get an A in Software I and did not get an A in Foundations I. This leads us to the second DeMorgan's law:

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q.$$

Constructing a truth table to verify the second DeMorgan's law is left as an exercise for the reader. Note that DeMorgan's laws can be extended to expressions with more than one conjunction and disjunction as follows:

$$\neg(P \wedge Q \wedge R) \equiv \neg P \vee \neg Q \vee \neg R, \qquad \neg(P \vee Q \vee R) \equiv \neg P \wedge \neg Q \wedge \neg R.$$

Example 1.22 Negate and simplify the following two compound propositions.

(a) $(P \vee Q) \wedge \neg R$. (b) $(P \vee Q) \vee (R \wedge \neg P)$.

Solution (a)
$$\begin{aligned} \neg((P \vee Q) \wedge \neg R) &\equiv \neg(P \vee Q) \vee \neg\neg R && \text{(DeMorgan's law)} \\ &\equiv \neg(P \vee Q) \vee R && \text{(Double negation law)} \\ &\equiv (\neg P \wedge \neg Q) \vee R. && \text{(DeMorgan's law).} \end{aligned}$$

(b) This item is left as an exercise for the reader.

Negating implications Suppose your (rich) roommate who is stingy made the request that "If you go to the supermarket, buy him organic food with your money". Then you realized that it is time to teach him an unforgettable lesson for having these miserly attitudes and asking you this. What should you do? Is the correct answer by not go to supermarket and not buy him organic food? No. In fact, the best way is "You go to the supermarket and you do not buy him organic food". This leads us to the law

$$\neg(P \to Q) \equiv P \wedge \neg Q, \tag{1.1}$$

which we call the implication negation law. The logical equivalence in (1.1) immediately follows by noting that

$$\begin{array}{rcll} \neg(P \to Q) & \equiv & \neg(\neg P \vee Q) & \text{(Implication law)} \\ & \equiv & (\neg\neg P) \wedge \neg Q & \text{(DeMorgan's law)} \\ & \equiv & P \wedge \neg Q. & \text{(Double negation law)} \end{array}$$

We can negate and simplify conditional statements either by directly applying the implication negation law, such as

$$\neg((P \wedge Q) \to R) \equiv P \wedge Q \wedge \neg R, \tag{1.2}$$

or by applying earlier logical equivalences, such as

$$\begin{array}{rcll} \neg(P \to (Q \vee R)) & \equiv & P \wedge \neg(Q \vee R) & \text{(Implication negation law)} \\ & \equiv & P \wedge \neg Q \wedge \neg R. & \text{(DeMorgan's law)} \end{array} \tag{1.3}$$

In Exercise 1.12, we learn how to negate exclusive disjunctions and double implications.

Universal sets We have seen that we can transform any propositional formula into an equivalent formula in DNF that uses only operators from the set $\{\wedge, \vee, \neg\}$.

Definition 1.13 *A set of operators, such as $\{\wedge, \vee, \neg\}$, that can be used to express any proposition is called a universal set.*

Note that the set of operators $\{\vee, \neg\}$ is universal. To see this, it is noted the proposition $P \vee Q$ can be expressed using just $\neg$ and $\wedge$. In fact, using double negation and DeMorgan's laws, we have

$$P \vee Q \equiv \neg\neg(P \vee Q) \equiv \neg(\neg P \wedge \neg Q).$$

So, we can for instance write $P \vee (\neg Q \wedge R) \equiv \neg[\neg P \wedge \neg(\neg Q \wedge R)]$.

Note also that the set of operators $\{\wedge, \neg\}$ is universal, while $\{\vee, \wedge\}$ is not.

Modeling using propositional logic

Propositional logic modeling is the process of abstracting a propositional logic problem from the real world. We have the following example.

Example 1.23

(a) Model the following statement using propositional logic tools.

"If it is raining and Paul does not have an umbrella, then he will get wet".

(*b*) Simplify the contrapositive of the propositional formula obtained in item (*a*), then translate it back to English.

Solution (*a*) Let P, Q and R be three propositional variables defined below.

$$\text{If } \overbrace{\text{it is raining}}^{P} \text{ and } \overbrace{\text{Paul does not have an umbrella}}^{\neg Q}, \text{ then } \overbrace{\text{he will get wet}}^{R}.$$

(Here, Q is the proposition "Paul has an umbrella"). Therefore, the statement can be symbolized as $(P \wedge \neg Q) \to R$.

(*b*) The contrapositive of the propositional formula $(P \wedge \neg Q) \to R$ is the propositional formula $\neg R \to \neg(P \wedge \neg Q)$. Using the DeMorgan's and double negation laws, this formula can be simplified to $\neg R \to (\neg P \vee Q)$. Translating back to English, we get $\text{If } \underbrace{\text{Paul did not get wet}}_{\neg R}, \text{ then } \underbrace{\text{it was not raining}}_{\neg P} \text{ or } \underbrace{\text{he had an umbrella}}_{Q}.$

■

The general scheme for solving a logical word problem is as follows. We construct a propositional logical model from a logical word problem statement in a problem domain. The model is then solved to obtain a logical conclusion, which can be finally translated back into the problem domain. See Figure 1.1.

We have the following examples.

Example 1.24 Negate the following statements.

(*a*) If Sara gets an A in Foundations I and COVID-19 disappears, then she will travel to San Francisco.

(*b*) If COVID-19 disappears, then Adam will travel to San Francisco or Los Angeles.

Solution (*a*) This statement can be symbolized as the propositional formula $(P \wedge Q) \to R$, where P is "Sara gets an A in Foundations I", Q is "COVID-19 disappears" and R is "Sara will travel to San Francisco". According the equivalence (1.2), the negation is $P \wedge Q \wedge \neg R$. Translating back to English, we get the statement "Sara got an A in Foundations I, COVID-19 disappeared and she did not travel to San Francisco", which is the negation of the original statement.

(*b*) Let P be the proposition "COVID-19 disappears", Q be the proposition "Adam will travel to San Francisco" and R be the proposition "Adam will travel to Los Angeles". The given statement can then be represented as the propositional formula $P \to (Q \vee R)$, which, using the logical equivalence (1.3), has the negation $P \wedge \neg Q \wedge \neg R$. Translating back to English, we get the statement "COVID-19 disappeared and Adam did not travel to San Francisco nor to Los Angeles", which is the negation of the original statement.

■

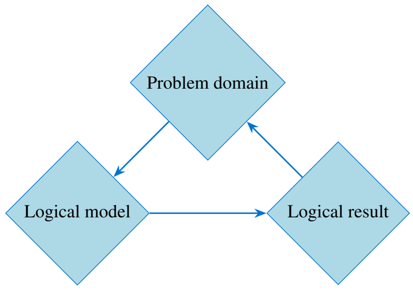

Figure 1.1: Conceptual relationships among problem domain, modeling and solution strategies.

L	M	N	$\neg L$	$N \vee M$	$M \to L$	$\neg L \wedge (N \vee M) \wedge (M \to L)$	(1.4)
T	T	T	F	T	T	F	T
T	T	F	F	T	T	F	T
T	F	T	F	T	T	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	F	T	F	T

Table 1.6: A truth table verifying that the conditional statement in (1.4) is a tautology.

Example 1.25 Consider the following propositional logic word problem:

$$\text{Problem statement: } \begin{cases} \text{Laila is not a Lebanese.} \\ \text{Nora is a New Zealander or Maria is a Macedonian.} \\ \text{If Maria is a Macedonian then Laila is a Lebanese.} \end{cases}$$

Can you conclude that Nora is a New Zealander? Justify your reasoning and conclusion by linking them to the propositional logic.

Solution Let L be "Laila is a Lebanese", M be "Maria is a Macedonian", and N be "Nora is a New Zealander", Then the problem statement can be formulated in the following propositional logical model.

$$\text{Problem model: } \begin{cases} \neg L, \\ N \vee M, \\ M \to L. \end{cases}$$

The desired conclusion is an affirmative answer to the following question.

$$\text{Is the propositional formula } [(\neg L) \wedge (N \vee M) \wedge (M \to L)] \longrightarrow N \text{ a tautology?} \tag{1.4}$$

Otherwise, we cannot conclude that Nora is indeed a New Zealander. It is seen from Table 1.6 that Nora is a New Zealander. ■

Propositional formulas in computer programs Propositions and logical connectives arise all the time in computer programs. As an example, consider the code snippet given[7] in Algorithm 1.1. In Algorithm 1.1, || denotes the logical OR operator and && denotes the logical AND operator. The condition in the "if-statement" can be simplified. Letting P be "x > 0" and Q be "y > 100", then the condition can be symbolized as $P \vee (\neg P \wedge Q)$. In Example 1.26(a) (see also Example 1.27(a)), we will prove that $P \vee (\neg P \wedge Q) \equiv P \vee Q$. This means that we can simplify the code snippet without changing the program's behavior. See Algorithm 1.2. Simplifying expressions in software can increase the speed of your program.

Deriving logical equivalences

We can derive (and prove) logical equivalences either by constructing corresponding truth tables or by applying earlier logical equivalences.

Using truth tables to prove logical equivalences If the truth values of two propositional formulas are identical under all possible interpretations [8], then they are logically equivalent.

Example 1.26 Prove or disprove each of the following logical equivalences.[9]

(a) $P \vee (\neg P \wedge Q) \equiv P \vee Q$. ($b$) $P \vee (Q \oplus R) \equiv (P \vee Q) \oplus (P \vee R)$.

Solution From the last two columns of Table 1.7, it is seen that the truth values of the propositional formulas $P \vee (\neg P \wedge Q)$ and $P \vee Q$ are identical under all interpretations, hence they are logically equivalent. This proves the desired result in item (a). Item (b) is left for the reader an exercise. ■

Algorithm 1.1: A code snippet

```
if (x > 0 || (x ≤ 0 && y > 100)) then
    ⋮
    (further instructions)
end
```

Algorithm 1.2: The code snippet in Algorithm 1.1 revisited

```
if (x > 0 || y > 100) then
    ⋮
    (further instructions)
end
```

[7] "If-statement" written in the code is formally introduced in Section 7.1.
[8] An interpretation assigns a truth value to each propositional variable.
[9] "Prove or disprove" means that either you choose to give a proof that the given statement is correct or you give a counterexample (assignments for the variables $P, Q, \ldots$) that shows the given statement is incorrect.

P	Q	$\neg P$	$\neg P \wedge Q$	$P \vee (\neg P \wedge Q)$	$P \vee Q$
T	T	F	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	F	T	F	F	F

Table 1.7: A truth table verifying $P \vee (\neg P \wedge Q) \equiv P \vee Q$.

Applying earlier equivalences to prove logical equivalences Recall that a direct proof is a sequence of statements which are either givens or deductions from previous statements, and whose last statement is the conclusion to be proved. In particular, we use a sequence of logical equivalences to prove the logical equivalence of two compound propositions. In propositional logic, many common logical equivalences exist and are often listed as laws or properties. In Table 1.8, we include a list of the most common such logical equivalences. Item (a) of Example 1.27 uses Table 1.8 to re-verify the logical equivalence in item (a) of Example 1.26.

Example 1.27 Use logical equivalences to show that:

(a) $P \vee (\neg P \wedge Q) \equiv P \vee Q$.

(b) $(P \wedge Q) \to P$ is a tautology.

(c) $[P \wedge (P \to Q)] \wedge \neg Q$ is a contradiction.

Solution We use direct proofs. The following sequence of logical equivalences proves the logical equivalence of $P \vee (\neg P \wedge Q)$ and $P \vee Q$.

$$\begin{aligned} P \vee (\neg P \wedge Q) &\equiv (P \vee \neg P) \wedge (P \vee Q) && \text{(Distributive law)} \\ &\equiv T \wedge (P \vee Q) && \text{(Tautology law)} \\ &\equiv P \vee Q. && \text{(Identity law)} \end{aligned}$$

This proves the desired result in item (a). For item (b), we prove that $(P \wedge Q) \to P$ is a tautology by showing that $(P \wedge Q) \to P$ and T are logically equivalent, which is seen from the sequence of equivalences.

$$\begin{aligned} (P \wedge Q) \to P &\equiv \neg(P \wedge Q) \vee P && \text{(Implication law)} \\ &\equiv (\neg P \vee \neg Q) \vee P && \text{(DeMorgan's law)} \\ &\equiv \neg P \vee (\neg Q \vee P) && \text{(Associative law)} \\ &\equiv \neg P \vee (P \vee \neg Q) && \text{(Commutative law)} \\ &\equiv (\neg P \vee P) \vee \neg Q && \text{(Associative law)} \\ &\equiv T \vee \neg Q \equiv T. && \text{(Tautology and domination laws)} \end{aligned}$$

This proves the desired result in item (b). Item (c) is left for the reader an exercise. ■

Name	Formula(s)
Identity laws	$P \wedge T \equiv P$
	$P \vee F \equiv P$
Domination laws	$P \wedge F \equiv F$
	$P \vee T \equiv T$
Idempotent laws	$P \vee P \equiv P$
	$P \wedge P \equiv P$
Tautology law	$P \vee \neg P \equiv T$
Contradiction law	$P \wedge \neg P \equiv F$
Double negation law	$P \equiv \neg(\neg P)$
DeMorgan's laws	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
Contrapositive law	$P \to Q \equiv \neg Q \to \neg P$
Implication law	$P \to Q \equiv \neg P \vee Q$
Implication negation law	$\neg(P \to Q) \equiv P \wedge \neg Q$
Double implication law	$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$
Commutative laws	$P \vee Q \equiv Q \vee P$
	$P \wedge Q \equiv Q \wedge P$
	$P \oplus Q \equiv Q \oplus P$
Associative laws	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
	$(P \oplus Q) \oplus R \equiv P \oplus (Q \oplus R)$
Distributive laws	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
	$P \wedge (Q \oplus R) \equiv (P \wedge Q) \oplus (P \wedge R)$
Exclusive distributive laws	$P \oplus Q \equiv (P \vee Q) \wedge \neg(P \wedge Q)$
	$P \oplus Q \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Table 1.8: A list of the most common logical equivalences.

P	Q	R	$f(P,Q,R)$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Table 1.9: A truth table defining a Boolean function $f(\cdot,\cdot,\cdot)$ on three variables.

1.4 Logical normal forms

From mathematics classes, we know that $f(x) = x^2$ defines a function that squares its one input, and $g(x, y) = 2x + y$ defines a function that doubles its first input and adds the second input. The domain and codomain of such functions are real numbers.

We can also define functions whose domain and codomain are $\{T, F\}$. Such functions are called Boolean functions.[10] For example, the truth table in Table 1.9 defines a Boolean function $f(\cdot,\cdot,\cdot)$ on three variables. In this section, we learn how to derive a propositional formula for the Boolean function $f(\cdot,\cdot,\cdot)$ represented by the truth table in Table 1.9. In particular, the Boolean disjunctive and conjunctive normal forms are given.

Disjunctive normal forms

To present the logical normal forms, we first have some definitions.

Definition 1.14 *A literal is an atomic proposition such as P, Q or R, or the negation of an atomic proposition such as $\neg P, \neg Q$ or $\neg R$.*

Definition 1.15 *A conjunctive clause is the conjunction of one or more literals.*

For example, $P, \neg Q \wedge R$ and $\neg P \wedge Q \wedge R \wedge \neg P$ are all conjunctive clauses.

Definition 1.16 *A proposition is in disjunctive normal form (DNF for short) if it is the disjunction of one or more conjunctive clauses.*

Informally, a proposition is in DNF if it is an ORing of AND clauses. For example, the propositions $\neg P$, $P \wedge \neg Q$, $\neg Q \vee R$ and $(P \wedge \neg Q) \vee \neg P \vee (P \wedge Q \wedge R)$ are all in DNF. In contrast, the propositions $\neg(P \vee Q)$ and $(P \wedge \neg Q) \vee (P \vee Q \vee R)$ are not in DNF.

[10]Named after George Boole, a 19$^{\text{th}}$ century logician.

P	Q	R	$f(P,Q,R)$	Conjunctive clauses
T	T	T	F	
T	T	F	T $\Longrightarrow$	$P \wedge Q \wedge \neg R$
T	F	T	F	
T	F	F	T $\Longrightarrow$	$P \wedge \neg Q \wedge \neg R$
F	T	T	F	
F	T	F	F	
F	F	T	T $\Longrightarrow$	$\neg P \wedge \neg Q \wedge R$
F	F	F	F	

Table 1.10: The truth table of Example 1.28.

When we derive explicit propositional formulas for Boolean functions, such as $f(\cdot,\cdot,\cdot)$ represented by the truth table in Table 1.9, we will see that the obtained formulas are in logical normal forms such as DNF. The two-step procedure in the following workflow, followed by two examples, will teach us to create such propositional formulas.

Workflow 1.1 *We create a propositional formula for a Boolean function defined by a truth table by following two steps:*

(*i*) *Create a connective clause for each row where the value of the function is true.*

(*ii*) *Create a disjunction of the conjunctive clauses obtained in (i).*

Example 1.28 Create a propositional formula using logical operators that is equivalent to the function f defined in the truth table given in Table 1.9.

Solution First, by creating a connective clause for each row where the value of the function is true, we get the conjunctive clauses shown in Table 1.10. Note that the clause $P \wedge Q \wedge \neg R$, for instance, evaluates to false for all other values of P, Q and R since it is a conjunction.

Next, by creating a disjunction of the conjunctive clauses obtained above, we obtain the formula

$$f(P,Q,R) = (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R),$$

which is in DNF. ■

Example 1.29 Write a truth table for each of the following propositions and use it to find an equivalent proposition in DNF.

(*a*) $\neg(P \to Q)$. (*b*) $(P \wedge Q) \vee \neg R$.

Solution (*a*) A truth table for the proposition $\neg(P \to Q)$ is given in Table 1.11. It is also seen from the table that $\neg(P \to Q) \equiv P \wedge \neg Q$ which is the equivalent proposition in DNF.

P	Q	$P \to Q$	$\neg(P \to Q)$	Conjunctive clauses
T	T	T	F	
T	F	F	T $\Longrightarrow$	$P \wedge \neg Q$
F	T	T	F	
F	F	T	F	

Table 1.11: The truth table of item (a) in Example 1.29.

P	Q	R	$P \wedge Q$	$\neg R$	$(P \wedge Q) \vee \neg R$	Conjunctive clauses
T	T	T	T	F	T $\Longrightarrow$	$P \wedge Q \wedge R$
T	T	F	T	T	T $\Longrightarrow$	$P \wedge Q \wedge \neg R$
T	F	T	F	F	F	
T	F	F	F	T	T $\Longrightarrow$	$P \wedge \neg Q \wedge \neg R$
F	T	T	F	F	F	
F	T	F	F	T	T $\Longrightarrow$	$\neg P \wedge Q \wedge \neg R$
F	F	T	F	F	F	
F	F	F	F	T	T $\Longrightarrow$	$\neg P \wedge \neg Q \wedge \neg R$

Table 1.12: The truth table of item (b) in Example 1.29.

(b) A truth table for the proposition $(P \wedge Q) \vee \neg R$ is given in Table 1.12. It is seen that

$$\begin{aligned}(P \wedge Q) \vee \neg R \quad &\equiv \quad (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R) \\ &\quad\ \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R),\end{aligned}$$

which is in DNF.

■

The three-step procedure in the following workflow, followed by an example, will learn us to convert any proposition into DNF.

Workflow 1.2 *We transform a proposition into DNF using a sequence of logical equivalences by following three steps:*

(i) *Convert any symbols that are not $\vee, \wedge$ or $\neg$ using logical equivalence laws. For instance, $P \to Q \equiv \neg P \vee Q$.*

(ii) *Use the DeMorgan's laws and the double negation law, as needed, so that any negation symbols are directly on the variables.*

(iii) *Use the distributive, associative and commutative laws, if needed.*

Example 1.30 Transform the proposition $(P \to (Q \to R)) \vee \neg(P \vee \neg(R \vee S))$ into DNF.

Solution The following sequence of logical equivalences transforms the given proposition into DNF.

$$\begin{aligned}
&(P \to (Q \to R)) \vee \neg(P \vee \neg(R \vee S)) \\
&\equiv (\neg P \vee (Q \to R)) \vee \neg(P \vee \neg(R \vee S)) && \text{(Implication law)} \\
&\equiv (\neg P \vee (\neg Q \vee R)) \vee \neg(P \vee \neg(R \vee S)) && \text{(Implication law)} \\
&\equiv (\neg P \vee (\neg Q \vee R)) \vee (\neg P \wedge \neg\neg(R \vee S)) && \text{(DeMorgan's law)} \\
&\equiv (\neg P \vee (\neg Q \vee R)) \vee (\neg P \wedge (R \vee S)) && \text{(Double negation law)} \\
&\equiv (\neg P \vee (\neg Q \vee R)) \vee ((\neg P \wedge R) \vee (\neg P \wedge S)) && \text{(Distributive law)} \\
&\equiv (\neg P) \vee (\neg Q) \vee R \vee (\neg P \wedge R) \vee (\neg P \wedge S).
\end{aligned}$$

■

Conjunctive normal forms

The conjunctive normal form and the disjunctive normal form can be viewed as the dual of each other. We have this definition.

Definition 1.17 *A proposition is in conjunctive normal form (CNF for short) if it is the conjunction of one or more disjunctive clauses, where each clause is a disjunction of one or more literals.*

Informally, a proposition is in CNF if it is an ANDing of OR clauses. For example, the following propositions are all in CNF:

$$\neg P,\ \neg Q \vee R,\ P \wedge \neg Q \text{ and } (P \vee \neg Q) \wedge \neg P \wedge (P \vee Q \vee R).$$

In contrast, the following propositions are not in CNF:

$$\neg(P \vee Q) \text{ and } (P \vee \neg Q) \wedge (P \vee Q \wedge R).$$

The three-step procedure in the following workflow, followed by an example, will learn us to convert a proposition into CNF.

Workflow 1.3 *We transform a propositional formula into CNF by following three steps:*

(*i*) *Create a truth table for the negation of the proposition.*

(*ii*) *Use the truth table from (i) to create an equivalent proposition in DNF.*

(*iii*) *Negate the proposition from (ii), and apply DeMorgan's law and the double negation law to convert the proposition to CNF.*

Example 1.31 Convert the proposition $(P \wedge Q) \vee \neg R$ into CNF.

Solution A truth table for the negation of the proposition $(P \wedge Q) \vee \neg R$ is given in Table 1.13.

P	Q	R	$P \wedge Q$	$\neg R$	$(P \wedge Q) \vee \neg R$	$\neg[(P \wedge Q) \vee \neg R]$	Conjunctive clauses
T	T	T	T	F	T	F	
T	T	F	T	T	T	F	
T	F	T	F	F	F	T $\Longrightarrow$	$P \wedge \neg Q \wedge R$
T	F	F	F	T	T	F	
F	T	T	F	F	F	T $\Longrightarrow$	$\neg P \wedge Q \wedge R$
F	T	F	F	T	T	F	
F	F	T	F	F	F	T $\Longrightarrow$	$\neg P \wedge \neg Q \wedge R$
F	F	F	F	T	T	F	

Table 1.13: The truth table of Example 1.31.

This shows that

$$\neg[(P \wedge Q) \vee \neg R] \equiv (P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R). \tag{1.5}$$

which is in DNF. Then

$$\begin{aligned} (P \wedge Q) \vee \neg R \;&\equiv\; \neg[\neg[(P \wedge Q) \vee \neg R]] && \text{(DNL)} \\ &\equiv\; \neg[(P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)] && \text{By (1.5)} \\ &\equiv\; \neg(P \wedge \neg Q \wedge R) \wedge \neg(\neg P \wedge Q \wedge R) \wedge \neg(\neg P \wedge \neg Q \wedge R) && \text{(DL)} \\ &\equiv\; (\neg P \vee \neg\neg Q \vee \neg R) \wedge (\neg\neg P \vee \neg Q \vee \neg R) \wedge (\neg\neg P \vee \neg\neg Q \vee \neg R) && \text{(DL)} \\ &\equiv\; (\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee Q \vee \neg R) && \text{(DNL)} \end{aligned}$$

where DNL and DL stand for the double negation law and DeMorgan's law, respectively. ■

1.5 The Boolean satisfiability problem

The Boolean satisfiability is the problem of determining if a given propositional formula is satisfiable or not; see for example [Knuth, 2022, Subsection 7.2.2.2]. We have the following definition.

Definition 1.18 *A propositional formula is said to be satisfiable if it evaluates to true for some values of its variables. If a propositional formula is not satisfiable, it is called unsatisfiable. An unsatisfiable formula evaluates to false on any values of its variables.*

For example, the proposition $(P \vee Q) \rightarrow \neg P$ is satisfiable because it evaluates to true when P is false and Q is true. Note that a satisfiable proposition is either a tautology or contingency, and that an unsatisfiable proposition is a contradiction.

The DNF satisfiability problem is the problem of determining whether or not there is any variable assignment that makes a DNF formula true. We have the following example.

Example 1.32 The following proposition

$$(\neg P \wedge Q) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg R)$$

which is in DNF, is satisfiable. To see this, note that the first clause evaluates to true when P is false and Q is true. Since the proposition is a disjunction of clauses, it is not necessary to evaluate the rest of the clauses. ■

Example 1.32 leads us to the following remark.

Remark 1.5 *A propositional formula in DNF is satisfiable if and only if at least one of its conjunctive clauses is satisfiable. A conjunctive clause is satisfiable if and only if for every atomic proposition P, the clause does not contain both P and $\neg P$ as literals.*

From Remark 1.5, it can be seen that the DNF satisfiability is "easy". In fact, we evaluate one conjunction at a time, if at least one conjunction is not a contradiction, the formula is satisfiable.

The CNF satisfiability problem is the problem of determining whether or not there is some variable assignment that makes a DNF formula true. We have the following example.

Example 1.33 The following proposition

$$(P \vee Q) \wedge (\neg Q \vee R \vee \neg S) \wedge (\neg P \vee S)$$

which is in CNF, is satisfiable. To see this, note that all clauses evaluate to true when P is true, Q is false, R is false, and S is true. Since the proposition is a conjunction of clauses, it is necessary to evaluate all clauses. ■

If the number of propositional variables is very small in a CNF formula, we can determine whether the formula is satisfiable or not is by using a truth table. We will leave proving (by constructing a truth table) whether the following proposition is satisfiable or not as an exercise for the reader.

$$(P \vee Q \vee R) \wedge (\neg P \vee \neg Q) \wedge (\neg P \vee \neg R) \wedge (\neg R \vee \neg Q).$$

In general, the CNF satisfiablity problem is not as easy as this looks in Example 1.33 or in the above exercise.

Definition 1.19 *A propositional formula is in k-CNF if it is the AND of clauses of ORs of exactly k variables or their negations.*

For example, the propositional formula $(P \vee Q) \wedge (\neg Q \vee \neg R) \wedge (\neg P \vee R)$ is in 2-CNF, and the propositional formula $(P \vee Q \vee R) \wedge (\neg Q \vee \neg R \vee \neg S) \wedge (\neg P \vee R \vee S)$ is in 3-CNF.

In fact, the 2-CNF satisfiability problem is a version of the satisfiability problem that can be formulated and solved using the so-called depth-first search method, which is the topic to be covered in Sections 9.4 and 9.5. However, it is worth mentioning that 3-CNF satisfiability is a "hard" problem. Further discussion about this can be found in any textbook on computational complexity theory.

1.6 Predicates and quantifiers

In the second part of this chapter, we study the predicate logic which extends propositional logic by adding predicates and allowing the presence of quantifiers. The predicate logic is also known as first-order logic. First-order logic quantifies only variables that range over individuals. Second-order logic and third-order logic admit quantifications over sets and sets of sets, respectively. In general, higher-order logic quantifies over sets that are nested deeply as necessary.

Predicates A predicate can be thought of as a function whose codomain is {T, F}.

As a matter of examples, the following functions are predicates.

- $P(x) = $ "$x > 4$" is a predicate on one variable. Note that, for instance, $P(5)$ is true because $5 > 4$, but $P(3)$ is false because $3 \not> 4$.
- $Q(x, y) = $ "$x \geq y$" is a predicate on two variables. Note that, for instance, $Q(5, 4)$ is true while $Q(4, 5)$ is false.

The above two examples illustrate the concept of a predicate. A formal definition looks like:

> **Definition 1.20** *A predicate is a statement whose truth value depends on the value of one or more variables.*

Quantifiers Defining quantifiers leads us to define the predicate logic.

> **Definition 1.21** *A quantifier is an expression, such as "every" and "there exists", that indicates the scope of a term to which it is attached.*

Many theorems, conjectures and definitions in mathematics use quantifiers. For example,

- "Every prime number[11] greater than 2 is odd".
- "There exists an even prime number".
- "For any integer value of n greater than 2, there are no three positive integers a, b, and c that satisfy the equation $a^n + b^n = c^n$".[12]

There are two fundamental kinds of quantification in the predicate logic:

(i) The universal quantification: The quantifier that we symbolize by "∀" and read as "every" or "for all" is called the universal quantifier.

(ii) The existential quantification: The quantifier that we symbolize by "∃" and read as "there is" or "there exists" is called the existential quantifier.

The following example helps us to always remember that the upside-down "A" stands for "All" and the backward "E" stands for " Exists". Recall that $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of all natural numbers.

[11] Recall that a prime number is a natural number which is greater than 1 and can be divided only by 1 and the number itself.

[12] This is called the Fermat's last theorem (also known as Fermat's conjecture) and was first conjectured by Pierre de Fermat around 1637.

Example 1.34 If $P(x) =$ "$x^2 > 4$", then

(*a*) "$\exists x \in \mathbb{N}, P(x)$" is a proposition that means "there is a natural number x such that $x^2 > 4$". This is a true predicate logic proposition.

(*b*) "$\forall x \in \mathbb{N}, P(x)$" is a proposition that means "for every natural number x we have $x^2 > 4$". This is a false predicate logic proposition. (To see its falsity, take $x = 1$).

In the two predicate logic propositions given in Example 1.34, the set $\mathbb{N}$ is called the universe of discourse. We have the following definition.

Definition 1.22 *The universe of discourse, which is also-called the domain of discourse (or simply the domain) and is denoted by D, is the set of the possible values of the variable(s) in the predicate.*

When we use quantifiers, it is important to state the universe of discourse. In Calculus courses, unless specified otherwise, the domain of a function is usually the set of real numbers $\mathbb{R}$. In propositional logic, the default domain is everything: All numbers, people, computers, cats, etc.

Let $P(x)$ be a predicate. The simplest form of a quantified formula is as follows.

$$\underbrace{\begin{Bmatrix} \forall \\ \exists \end{Bmatrix}}_{\text{a quantifier}} \quad \underbrace{x \in D,}_{\substack{\text{a variable} \\ \text{in a domain}}} \quad \underbrace{P(x).}_{\text{a predicate}}$$

More specifically,

- we write "$\forall x \in D, P(x)$" to say that a predicate, $P(x)$, is true for all values of x in some set D,
- we write "$\exists x \in D, P(x)$" to say that a predicate, $P(x)$, is true for at least one value of x in some set D.

For example, the proposition "$\forall n \in \mathbb{N}, n < n^2$", which means that "Every natural number is less than its own square", is false, while the proposition "$\exists n \in \mathbb{N}, n = n^2$", which means that "Some natural number is equal to its own square", is true because $n = 1$ is a witness. Table 1.14[13] shows when the truth and falsity of quantifiers are asserted.

Quantifiers	Universal quantifier	Existential quantifier
Formulas	$\forall x \in D, P(x)$	$\exists x \in D, P(x)$
When true?	If $P(x)$ is true for every x	If there is an x s.t. $P(x)$ is true
When false?	If there is an x s.t. $P(x)$ is false	If $P(x)$ is false for every x

Table 1.14: The truth and falsity of quantifiers.

[13]In Table 1.14, "s.t." refers to "such that".

Some propositions have multiple quantifiers and/or multiple predicates. In this part, we study this concretely.

Multiple quantifiers More than one quantifier can be used in a proposition. Let $P(x, y)$ be a predicate on two variables. The simplest forms of a quantified formula with multiple quantifiers are as follows.

$$\forall x \in D, \forall y \in D, P(x, y),$$
$$\exists x \in D, \exists y \in D, P(x, y).$$

We have the following examples.

Example 1.35 Let $P(x, y) =$ "$x + y = 5$". Then:

(a) The proposition "There exist two natural numbers whose sum is 5", or equivalently the more detailed proposition "There is a natural number x and a number natural y such that $x + y = 5$", can be symbolized as $\exists x \in \mathbb{N}, \exists y \in \mathbb{N}, P(x, y)$. This is a true proposition.

(b) The proposition "The sum of every pair of natural numbers is 5", or equivalently the more detailed proposition "For any two natural numbers x and y we have $x + y = 5$", can be symbolized as $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, P(x, y)$. This is a false proposition. To see its falsity, take the pair of numbers 1 (to be x) and 2 (to be y).

■

Example 1.36 Let $P(x, y) =$ "$x + y = 2$" and $Q(x, y) =$ "$xy \geq 0$". Decide whether each of the following propositions is true or false. Justify your answer.

(a) $\exists x \in \mathbb{N}, \exists y \in \mathbb{N}, P(x, y)$.

(b) $\exists x \in \mathbb{N}, \exists y \in \mathbb{N}, P(x, y) \wedge x \neq y$.

(c) $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, Q(x, y)$.

(d) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, Q(x, y)$.

Solution The proposition in item (a) is true because x and y can both be 1. For item (b), note that "$\neq$" has higher precedence than "$\wedge$". The only assignment for x and y that makes the proposition in item (b) true is $x = y = 1$. Thus, the proposition in item (b) is false. Items (c) and (d) are left as an exercise for the reader. ■

Multiple predicates Some propositions have more than one predicate. Let $P(x)$ and $Q(x)$ be two predicates, the simplest forms of a quantified formula with multiple predicates are as follows.

$$\forall x \in D, P(x) \circ Q(x),$$
$$\exists x \in D, P(x) \circ Q(x),$$

where "$\circ$" is any binary logical operator such as $\wedge, \vee, \oplus, \rightarrow$ and $\leftrightarrow$. For example, the proposition "Some natural number is equal to its own square and is equal to its own cube". This can be symbolized as $\exists n \in \mathbb{N}, n = n^2 \wedge n = n^3$.

Note that "$=$" has higher precedence than "$\wedge$". So, $\exists n \in \mathbb{N}, n = n^2 \wedge n = n^3$ means $\exists n \in \mathbb{N}, (n = n^2) \wedge (n = n^3)$. In general, mathematical symbols such as addition, subtraction, multiplication, division, equality and inequality have higher precedence than logical operators such as negation, conjunction, disjunction, implication and double implication.

When a quantifier is used on the variable x we say that this occurrence of x is bound. When the occurrence of a variable is not bound by a quantifier or set to a particular value, the variable is said to be free. The part of a logical expression to which a quantifier is applied is the scope of the quantifier. So, a variable is free if it is outside the scope of all quantifiers. For example, if we consider the statement $(\forall x \in D, P(x)) \wedge Q(x)$, the x in $P(x)$ is bound by the universal quantifier, while the x in $Q(x)$ is free. The scope of the universal quantifier is only $P(x)$. In fact, the statement $(\forall x \in D, P(x)) \wedge Q(x)$ is not a proposition since there is a free variable.

The quantifiers "$\forall$" and "$\exists$" have higher precedence than logical operators. For example, $\forall x \in D, P(x) \wedge Q(x)$ means $(\forall x \in D, P(x)) \wedge Q(x)$, and it does not mean $\forall x \in D, (P(x) \wedge Q(x))$.

Some of the questions that arise now are the following.

- Is $\forall x \in D, (P(x) \wedge Q(x))$ logically equivalent to $\forall x \in D, P(x) \wedge \forall x \in D, Q(x)$?
- Is $\forall x \in D, (P(x) \vee Q(x))$ logically equivalent to $\forall x \in D, P(x) \vee \forall x \in D, Q(x)$?

Before answering these questions, we need the following definition.

Definition 1.23 *Two predicate logic propositions S and T are logically equivalent if they have the same truth value regardless of the interpretation.* [a]

[a] That is regardless of the meaning that is attributed to each propositional function, and regardless of the domain of discourse.

Proving that the proposition $\forall x \in D, (P(x) \wedge Q(x))$ is logically equivalent to $\forall x \in D, P(x) \wedge \forall x \in D, Q(x)$ is left as an exercise for the reader. To disprove that $\forall x \in D, (P(x) \vee Q(x))$ is logically equivalent to $\forall x \in D, P(x) \vee \forall x \in D, Q(x)$, we give the following counterexample to the assertion that they have the same truth value for all possible interpretations. Let D denote the set of people in the world, $P(x)$ ="x is male", and $Q(x)$ ="x is female". Then $\forall x \in D, (P(x) \vee Q(x))$ is true, while $\forall x \in D, P(x) \vee \forall x \in D, Q(x)$ is false. In fact, the first proposition means that every person in the world is a male or a female, while the second one means that every person in the world is a male or every person in the world is a female. Note that the formula $\forall x \in D, P(x) \vee \forall x \in D, Q(x)$ can also be written as $\forall x \in D, P(x) \vee \forall y \in D, Q(y)$ or as $\forall x, y \in D, (P(x) \vee Q(y))$.

Consider also the following two assertions.

- $\exists x \in D, (P(x) \wedge Q(x))$ is logically equivalent to $\exists x \in D, P(x) \wedge \exists x \in D, Q(x)$.
- $\exists x \in D, (P(x) \vee Q(x))$ is logically equivalent to $\exists x \in D, P(x) \vee \exists x \in D, Q(x)$.

The first assertion is false, and a counterexample is $D = \mathbb{Z}$, $P(x)$ = "$x < 0$", and $Q(x)$ ="$x \geq 0$". Stating whether the second assertion is true or false and providing a sentence or two of justification are left as an exercise for the reader.

Mixing quantifiers Some propositions have two different kinds of quantifiers. Let $P(x, y)$ be a predicate on two variables. The simplest forms of a quantified formula with mixed quantifiers are as follows.

$$\begin{aligned} &\forall x \in D, \exists y \in D, P(x, y), \\ &\exists x \in D, \forall y \in D, P(x, y). \end{aligned} \tag{1.6}$$

When quantifiers of the same kind are used in a proposition, the order of the quantified variables does not matter. For example, let $P(x, y)$ be a predicate on two variables, the following two propositions have always the same meaning regardless of the predicate $P(x, y)$.

$$\exists x \in D, \exists y \in D, P(x, y),$$
$$\exists y \in D, \exists x \in D, P(x, y).$$

When both universal and existential quantifiers are used in the same proposition, the order can matter. For example, the first proposition in (1.6) and the proposition

$$\exists y \in D, \forall x \in D, P(x, y)$$

can have different meanings. Also, the second proposition in (1.6) and the proposition

$$\forall y \in D, \exists x \in D, P(x, y)$$

can have different meanings. For more illustration, we give the following example.

Example 1.37 Let $P(x, y) =$ "$x + y = 0$". Then:

(a) The proposition $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, P(x, y)$ means that for any real number x there exists a real number y such that $x + y = 0$. This is clearly a true proposition because $y = -x$ is a witness.

(b) The proposition $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, P(x, y)$ means that there is a real number x such that for every real number y we have $x + y = 0$. This is clearly a false proposition because there does not exist a real number whose sum with every real number is equal to 0.

■

Some propositions have two or more domains of discourse, one for each kind of quantifiers, as we will see in the following example.

Example 1.38 Symbolize Goldbach's conjecture which states that "Every even integer greater than 2 is the sum of two primes".

Solution Goldbach's conjecture can be restated as "For every even integer n greater than 2, there exist primes p and q such that $n = p + q$".

Let `Evens` be the set of even integers greater than 2 and `Primes` be the set of prime numbers. Then the Goldbach's conjecture can be symbolized as

$$\forall n \in \texttt{Evens}, \exists p \in \texttt{Primes}, \exists q \in \texttt{Primes}, n = p + q.$$

This answers the question raised in the example. ■

Negating quantified statements

In this part, we study how to negate quantified statements.

The negation of a universal statement is an existential statement with the predicate negated. That is,

$$\neg(\forall x, P(x)) \equiv \exists x, \neg P(x). \tag{1.7}$$

Often, a proposition such as $\neg(\forall x, P(x))$ is simply written as $\neg\forall x, P(x)$. As a direct application, we have the following example.

Example 1.39 The proposition "Every prime number is odd" is false because 2 is a prime number that is not odd. So its negation, which is the proposition "There exists a prime number that is not odd", is true.

Symbolically, to see why the second proposition is the negation of the first one, let $\forall x$ be "For every prime number x" and $P(x)$ be "x is an odd number". The negation of the proposition "Every prime number is odd" is expressed as $\neg\forall x, P(x)$ or equivalently using (1.7) $\exists x, \neg P(x)$. This statement, when subjected to a process of linguistic transformation, is subsequently rendered into the English language as "There exists a prime number that is not odd". ■

The negation of an existential statement is a universal statement with the predicate negated. That is,

$$\neg(\exists x, P(x)) \equiv \forall x, \neg P(x). \tag{1.8}$$

Often, a proposition such as $\neg(\exists x, P(x))$ is simply written as $\neg\exists x, P(x)$. As a direct application, we have the following example.

Example 1.40 The proposition "There is a negative integer whose square is negative" is clearly false. So its negation, which is the proposition "The square of every negative integer is nonnegative", is true.

Symbolically, to see why the second proposition is the negation of the first one, let $\exists x$ be "There is a negative integer x" and $P(x)$ be "The square of x is negative". The negation of the proposition "Every prime number is odd" is expressed as $\neg\exists x, P(x)$ or equivalently using (1.8) $\forall x, \neg P(x)$, which is translated back to English as "The square of every negative integer is nonnegative". ■

Negating propositions with multiple quantifiers The process of negating quantified statements is not limited solely to those propositions that feature a single quantifier; it can be expanded and applied to more complex propositions that involve multiple quantifiers. To illustrate this, consider a scenario where we have a predicate denoted as $P(x, y, z)$, which operates on three distinct variables. To negate the proposition $\forall x \in D, \exists y \in D, \forall z \in D, P(x, y, z)$ effectively, we employ the following sequence of logical equivalences.

$$\begin{aligned}
\neg\left(\forall x \in D, \exists y \in D, \forall z \in D, P(x, y, z)\right) &\equiv \exists x \in D, \neg\left(\exists y \in D, \forall z \in D, P(x, y, z)\right) \\
&\equiv \exists x \in D, \forall y \in D, \neg\left(\forall z \in D, P(x, y, z)\right) \\
&\equiv \exists x \in D, \forall y \in D, \exists z \in D, \neg P(x, y, z).
\end{aligned}$$

Example 1.41 The proposition $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1$ means that "Every real number has a multiplicative inverse". This is false because picking $x = 0$, then for any $y \in \mathbb{R}$, $xy \neq 1$. To negate the proposition $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1$, we can proceed as follows.

$$\begin{aligned}
\neg\left(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1\right) &\equiv \exists x \in \mathbb{R}, \neg\left(\exists y \in \mathbb{R}, xy = 1\right) \\
&\equiv \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \neg\left(xy = 1\right) \\
&\equiv \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy \neq 1,
\end{aligned}$$

which is the desired negation. ■

Expressing propositions using only one type of quantifiers On certain occasions, for the sake of simplicity and ease of expression, it becomes necessary to formulate propositions or statements using just a single type of quantifier, which can be either the universal quantifier or the existential quantifier. This approach to proposition construction simplifies the language and allows for more straightforward and concise communication of ideas or concepts without the added complexity that may arise from using both types of quantifiers within a single proposition.

In fact, every universal statement can be expressed as an existential statement as follows.

$$\forall x \in D, P(x) \equiv \neg\neg\forall x \in D, P(x) \equiv \neg\exists x \in D, \neg P(x). \tag{1.9}$$

As a direct application, we have the following example.

Example 1.42 The proposition "Every US senator is at least 30 years old" is the same as the proposition "There does not exist a US senator who is not at least 30 years old".

Symbolically, to see why the first proposition is equivalent to the second one, let $\forall x$ be "Every US senator x" and $P(x)$ be "x is at least 30 years old". The proposition "Every US senator is at least 30 years old" is expressed as

$$\forall x \in D, P(x) \quad \text{or equivalently using (1.9)} \quad \neg\exists x \in D, \neg P(x),$$

which is translated back to English as "There does not exist a US senator who is not at least 30 years old". ■

Similarly, every existential statement an be expressed as a universal statement as follows.

$$\exists x \in D, P(x) \equiv \neg\neg\exists x \in D, P(x) \equiv \neg\forall x \in D, \neg P(x). \tag{1.10}$$

As a direct application, we have the following example.

Example 1.43 The proposition "There exists a white cat" is the same as the proposition "Not every cat has color different than white".

Symbolically, to see why the first proposition is equivalent to the second one, let $\exists x$ be "There exists cat x" and $P(x)$ be "x has the color white". The proposition "There exists a white cat" is expressed as

$$\exists x \in D, P(x) \quad \text{or equivalently using (1.10)} \quad \neg\forall x \in D, \neg P(x),$$

which is rendered in English as "Not every cat has color different than white". ■

Note that the above procedure can be extended to propositions with more than one quantifier. For example, let $P(x, y, z)$ be a predicate on three variables, then

$$\begin{aligned} \forall x \in D, \exists y \in D, \forall z \in D, P(x, y, z) &\equiv \forall x \in D, \neg\neg\exists y \in D, \forall z \in D, P(x, y, z) \\ &\equiv \forall x \in D, \neg\forall y \in D, \neg\forall z \in D, P(x, y, z). \end{aligned}$$

1.7 Symbolizing statements of the form "All P are Q"

In this section, we study modeling of the statements of the form "All P are Q" based on the predicate logic. Perhaps the best way to understand how to symbolize this form of predicate logic propositions is to give illustrative examples.

The proverb saying "All that glitters is not gold" means that not everything that is shiny and superficially attractive is valuable, i.e., appearances can be deceiving. On the contrary, Led Zeppelin[14] in the song "Stairway to heaven" said "All that glitters is gold". In the following example, inspired by Exercise 3 in [Pugh, 2003, Chapter 1], we express the statement "All that glitters is gold" and its negation using quantifiers.

Example 1.44

(*a*) Symbolize the statement "All that glitters is gold".

(*b*) Symbolize the statement "All that glitters is gold" using only an existential quantifier.

(*c*) Symbolize the negation of the statement "All that glitters is gold" and translate it back to English.

(*d*) Symbolize the statement "Nothing that glitters is gold".

Solution. We let D be the set of all elements, whether they are gold or not, whether they glitter or not. We also define the two predicates:

$$GLT(x) = \text{“}x \text{ glitters''} \quad \text{and} \quad GLD(x) = \text{“}x \text{ is gold''}.$$

(*a*) The statement "All that glitters is gold" means that if "If x glitters, then x is gold". This is an implication, so the correct way to express it as

$$\forall x \in D, (GLT(x) \longrightarrow GLD(x)). \tag{1.11}$$

By the implication law, formulation in (1.11) is equivalent to the following formulation.

$$\forall x \in D, (\neg GLT(x) \vee GLD(x)). \tag{1.12}$$

It is worth mentioning that a common error is to symbolize the statement "All that glitters is gold" as $\forall x \in D, (GLT(x) \wedge GLD(x))$, which is not correct because it means that everything in the universe both glitters and is gold. Another error that some people make is to symbolize "All that glitters is gold" as $\forall x \in GLT(x), GLD(x)$, which is not correct because "$\in$" is a set operator but $GLT(x)$ is a predicate, not a set.

(*b*) From (1.12), it is seen that the statement "All that glitters is gold" is symbolized as $\forall x \in D, (\neg GLT(x) \vee GLD(x))$. The following sequence of logical equivalences expresses

[14] Led Zeppelin were an English rock band formed in London in 1968.

this formula using only an existential quantifier.

$$\begin{aligned}
\forall x \in D, (\neg GLT(x) \vee GLD(x)) &\equiv \neg\neg\forall x \in D, (\neg GLT(x) \vee GLD(x)) \\
&\equiv \neg\exists x \in D, \neg(\neg GLT(x) \vee GLD(x)) \\
&\equiv \neg\exists x \in D, (\neg\neg GLT(x) \wedge \neg GLD(x)) \\
&\equiv \neg\exists x \in D, (GLT(x) \wedge \neg GLD(x)).
\end{aligned}$$

Here, the first equivalence follows from the double negation law, the second equivalence follows from (1.7), and the last two follow from the DeMorgan's law and double negation law.

(*c*) From (1.11), the negation of the statement "All that glitters is gold" is symbolized as $\neg\forall x \in D, (\neg GLT(x) \vee GLD(x))$. Using the rules of propositional logic, we have

$$\begin{aligned}
\neg\forall x \in D, (\neg GLT(x) \vee GLD(x)) &\equiv \exists x \in D, \neg(\neg GLT(x) \vee GLD(x)) \\
&\equiv \exists x \in D, (\neg\neg GLT(x) \wedge \neg GLD(x)) \\
&\equiv \exists x \in D, (GLT(x) \wedge \neg GLD(x)),
\end{aligned}$$

which is translated back to English as"There is something in the universe that glitters but is not gold".

(*d*) Using the predicates defined above, the statement "Nothing that glitters is gold" is symbolized as $\forall x \in D, (GLT(x) \to \neg GLD(x))$.

This answers the questions raised in the example. ■

In the following example, we encounter a statement asserting the absence of a specific condition. In the first part, we are tasked with transforming this statement into the formal structure of "All P are Q". The second part takes a different turn as we examine the concept of logical negation and double negation.

Example 1.45 Consider the statement "There are no orange sharks".

(*a*) Symbolize the given statement after writing it in the form "All P are Q".

(*b*) Can you obtain the same formula obtained in item (*a*) by symbolizing after negating the negation of the given statement? Justify your answer.

Solution. We let D be the set of all creatures (or species), whether they are sharks or not, whether they are orange-colored or not. We also define the two predicates:

$$SRK(x) = \text{“}x \text{ is a shark”} \quad \text{and} \quad ORG(x) = \text{“}x \text{ is orange”}.$$

(*a*) The statement "There are no orange sharks" is equivalent to the implication "If something is a shark, then it is not orange". So, a correct way to symbolize the given statement is $\forall x \in D, (SRK(x) \to \neg ORG(x))$.

(*b*) We can obtain the same formula by negating the negation of the given statement. In fact, another way to get the same formula is to symbolize the negation of the statement "There is an orange shark" as follows.

$$\begin{aligned}\neg\exists x \in D, (SRK(x) \wedge ORG(x)) &\equiv \forall x \in D, \neg(SRK(x) \wedge ORG(x))\\ &\equiv \forall x \in D, (\neg SRK(x) \vee \neg ORG(x))\\ &\equiv \forall x \in D, (\neg\neg SRK(x) \to \neg ORG(x))\\ &\equiv \forall x \in D, (SRK(x) \to \neg ORG(x)).\end{aligned}$$

This justifies our formulation and hence answers the questions raised in the example. ■

Example 1.46 Symbolize the following statements using the predicate logic.

(*a*) "There is at most one piano".

(*b*) "There are at least two pianos".

(*c*) "There are at most two pianos".

(*d*) "There are exactly two pianos".

Solution. We let D be the set of all musical instruments, whether they are pianos or not. We also define the predicate: $P(x)$ = "x is a piano".

(*a*) The statement "There are at least two pianos" is equivalent to the implication "If there are two musical instruments that are pianos, then they must be the same instrument". This can be symbolized as

$$\forall x \in D, \forall y \in D, (P(x) \wedge P(y) \longrightarrow x = y),$$

or equivalently, using the implication and DeMorgan's laws, as

$$\forall x \in D, \forall y \in D, (\neg P(x) \vee \neg P(y) \vee x = y).$$

(*b*) An incorrect way to symbolize the statement "There are at least two pianos" is the following. (Why is this incorrect?)

$$\exists x \in D, \exists y \in D, (P(x) \wedge P(y)).$$

To obtain the correct formula, note that the statement "There are at least two pianos" is the negation of the statement "There is at most one piano". Using item (*a*) and the rules of propositional logic, we have

$$\begin{aligned}&\neg\forall x \in D, \forall y \in D, (\neg P(x) \vee \neg P(y) \vee x = y)\\ &\qquad\equiv \exists x \in D, \neg\forall y \in D, (\neg P(x) \vee \neg P(y) \vee x = y)\\ &\qquad\equiv \exists x \in D, \exists y \in D, \neg(\neg P(x) \vee \neg P(y) \vee x = y)\\ &\qquad\equiv \exists x \in D, \exists y \in D, (P(x) \wedge P(y) \wedge x \neq y).\end{aligned}$$

Thus, the statement "There are at least two pianos" is symbolized as $\exists x \in D, \exists y \in D, (P(x) \wedge P(y) \wedge x \neq y)$. Note that the obtained formula explicitly says that x and y are not the same instrument. In the earlier incorrect formula, x and y could refer to the same instrument.

(*c*) The statement "There are at most two pianos" is symbolized as

$$\forall x \in D, \forall y \in D, \forall z \in D, ((P(x) \wedge P(y) \wedge P(z)) \longrightarrow (x = y) \vee (x = z) \vee (y = z)).$$

(*d*) The statement "There are exactly two pianos" means that "There are at least two pianos and there are at most two pianos". From items (*b*) and (*c*), this can be symbolized as

$$[\exists x \in D, \exists y \in D, (P(x) \wedge P(y) \wedge x \neq y)] \ \wedge$$
$$[\forall x \in D, \forall y \in D, \forall z \in D, P(x) \wedge P(y) \wedge P(z) \to (x = y) \vee (x = z) \vee (y = z)]\,.$$

An alternative way to symbolize the statement "There are exactly two pianos" is as follows.

$$\exists x \in D, \exists y \in D, \Big((x \neq y) \wedge \Big[\forall z \in D, \Big(P(z) \longleftrightarrow [(x = z) \vee (y = z)]\Big)\Big]\Big).$$

■

Finally, it is worth mentioning that sometimes statements of the form "All P are Q" are expressed using quantifiers and predicates with more than one variable. For example, to symbolize the statement "Every mail message larger than two megabytes will be compressed", we define the predicates $ML(x, k)$ = "mail message x is larger than k megabyte" and $MC(x)$ = "mail message x will be compressed". Then the given statement is symbolized as follows.

$$\forall x \in D, \Big(ML(x, 2) \longrightarrow MC(x)\Big).$$

Here, the domain of discourse D is the set of all mail messages.

Exercises

1.1 Choose the correct answer for each of the following multiple-choice questions/items.

(*a*) True or False: The statement "$x + 2 = 7$ if $x = 2$" is a proposition.

(*i*) True. (*ii*) False.

(*b*) Let P be the sentence "Q is false", Q be the sentence "R is false", and R be the sentence "P is false". Then P is

(*i*) a proposition.
(*ii*) not a proposition, but a paradox.
(*iii*) not a proposition, but an anti-paradox.
(*iv*) not a proposition, neither a paradox nor an anti-paradox.

(*c*) Let P represent a true statement, and let Q and R represent false statements. The truth value of the compound statement $\neg(\neg P \wedge \neg Q) \vee (\neg R \vee \neg P)$.

(*i*) True. (*ii*) False.

(*d*) Let P be the statement "Kids are happy" and Q be the statement "Parents are happy". If we translate the compound proposition $\neg(P \vee \neg Q)$ into words, we get (*i*) It is not the case that kids are happy or parents are not happy.

(*ii*) Kids are not happy and parents are not happy.

(*iii*) It is not the case that kids are happy and parents are not happy.

(*iv*) Kids are not happy or parents are not happy.

(*e*) True or False: If $6 < 1$, then $11 < 4$.

(*i*) True. (*ii*) False.

(*f*) A sufficient condition that a triangle T be a right triangle is that its three sides satisfy a Pythagorean triple. An equivalent statement is

(*i*) If T is a right triangle then its three sides satisfy a Pythagorean triple.

(*ii*) If the three sides of a triangle T satisfy a Pythagorean triple, then T is a right triangle.

(*iii*) If the three sides of a triangle T do not satisfy a Pythagorean triple, then T is not a right triangle.

(*iv*) T is a right triangle only if its three sides satisfy a Pythagorean triple.

(*g*) Consider the following argument:

If I am thirsty, then I will drink a glass of water.

I am not thirsty.

I will not drink a glass of water.

Then this argument is

(*i*) valid by the implication.

(*ii*) invalid by the fallacy of the inverse.

(*iii*) valid by the contrapositive.

(*iv*) invalid by the fallacy of the converse.

(*h*) The total number of rows in the truth table for the compound proposition $(P \vee Q) \wedge (R \vee \neg S) \rightarrow T$ is

(*i*) 25. (*ii*) 15. (*iii*) 23. (*iv*) 32.

(*i*) Consider the statement, “If n is divisible by 12, then n is divisible by 2 and by 3 and by 4”. This statement is equivalent to the statement

(*i*) If n is not divisible by 12, then n is divisible by 2 or divisible by 3 or divisible by 4.

(*ii*) If n is not divisible by 12, then n is not divisible by 2 or not divisible by 3 or not divisible by 4.

(*iii*) If n is divisible by 2 and divisible by 3 and divisible by 4, then n is divisible by 12.

(*iv*) If n is not divisible by 2 or not divisible by 3 or not divisible by 4, then n is not divisible by 12.

(*j*) Consider the statement "Given that people who are in need of refuge and consolation are apt to do odd things, it is clear that people who are apt to do odd things are in need of refuge and consolation." This statement, of the form $(P \to Q) \to (Q \to P)$, is logically equivalent to

(*i*) People who are in need of refuge and consolation are not apt to do odd things.

(*ii*) People are apt to do odd things if and only if they are in need of refuge and consolation.

(*iii*) People who are apt to do odd things are in need of refuge and consolation.

(*iv*) People who are in need of refuge and consolation are apt to do odd things.

(*k*) Let M = "Adam is a Math major", P = "Adam is a Physics major", A = "Adam's wife is an Astronomy major", K = "Adam's wife has read Al-Khwarizmi's algebra", and H = "Adam's wife has read Al-Haytham's optics". One of the following propositions expresses the statement "Adam is a Physics major and a Math major, but his wife is an Astronomy major who has not read both Al-Haytham's optics and Al-Khwarizmi's algebra", which is

(*i*) $P \wedge M \wedge (A \vee (\neg K \vee \neg H))$.

(*ii*) $P \wedge M \wedge A \wedge (\neg K \wedge \neg H)$.

(*iii*) $P \wedge M \wedge A \wedge (\neg K \vee \neg H)$.

(*iv*) $P \wedge M \wedge (A \vee (\neg K \wedge \neg H))$.

(*l*) The negation of the proposition "If 3 is positive, then -3 is negative" is the proposition

(*i*) "If 3 is positive, then -3 is also positive".

(*ii*) "If 3 is negative, then -3 is positive".

(*iii*) "3 is positive and -3 is positive".

(*iv*) "3 is positive and -3 is nonnegative".

(*m*) One of the following is not logically equivalent to $(P \wedge Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q)$, which is

(*i*) $\neg P \wedge \neg Q$. (*ii*) $P \vee Q$. (*iii*) $Q \vee (P \wedge \neg Q)$. (*iv*) $P \vee (Q \wedge \neg P)$.

(*n*) The contradiction law is an example of

(*i*) a tautology.

(*ii*) a contradiction.

(*iii*) a contingency.

(*iv*) none of the above.

(*o*) One of the following compound propositions is in both DNF and CNF, which is

(*i*) $(P \vee Q) \wedge \neg R$.

(*ii*) $(P \wedge Q) \vee \neg R$.

(*iii*) $P \wedge Q \wedge \neg R$.

(*iv*) None of the above.

(p) True or False: A propositional formula in CNF is satisfiable if and only if at least one of its disjunctive clauses is satisfiable.

(i) True. (ii) False.

1.2 Which of the following sentences are propositions? What are the truth values of those that are propositions? (Here, in items (c)-(f), x and y are any real numbers and θ is any angle in the standard position).

(a) $2 + 3 = 5$.

(b) $2 + 3$.

(c) $x + 2 = 11$.

(d) $x + y = y + x$.

(e) $\cos^2 \theta = 1$.

(f) $\cos^2 \theta + \sin^2 \theta = 1$.

(g) Which of the following sentences are propositions?

(h) John F. Kennedy is the 35th president of the United States.

(i) Do not be in close contact with a sick person with COVID-19.

(j) Repeat your answer to item (l).

(k) Your answer to item (l) is incorrect.

(l) The truth value of the statement in item (h) is true.

(m) The truth value of the statement in item (i) is true.

(n) The truth value of the statement in this item is false.

1.3 What is the negation of each of the following propositions?

(a) Today is Thursday.

(b) There is no pollution in New Jersey.

(c) $2 + 1 = 3$.

(d) Sara's first answer to item (l) in Exercise 1.1 was incorrect.

(e) The summer in Santiago is not hot.

(f) The summer in Santiago is hot but bearable.

(g) The summer in Santiago is not hot or it is not humid.

(h) If the sun is shining in Santiago's sky, then I will go to the nearest beach.

(i) If the sun is shining in Santiago's sky, then I will go to the nearest beach and do a little physical exercise.

1.4 Let P and Q be propositions

P : I bought a lottery ticket this week.

Q : I won the million-dollar jackpot on Friday.

Express each of the following propositions as an English sentence.

(*a*) $\neg P$. (*b*) $P \vee Q$. (*c*) $P \wedge Q$. (*d*) $\neg P \wedge \neg Q$.

1.5 Let P and Q denote the statements "It is below freezing" and "It is snowing", respectively. Formulate the following statements using P and Q and logical connectives.

(*a*) It is below freezing and snowing.

(*b*) It is below freezing but not snowing.

(*c*) It is not below freezing and it is not snowing.

(*d*) It is either snowing or below freezing (or both).

1.6 Consider the implication: "A passing score on the final exam is required in order to receive a passing grade for the course". Restate this statement as five implications, all have the same meaning, and each using one of the five equivalent forms listed in Remark 1.3.

1.7 The following implications are mainly about the twin primes conjecture.[15] Decide whether each of the these implications is true or false. Justify your answer.

(*a*) If the twin primes conjecture is unsolved, then the twin primes conjecture is conjectured by a monkey.

(*b*) If the twin primes conjecture is true, then the Pythagoras' theorem is true.

(*c*) If the twin primes conjecture is false, then the Pythagoras' theorem is true.

(*d*) If the twin primes conjecture is either proven or disproven, then the twin primes conjecture is solved.

(*e*) If the twin primes conjecture is true, then the twin primes conjecture is true.

(*f*) If the twin primes conjecture is either true or false, then the twin primes conjecture is solved.

(*g*) If the twin primes conjecture is either true or false, then the twin primes conjecture is unsolved.

1.8 Prove the result of the following theorem. Start by restating the theorem in IF-THEN form.

Theorem I: The sum of two odd integers is even.

1.9 Write the contrapositive, converse, and inverse of the proposition: "If the Sun is shrunk to the size of your head, then the Earth will be the size of the pupil of your eye".

[15] Twin primes are primes that are two steps apart from each other on the number line, such as 3 and 5, 5 and 7, 29 and 31, and so on. The twin primes conjecture (also known as the Polignac's conjecture) states that there are infinitely many twin primes. This conjecture is still a mysterious unsolved problem in the field of Number Theory.

1.10 Give, if possible, an example of a true conditional statement for which

(a) the converse is true.

(b) the contrapositive is false.

1.11 Let x be a positive integer. Prove the results of the following two theorems. Use the low of contrapositive.

Theorem I: If x^2 is even, then x is even.
Theorem II: x^2 is odd iff x is odd.

1.12 Negate and simplify the following two compound propositions.

(a) $P \leftrightarrow Q$. (Hint: Use the logical equivalence $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$).

(b) $P \oplus Q$. (Hint: Use the logical equivalence $P \oplus Q \equiv (P \vee Q) \wedge \neg(P \wedge Q)$).

1.13 Use truth tables to prove that $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$.

1.14 Construct truth tables for the following compound propositions.

(a) $P \wedge \neg P$.

(b) $(P \vee \neg Q) \to Q$.

(c) $(P \vee Q) \to (P \wedge Q)$.

(d) $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$.

(e) $P \oplus P$.

(f) $P \oplus \neg Q$.

(g) $\neg P \oplus \neg Q$.

(h) $(P \oplus Q) \wedge (P \oplus \neg Q)$.

(i) $P \to \neg Q$.

(j) $\neg P \leftrightarrow Q$.

(k) $(P \to Q) \vee (\neg P \to Q)$.

(l) $(P \leftrightarrow Q) \vee (\neg P \leftrightarrow Q)$.

(m) $(P \wedge Q) \vee R$.

(n) $(P \wedge Q) \wedge R$.

(o) $(P \vee Q) \vee R$.

(p) $(P \wedge Q) \vee \neg R$.

(q) $P \to (\neg Q \vee R)$.

(r) $(P \to Q) \vee (\neg P \to R)$.

(s) $(P \to Q) \wedge (\neg P \to R)$.

(t) $(P \leftrightarrow Q) \vee (\neg Q \leftrightarrow R)$.

1.15 Consider the following propositional logic word problem: Either Lillian is forceful or she is creative. If Lillian is forceful, then she will be a good executive. It is not possible that Lillian is both efficient and creative. If she is not efficient, then either she is forceful or she will be a good executive. Can you conclude that Lillian will be a good executive? Your reasoning and conclusion should be justified by linking them to the propositional logic.

1.16 List the names of the logical equivalence laws from Table 1.8 that are used to prove Item (c) of Example 1.27.

1.17 Determine if each of the following implications is a tautology, contradiction or contingency. Justify your answer without using truth tables.

(a) $[\neg P \wedge (P \vee Q)] \to Q$.

(b) $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$.

(c) $[P \wedge (P \to Q)] \to Q$.

(d) $[(P \vee Q) \wedge (P \to R) \wedge (Q \to R)] \to R$.

1.18 Without using truth tables, prove that the following propositions are logically equivalent.

(a) $P \leftrightarrow Q$.

(b) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

(c) $\neg(P \oplus Q)$.

Hint: You can show that the propositions (a), (b) and (c) are logically equivalent by showing that the propositions in (a) and (b) are logically equivalent and that those in (b) and (c) are logically equivalent.

1.19 Construct a *disjunctive normal form* having the following truth table.

P	Q	R	Statement
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	F

1.20 For each of the following statements, find an equivalent statement in conjunctive normal form.

(a) $\neg(A \vee B)$.

(b) $\neg(A \wedge B)$.

(c) $A \vee (B \wedge C)$.

1.21 Give an example of a proposition in DNF with three variables and four distinct clauses that is unsatisfiable.

1.22 Is the propositional formula $P \wedge Q \wedge (\neg R \vee S \vee \neg T)$ satisfiable? Explain.

1.23 Choose the correct answer for each of the following multiple-choice questions/items.

(a) The propositions in items (c) and (d) of Example 1.36 are

(i) both true.

(ii) both false.

(iii) true and false, respectively.

(iv) false and true, respectively.

(b) Expressing the proposition $\exists x \in D, \forall y \in D, \forall z \in D, P(x, y, z)$ using only the existential quantifier, we get

(*i*) $\exists x \in D, \neg\exists y \in D, \neg\exists z \in D, \neg P(x,y,z)$.
(*ii*) $\exists x \in D, \neg\exists y \in D, \neg\exists z \in D, P(x,y,z)$.
(*iii*) $\exists x \in D, \exists y \in D, \neg\exists z \in D, \neg P(x,y,z)$.
(*iv*) $\exists x \in D, \neg\exists y \in D, \exists z \in D, \neg P(x,y,z)$.

(*c*) Let V be the set of graph vertices and C be the set of all colors (see Sections 4.1 and 4.4). Let also edge(u,v) mean that "vertex v is adjacent to vertex u", and color(v,x) mean that "vertex v has color x". Which one of the following is the correct formulation for the statement "Adjacent vertices do not have the same color"?

(*i*) $\forall v \in V, \forall u \in V, \forall x \in C, (\text{edge}(v,u) \wedge \text{color}(v,x) \wedge \neg\text{color}(u,x))$.
(*ii*) $\forall v \in V, \forall u \in V, \forall x \in C, (\text{edge}(v,u) \wedge \text{color}(v,x) \rightarrow \neg\text{color}(u,x))$.
(*iii*) $\forall v \in V, \forall u \in V, \forall x \in C, (\text{edge}(v,u) \wedge \text{color}(v,x) \vee \neg\text{color}(u,x))$.
(*iv*) $\forall v \in V, \forall u \in V, \forall x \in C, (\text{edge}(v,u) \wedge \text{color}(v,x) \vee \text{color}(u,x))$.

(*d*) Let F be the set of all (direct or indirect) flights, and define the predicate

$$DF(u,x,y) = \text{"There is a direct flight } u \text{ from a city } x \text{ to a city } y\text{"}.$$

One of the following statements is a correct translation for the quantified statement $\neg\exists x \in F, DF(x, \text{Columbus, Rochester})$, which is

(*i*) There is no flight between Columbus and Rochester.
(*ii*) There is no flight from Columbus to Rochester.
(*iii*) There is no direct flight between Columbus and Rochester.
(*iv*) All flights from Columbus to Rochester are indirect.

(*e*) The negation of "Nothing that glitters is gold" is

(*i*) "All that glitters is gold".
(*ii*) "There is something that glitters that is gold".
(*iii*) "It is not all that glitters that is gold".
(*iv*) "All that glitters that is not gold".

(*f*) Let S be the set of all shapes, and define the three predicates $FS(x)$ = "x is a footprint shape", $C(x)$ = "x is circular-shaped", and $E(x)$ = "x is elliptical-shaped". One of the following is the correct formulation for the statement "Each footprint shape is either circular or elliptical, but not both", which is

(*i*) $\forall x \in S, (F(x) \longrightarrow [C(x) \vee E(x)])$.
(*ii*) $\forall x \in S, (F(x) \longrightarrow [C(x) \rightarrow \neg E(x)])$.
(*iii*) $\forall x \in S, (F(x) \longrightarrow [C(x) \leftrightarrow \neg E(x)])$.
(*iv*) $\forall x \in S, (F(x) \longrightarrow [C(x) \oplus \neg E(x)])$.

1.24 Let the domain of discourse consist of the non-zero integers, i.e., $D = \mathbb{Z} - \{0\}$. Give the truth value of each of the following quantified statements. Find a value of x which supports your conclusion.

(*a*) $\forall x \in D, x < 2x.$

(*b*) $\exists x \in D, x + x = x - x.$

(*c*) $\exists x \in D, x^2 = 2x.$

(*d*) $\forall x \in D, \frac{x}{2x} < x.$

(*e*) $\exists x \in D, \frac{x}{x} = x.$

1.25 Consider the following two propositions.

$$\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, P(x, y), \tag{1.13}$$

$$\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, P(x, y). \tag{1.14}$$

Using the above two propositions, give a predicate $P(x, y)$ that makes (or explain why it is not possible):

(*a*) Propositions (1.13) and (1.14) are both true.

(*b*) Propositions (1.13) and (1.14) are both false.

(*c*) Proposition (1.13) is true and Proposition (1.14) is false.

(*d*) Proposition (1.13) is false and Proposition (1.14) is true.

1.26 Negate the false proposition $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 0$.

1.27 Symbolize the following statements using the predicate logic. Use the four predicates $A(x)$ = "x is an architect", $C(x)$ = "x is a constructor", $I(x)$ = "x is invited", and $L(x)$ = "x is late".

(*a*) There is at least a person who is on time.

(*b*) There is at least an invited person who is neither a constructor nor an architect.

(*c*) All architects and constructors invited to the party are late.

1.28 Let D be the universe of all living things, and define the three predicates: Child(x) = " x is a child", Mime(y) = "y is a mime", and Like(x, y) = "x likes y". Symbolize the negation of the quantified statement "Some children do not like mimes".

1.29 Symbolize the following statements using the predicate logic.

(*a*) For each owner, there is at least a house with this owner.

(*b*) Adjacent houses do not have the same owner.

(*c*) Each house has at most one owner.

1.30 Symbolize the following statements using the predicate logic.

(*a*) Friends of friends are friends.

(*b*) Friendless people smoke.

(*c*) Smoking causes cancer.

(*d*) If two people are friends, either both smoke or neither does.

Notes and sources

Mathematical logic has a rich history dating back to ancient civilizations. The origins of mathematical logic can be traced to the work of ancient Greek philosophers like Aristotle in the 4th century BCE. However, the modern formalization of mathematical logic began in the 19th and early 20th centuries with the groundbreaking contributions of mathematicians and logicians such as George Boole, Augustus De Morgan, and Gottlob Frege. Boole's book Boole [1854] laid the groundwork for Boolean algebra, a crucial component of mathematical logic. Additionally, Frege's book Frege and Angelelli [1977] provided a rigorous symbolic language for logical propositions and introduced the concept of predicate logic. These early contributions set the stage for the further advancement of mathematical logic in the 20th century.

This chapter offered a comprehensive introduction to the fundamental branches of mathematical logic, which served as the basis for reasoning in mathematics, computer science, and philosophy. More precisely, it provided a valuable resource for anyone who sought to gain a firm grasp of propositional and predicate logic, from their foundational concepts to practical applications. Before transitioning to the topic of bridging propositional logic with predicate logic, a shift occurred in the chapter's focus to address a central problem within the realm of computer science, namely, the satisfiability problem.

As we conclude this chapter, it is worth noting that the cited references and others, such as Cusack and Santos [2021], Pinter [2014], Rosen [2002], Kunen [1983], Fraenkel et al. [1973], Scheurer [1994], Jech [1997], Schwartz et al. [2011], Krajícek [1995], Katsumata [2005], Moss [1981], Tütüncü et al. [2003], Clark [1980], Al-Yahya et al. [2023], Arvind and Guruswami [2022], Yin et al. [2021], Dovier [2019], Knuth [1997], also serve as valuable sources of information pertaining to the subject matter covered in this chapter. Exercises 1.14, 1.17 and 1.18 are due to Rosen [2002]. Exercise 1.30 is due to Domingos and Lowd [2009].

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CHAPTER 2

SET-THEORETIC STRUCTURES

Chapter overview: This chapter introduces some set-theoretic concepts that serve as the basis for various mathematical structures and algorithms in computer science. More specifically, the chapter covers mathematical induction, sets with their properties, relations, equivalence relations, ordering relations, partition, and functions with their characteristics. The material covered in this chapter not only equips readers with essential mathematical tools but also prepares them for more advanced studies in mathematics and computer science. The chapter concludes with a set of exercises to encourage readers to apply their knowledge and enrich their understanding.

Keywords: Mathematical induction, Sets, Relations, Functions

Contents

B. Alzalg, *Combinatorial and Algorithmic Mathematics: From Foundation to Optimization*

Set theory concepts that we study in this chapter provide the fundamental underpinning for a wide range of mathematical structures and algorithms within the field of computer science. The chapter begins with mathematical Induction, which is widely used in establishing the validity of statements, laying the groundwork for sound mathematical reasoning.

2.1 Induction

Mathematical induction is a powerful and elegant technique for proving certain types of mathematical statements. As explained in Figure 2.1, there are two steps involved in knocking over a row of dominoes. These steps are the same as the steps in a proof by induction. See Figure 2.1 which shows the idea of induction.

We have an infinite number of claims that we wish to prove: Claim (1), Claim (2), Claim (3), . . . , Claim (n), ... We have the following workflow principle.

Workflow 2.1 *If we can perform the following two steps, then we are assured that all these claims are true:*

(i) *Prove that Claim (1) is true.*

(ii) *Prove that, for every natural number k, if Claim (k) is true, then Claim ($k+1$) is true.*

In Workflow 2.1, (i) represents the knocking over the first domino, and (ii) shows that if the k^{th} domino is knocked over, then it follows that the $(k+1)^{\text{st}}$ domino will be knocked over.

The process of toppling a sequence of dominoes entails a two-fold operation: Initially, the first domino must be tipped over, and secondly, it is essential to ensure that when domino k succumbs to gravity and topples, it effectively instigates the fall of domino $k+1$. The steps involved in this domino scenario notably parallel the fundamental steps of a proof by induction, where an initial base case is established, and a recursive progression is established to demonstrate that a given statement holds true for all values within a defined sequence. This intriguing analogy draws a striking resemblance between these seemingly unrelated actions, highlighting the unifying principles of causation and sequential causality.

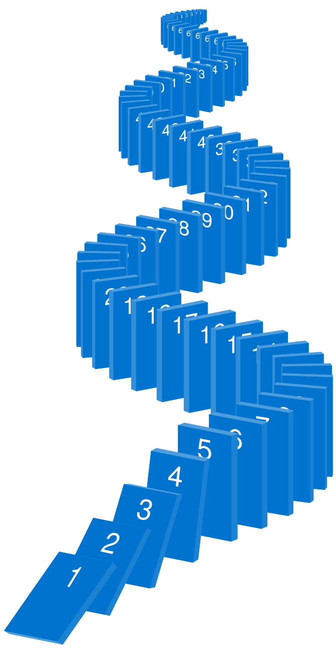


Figure 2.1: Falling dominoes.

To apply the principle of mathematical induction to sets of natural numbers, let S be a set of natural numbers (in other words, S is a subset of $\mathbb{N}$) such that

(i) $1 \in S$, and (ii) for each $k \in \mathbb{N}, k \in S \to k+1 \in S$,

then $S = \mathbb{N}$. The intuitive justification is as follows: By (i), we know that $1 \in S$. Now apply (ii) with $k = 1$, we have $2 = 1 + 1 \in S$. Now, apply (ii) with $k = 2$, we get $3 = 2 + 1 \in S$. And so forth.

Principle of induction for predicates We have the following principle.

Principle 2.1 *Let $P(n)$ be a predicate whose domain is $\mathbb{N}$ such that*

(i) *$P(1)$ is true, and*

(ii) *if $P(k)$ is true, then $P(k+1)$ is true, for all $k \in \mathbb{N}$.*

If (i) and (ii) are true, then $P(n)$ is true for all $n \in \mathbb{N}$.

In Principle 2.1, we call (i) the base case (basis step), and (ii) the inductive step, which consists the inductive hypothesis and the inductive conclusion.

Example 2.1 Use mathematical induction to prove that the sum of the first n positive integers is $n(n+1)/2$.

Solution By induction, let $P(n)$ be

$$\text{“}1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}\text{”}. \tag{2.1}$$

for $n \in \mathbb{N}$. First, we look at the base case for $n = 1$ to see whether or not $P(1)$ is true. Substituting $n = 1$ for both sides of (2.1), we get $1 = 1(1+1)/2 = 1$.

Next, we look at the inductive step for $n = k$ to see whether or not $P(k+1)$ is true if the inductive hypothesis $P(k)$

$$\text{“}1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}\text{”} \tag{2.2}$$

is true. Using the inductive hypothesis (2.2), we have

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Thus, if $P(k)$ is true, then $P(k+1)$ is true for any k greater than or equal to 1 which proves the inductive conclusion.

Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true. In other words, the equality in (2.1) holds. The proof is complete. ■

Corollary 2.1 *As one of the most known series, we have*[a]

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1),$$

which is well known in mathematics as the arithmetic series.

[a]The symbol $\sum$ is the capital Greek letter "sigma". We write $\sum_{k=1}^{n} k$ (read "the sum of k from k equals 1 to k equals n") to indicate the sum $1 + 2 + \cdots + n$.

Example 2.2 Use mathematical induction to prove that the sum of the first n positive odd integers is n^2.

Solution By induction on n, let $P(n)$ be

$$\text{“}1 + 3 + 5 + \cdots + (2n-1) = n^2\text{”} \tag{2.3}$$

for $n \in \mathbb{N}$. First, we look at the base case for $n = 1$ to see whether or not $P(1)$ is true. Substituting $n = 1$ for both sides of (2.3), we get $2n - 1 = 2(1) - 1 = 1$.

Now, we look at the inductive step for $n = k$ to see whether or not $P(k+1)$ is true if the inductive hypothesis $P(k)$

$$\text{“}1 + 3 + 5 + \cdots + (2k-1) = k^2\text{”} \tag{2.4}$$

is true. Using the inductive hypothesis (2.4), we have

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k-1) + (2(k+1)-1) &= k^2 + (2(k+1)-1) \\ &= k^2 + (2k+1) = (k+1)^2. \end{aligned}$$

Thus, if $P(k)$ is true, then $P(k+1)$ is true for any k greater than or equal to 1 which proves the inductive conclusion.

Therefore, by induction, for each $n \in \mathbb{N}$, $P(n)$ is true. In other words, the equality in (2.3) holds. This completes the proof. ■

Induction proves recursion A recurrence is a well-defined mathematical function written in terms of itself.

For example, it can be shown (see Exercise 2.5), that the following recurrence

$$T(1) = 1, \quad T(n) = 3T(n-1) + 4, \quad n = 1, 2, 3, \ldots, \tag{2.5}$$

describes the function

$$T(n) = 3^n - 2, \quad n = 0, 1, 2, \ldots . \tag{2.6}$$

Recurrences will be studied extensively in Chapter 5. In this part, we learn briefly how to use induction to verify solutions of recurrence formulas.

The formula (2.6) is called the solution of the recurrence relation (2.5) and can be obtained by using the method of mathematical induction. A simpler example is the following.

Example 2.3 Use the induction method to show that the recurrence

$$T(n) = \begin{cases} 1, & \text{if } n = 0; \\ T(n-1) + 1, & \text{if } n = 1, 2, 3, \ldots, \end{cases} \tag{2.7}$$

has the solution

$$T(n) = n + 1, \quad n = 0, 1, 2, \ldots.$$

Solution By induction on n. First, the base case for $n = 0$ is seen to be true as $T(0) = 1 = 0 + 1$, where the first equality follows from (2.7). Now, assume that the inductive hypothesis holds for $n = k$, i.e., $T(k) = k + 1$. Then

$$T(k+1) = T(k) + 1 = (k+1) + 1 = k + 2.$$

Therefore, if $T(k) = k + 1$, then $T(k+1) = k + 2$ for any integer $k \geq 0$. This proves the inductive conclusion. Thus, by induction on n, the equality in (2.7) holds. ■

2.2 Sets

Much of mathematical discrete structures are written in terms of sets. The concept of a set is used throughout mathematics, and its formal definition closely matches our intuitive understanding of the word.

Definition 2.1 *A set is an unordered collection of distinct objects.*

We can build sets containing any objects that we like, but usually, we consider sets whose elements have some property in common.

Definition 2.2 *The objects comprising a set are called its elements or members.*

Note that sets do not contain duplicates and that the order of elements in a set is not significant. A finite set is a set that has a finite number of elements. An infinite set is a set with an infinite number of elements. By convention, a set can be defined by enumerating its components in curly brackets. We have the following example.

Example 2.4 The following are examples of sets.

- Vowels = {a, e, i, o, u}.
- Weekend = {Saturday, Sunday}.
- Days = $\{1, 2, 3, \ldots, 365\}$.
- $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- Truth-Values = {true, false}.

Note that the above sets are finite, except $\mathbb{N}$ and $\mathbb{Z}$ which are infinite sets. ■

Set membership To express the fact that an object x is a member of a set A, we write $x \in A$. Note that an expression with this form is either true or false; the thing is either a member or not. For instance, in Example 2.4, the statements $\mathrm{u} \in \mathrm{Vowels}, 10 \in \mathbb{N}, 21 \in \mathrm{Days}$ are true, but the statement $-21 \in \mathrm{Days}$ is false.

Note that an expression such as $\{1,2,3\} \in \{4,5,6\}$ is not allowed, but expressions such as $\{1,2,3\} \in \{\{6,7,8\},\{1,2,3\}\}$ and $\{1,2,3\} \in \{\{6,2,1\}\}$ are allowed. (Why? And which one is true?).

Cardinality of sets The cardinality of a set A, denoted by $|A|$, is the number of elements in it. For instance, in Example 2.4, $|\mathrm{Vowels}| = 4$ and $|\mathrm{Days}| = 365$. A finite set is a set that has a finite number of elements. Therefore, in Example 2.4, Vowels and Days are finite sets, but $\mathbb{N}$ and $\mathbb{Z}$ are infinite sets.

Set equality Two arbitrary sets A and B are equal, denoted by $A = B$, if A and B contain precisely the same members. We emphasize that the order in which values occur in the set is not significant. For instance, in Example 2.4, the equality $\{1,2,3\} = \{2,3,1\}$ is true, but the equality $\mathrm{Days} = \mathbb{N}$ is false.

Subsets and proper subsets Let A and B be arbitrary sets of the same type. Then A is said to be a subset of B, denoted by $A \subseteq B$, if every member of A is also a member of B. For example, the expressions $\{1,2,3\} \subseteq \mathbb{N}$, $\mathbb{N} \subseteq \mathbb{Z}$ and $\{1,2,3\} \subseteq \{1,2,3\}$ are true, but the expressions $\{1,2,3,4\} \subseteq \{1,2,3\}$ and $\mathbb{Z} \subseteq \mathbb{N}$ are false. Note that equal sets are subsets of each other. That is, $A = B \leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$.

A set A is said to be a proper subset of a set B, denoted by $A \subset B$, if $A \subseteq B$ and $A \neq B$. For instance, in Example 2.4, the expression $\{\mathrm{Saturday}\} \subset \mathrm{Weekend}$ is true, but the expression $\{\mathrm{Monday}\} \subset \mathrm{Weekend}$ is false.

The empty set The empty set, denoted by $\{\ \}$ or $\emptyset$, is a special set that has the property of having no members. Note that nothing is a member of the empty set. That is, if x is any element, then the expression $x \in \emptyset$ must be false. Therefore $|\emptyset| = 0$, while $|\{\emptyset\}| = 1$ for instance. Note also that the empty set is a subset of every set. That is, if A is an arbitrary set, then $\emptyset \subseteq A$. This includes of course $\emptyset \subseteq \emptyset$.

The powerset operator Let A be an arbitrary set. The set of all subsets of A is called the powerset of A and is denoted as $\mathcal{P}(A)$. So, $B \in \mathcal{P}(A) \longleftrightarrow B \subseteq A$.

For example, if $A = \{1\}, B = \{1,2\}$ and $C = \{a,b,c\}$, then $\mathcal{P}(A) = \{\emptyset,\{1\}\}$, $\mathcal{P}(B) = \{\emptyset,\{1\},\{2\},\{1,2\}\}$, and $\mathcal{P}(C) = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$. Note that $\mathcal{P}(A) = 2, |\mathcal{P}(B)| = 4 = 2^2$ and $|\mathcal{P}(C)| = 8 = 2^3$. In general, for any finite set A, one can prove that $|\mathcal{P}(A)| = 2^{|A|}$. Proving this is left as an exercise for the reader (see Exercise 2.4). See also Example 6.5.

Manipulating sets Venn diagrams employ intersecting circles or other geometric forms to visualize the logical relationships among two or more sets. See Figure 2.2 which shows Venn diagrams for the most common combined sets obtained from two sets A and B, such as the union of A and B, denoted by $A \cup B$, the intersection of A and B, denoted by $A \cap B$, the complement of A, denoted by A', the difference of A and B, denoted by $A - B$, and the symmetric difference of A and B, denoted by $A \,\Delta\, B$.

Table 2.1 shows a list of most common laws of algebra of sets.

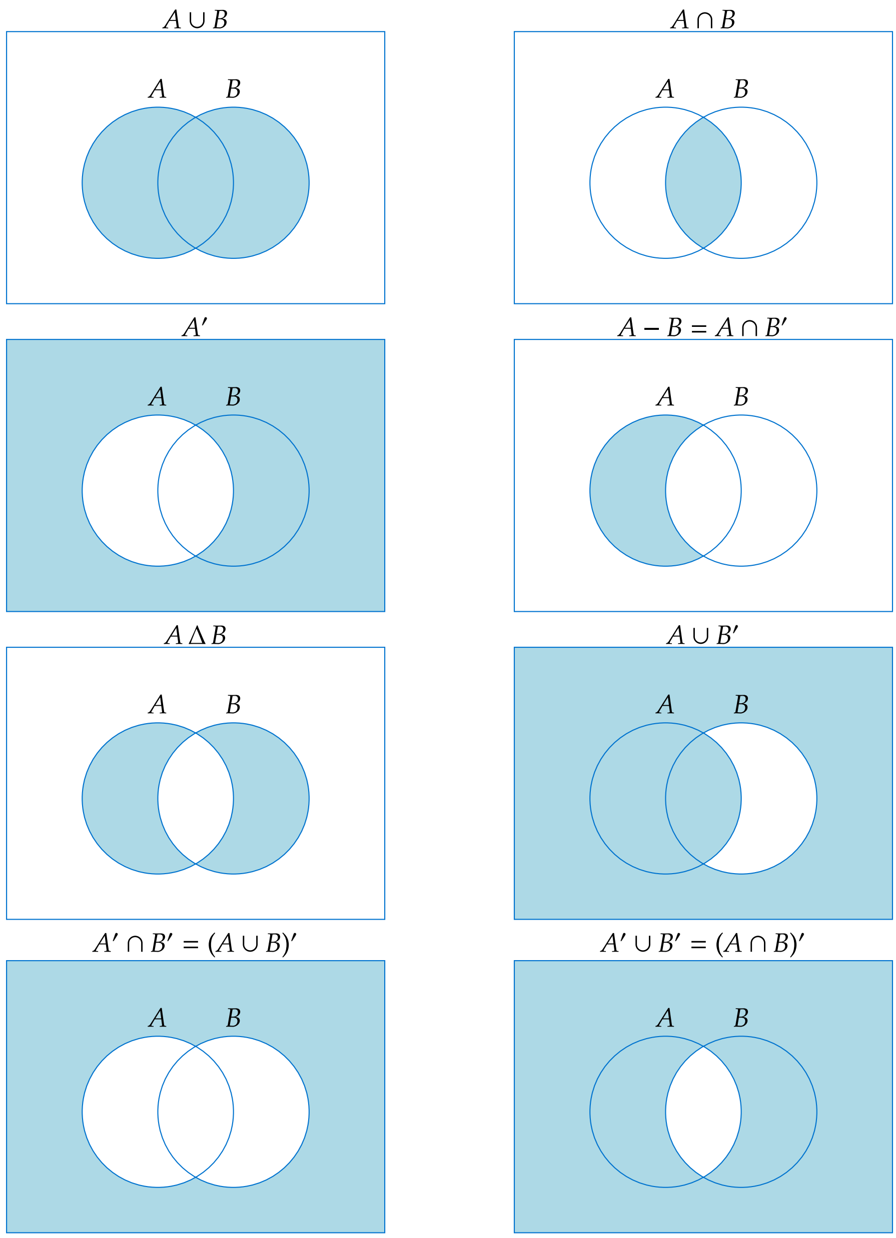


Figure 2.2: Venn diagrams for combined sets obtained from two sets A and B.

For example,

- $\{a, e, i\} \cup \{o, u\} = \{a, e, i, o, u\}$,
- $\{a, e\} \cup \{e, i\} = \{a, e, i\}$,
- $\emptyset \cup \{a, e\} = \{a, e\}$,
- $\{a, e, i\} \cap \{o, u\} = \emptyset$,
- $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$,
- $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$,
- $\{a, e, i, o, u\} - \{o\} = \{a, e, i, u\}$,
- $\emptyset - \{a, e, i, u\} = \emptyset$.

Name	Formula(s)
Identity laws	$A \cup A = A$
	$A \cap A = A$
Domination laws	$A \cap \emptyset = \emptyset$
	$A \cup S = S$
Idempotent laws	$A \cap A = A$
	$A \cup A = A$
Tautology law	$A \cup A' = S$
Contradiction law	$A \cap A' = \emptyset$
Double negation law	$A = (A')'$
DeMorgan's laws	$(A \cup B)' = A' \cap B'$
	$(A \cap B)' = A' \cup B'$
	$A - (B \cup C) = (A - B) \cap (A - C)$
	$A - (B \cap C) = (A - B) \cup (A - C)$
Contrapositive law	$(A \subseteq B) \equiv (B' \subseteq A')$
Commutative laws	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive laws	$(A \cup B) \cap C = (A \cap B) \cup (A \cap C)$
	$(A \cap B) \cup C = (A \cup B) \cap (A \cup C)$
More laws	$(A - B) \cap B = \emptyset$
	$A - B) \cup B = A \cup B$
	$A - (B \cap C) = (A - B) \cup (A - C)$
	$A - (B \cup C) = (A - B) \cap (A - C)$
	$A \cap (B - C) = (A \cap B) - (A \cap C)$
	$A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$
	$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

Table 2.1: A list of the most common laws of algebra of sets. Here, A, B and C are finite sets in a space S.

The Cartesian product of two sets A and B, denoted $A \times B$, is the set of all possible ordered pairs where the elements of A are first and the elements of B are second. For example, $\{a, b, c\} \times \{1, 2\} = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.

Sets defined by predicates The set of all elements x in a domain D such that a predicate $P(x)$ is true is denoted as $\{x \in D : P(x)\}$ and is called the set-builder notation. For example, if

$P(x)$ is

$$\text{“}x \text{ is an even integer between 1 and 11”}$$

then the set $\{x \in \mathbb{Z} : P(x)\} = \{2,4,6,8,10\}$. For another example, the set $\{n \in \mathbb{N} : (n \geq 1) \wedge (n < 4)\}$ is the set $\{1,2,3\}$ in the set-builder notation. We also have the following example.

Example 2.5 Redefine each of the following sets using the set-builder notation.

(*a*) $A = \{-1, \pi\}$. (*b*) $B = (-3, 5]$. (*c*) $\emptyset$. (*d*) $\mathbb{R}$.

Solution It is easy to see that the given sets can be redefined as follows.

(*a*) $A = \{x \in \mathbb{R} : (x+1)(x-\pi) = 0\}$.

(*b*) $B = \{x \in \mathbb{R} : (-3 < x) \wedge (x \leq 5)\}$.

(*c*) $\emptyset = \{x \in \mathbb{Z} : x + \pi = 0\}$.

(*d*) $\mathbb{R} = \{x \in \mathbb{R} : x^2 + 1 > 0\}$.

The desired answer is obtained. ■

Let S be a space that contains the sets A and B. We can write the union, intersection, complement, difference, and Cartesian product introduced above in set-builder notation as follows.

$$\begin{array}{lcl} A \cup B & = & \{x \in S : x \in A \ \vee \ x \in B\}, \\ A \cap B & = & \{x \in S : x \in A \ \wedge \ x \in B\}, \\ A' & = & \{x \in S : x \notin A\}, \\ A - B & = & \{x \in S : x \in A \ \wedge \ x \notin B\}, \\ A \,\Delta\, B & = & \{x \in S : x \in A - B \ \vee \ x \in B - A\}, \\ A \times B & = & \{(a,b) : a \in A \text{ and } b \in B\}. \end{array}$$

2.3 Relations

The concept of a relation is important in mathematics and computer science. For example, relations are used in the definition of functions which will be introduced in Section 2.5. Relations can be also used in the definition of directed graphs which will be introduced in Section 4.5.

Given a set of objects, we may want to say that certain pairs of objects are related in some way. For example, we may say that two students are related if they are attending the same university or working for your company, or if they are from the same hometown. Mathematically speaking, we have the following definition.

Definition 2.3 *A binary relation $\mathcal{R}$ on two sets A and B is a subset of the Cartesian product $A \times B$. The notation $a \,\mathcal{R}\, b$ is read a is $\mathcal{R}$-related (or simply related) to b and it means that $(a,b) \in \mathcal{R}$. If $(a,b) \notin \mathcal{R}$, we write $a \not\mathcal{R}\, b$. A relation from A to A is called a relation on A.*

There are several ways to represent a relation in a usable form. For example, we can list the ordered pairs inside set brackets, construct its corresponding table, or plot its corresponding rectangular coordinate graph. We can also describe a relation with an expression such as an equality or inequality. We have the following example.

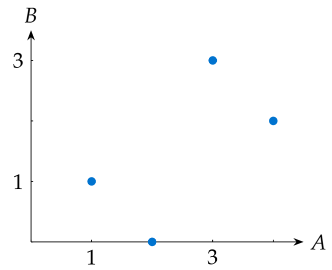
Example 2.6

Let $A = \{0, 1, 2, 3, 4\}$ and $B = \{0, 1, 2, 3\}$. We represent a binary relation $\mathcal{R}$ from the set A to the set B in three different ways. We first describe $\mathcal{R}$ by explicitly listing its ordered pairs inside set brackets:

$$\mathcal{R} = \{(1, 1), (2, 0), (3, 3), (4, 2)\}.$$

In table form, $\mathcal{R}$ can be represented by the table shown to the right. In this tabular representation, each ordered pair is illustrated, providing a structured and organized means of visualizing the relation $\mathcal{R}$. In graphical representation, $\mathcal{R}$ can be represented by the graph in the figure shown to the right.

1	2	3	4
1	0	3	2



In general, there are many different relations from a set A to a set B, because every subset of $A \times B$ is a relation from A to B. This includes the empty set $\emptyset$, which is called the empty relation from A to B, and includes the set $A \times B$, which is called the full relation (or universal relation) from A to B.

Definition 2.4 *The domain of the binary relation $\mathcal{R}$ from A to B is the set*

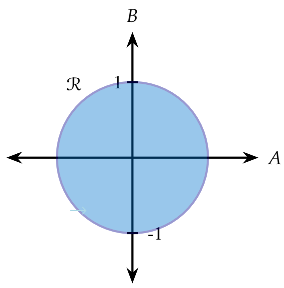
$$Dom(\mathcal{R}) = \{x \in A : \text{ there exists } y \in B \text{ such that } x \,\mathcal{R}\, y\}.$$

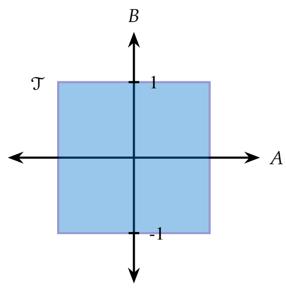
The range of the relation $\mathcal{R}$ is the set

$$Rng(\mathcal{R}) = \{y \in B : \text{ there exists } x \in A \text{ such that } x \,\mathcal{R}\, y\}.$$

Example 2.7 The graphs of the following binary relations are shown below.

$$\mathcal{R} = \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : x_1^2 + x_2^2 \leq 1\right\}, \quad \mathcal{T} = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : \max\{|x_1|, |x_2|\} \leq 1\}.$$





$\text{Dom}(\mathcal{R}) = \text{Rng}(\mathcal{R}) = [-1, 1].$ $\text{Dom}(\mathcal{T}) = \text{Rng}(\mathcal{T}) = [-1, 1].$

Definition 2.5 *An n-ary relation on sets $A_1, A_2, \ldots, A_n$ is a subset of the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$.*

As an example, the following 3-dimensional unit sphere is a ternary relation on $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

$$\mathcal{S} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} x_i^2 \leq 1 \right\}$$

Equivalence relations

Relations that exhibit and adhere to the three specific properties outlined in the forthcoming definition are of particular importance and offer substantial value within the context or domain in which they are applied or considered.

Definition 2.6 *Let A be a set and $\mathcal{R}$ be a relation on A. Then*

(i) *$\mathcal{R}$ is said to be reflexive if for all $x \in A, x \,\mathcal{R}\, x$, and irreflexive otherwise.*

(ii) *$\mathcal{R}$ is said to be symmetric if for all $x, y \in A$ if $x \,\mathcal{R}\, y$, then $y \,\mathcal{R}\, x$, and asymmetric otherwise.*

(iii) *$\mathcal{R}$ is said to be transitive if for all $x, y, z \in A$ if $x \,\mathcal{R}\, y$ and $y \,\mathcal{R}\, z$, then $x \,\mathcal{R}\, z$.*

In light of Definition 2.6 and using 1.7, if $\mathcal{R}$ is a relation on a set A, then

- $\mathcal{R}$ is not reflexive if there is some $x \in A$ such that $x \not\mathcal{R}\, x$.
- $\mathcal{R}$ is not symmetric if there are some $x, y \in A$ such that $x \,\mathcal{R}\, y$, and $y \not\mathcal{R}\, x$.
- $\mathcal{R}$ is not transitive such that there are some $x, y, z \in A$ such that $x \,\mathcal{R}\, y$ and $y \,\mathcal{R}\, z$ but $x \not\mathcal{R}\, z$.

Example 2.8 Let $A = \{1, 2, 3\}$, and define the following relations on A:

$$\begin{array}{lcl}
\mathcal{R} & = & \{(1,1),(2,2),(1,2),(2,1)\}, \\
\mathcal{S} & = & \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3)\}, \\
\mathcal{T} & = & \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\}, \\
\mathcal{U} & = & \{(1,1),(2,2),(3,3),(1,2),(2,1)\}, \\
\mathcal{V} & = & \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}.
\end{array}$$

Based on Definition 2.6, it can be seen that

- $\mathcal{R}$ is not reflexive because $3 \not\mathcal{R}\, 3$.
- $\mathcal{S}$ is reflexive, but not symmetric because $1 \,\mathcal{S}\, 3$ and $3 \not\mathcal{S}\, 1$.
- $\mathcal{T}$ is reflexive and symmetric, but not transitive because $2 \,\mathcal{T}\, 1$ and $1 \,\mathcal{T}\, 3$ but $2 \not\mathcal{T}\, 3$.
- $\mathcal{U}$ and $\mathcal{V}$ are reflexive, symmetric, and transitive on A. The next definition calls $\mathcal{U}$ and $\mathcal{V}$ equivalence relations.

■

Relations on $\mathbb{N}$:	"="	"$\leq$"	"<"
Is the relation reflexive?	Yes	Yes	No
Is the relation symmetric?	Yes	No	No
Is the relation transitive?	Yes	Yes	Yes
Is it an equivalence relation?	Yes	No	No

Table 2.2: Some relations on the natural numbers.

Definition 2.7 *A relation on a set A is called an equivalence relation on A if it is reflexive on A, symmetric, and transitive.*

Example 2.9 Among the three relations "=", "$\leq$" and "<" on $\mathbb{N}$, "=" is the only equivalence relation on $\mathbb{N}$. See Table 2.2. ■

An equivalence class is a set of elements that are considered to be similar or equivalent. We have the following definition.

Definition 2.8 *Let $\mathcal{R}$ be an equivalence relation on a set A. For $x \in A$, the equivalence class of x determined by $\mathcal{R}$ is the set*

$$x/\mathcal{R} = \{y \in A : x\,\mathcal{R}\,y\}.$$

This is read "the class of x modulo $\mathcal{R}$" or "x mod $\mathcal{R}$". When $\mathcal{R}$ is known from the context, equivalence class $x/\mathcal{R}$ is also denoted by $[x]$.

The set of all equivalence classes is called A module $\mathcal{R}$ and is denoted

$$A/\mathcal{R} = \{x/\mathcal{R} = [x] : x \in A\}.$$

The following are examples of equivalence classes.

Example 2.10 In Example 2.8, we have seen that $\mathcal{U} = \{(1,1),(2,2),(1,2),(2,1),(3,3)\}$ is an equivalence relation on $A = \{1,2,3\}$. It is also seen that

$$1/\mathcal{U} = 2/\mathcal{U} = \{1,2\} \text{ and } 3/\mathcal{U} = \{3\}.$$

Thus $A/\mathcal{U} = \{\{1,2\},\{3\}\}$.

In Example 2.8, we have also seen that $\mathcal{V}$ is an equivalence relation on A. Finding $1/\mathcal{V}, 2/\mathcal{V}$, $3/\mathcal{V}$ and $A/\mathcal{V}$ is left as an exercise for the reader. ■

Example 2.11 The relation "$\diamond$" on the set of all integers, $\mathbb{Z}$, defined as $x \diamond y$ if and only if $x^2 = y^2$ is an equivalence relation on $\mathbb{Z}$. In this example, we have $[1] = \{1,-1\}$, $[-2] = \{-2,2\}$, and $[0] = \{0\}$. In fact, for any $z \in \mathbb{Z}$, we have $[z] = \{z,-z\}$. Thus

$$\mathbb{Z}/\diamond = \{0,\{\pm1\},\{\pm2\},\{\pm3\},\ldots\}.$$

■

Example 2.12 (Congruence classes) Given an integer $n > 1$, called a modulus, two integers x and y are said to be congruent modulo n, denoted by $x \equiv y \pmod{n}$ or simply $x \equiv_n y$, if n is a divisor of their difference (i.e., if there is an integer k such that $a - b = kn$). For example, $76 \equiv_{12} 52$ because $76 - 52 = 24$ which is 2×12.

Congruence modulo n is a relation called a congruence relation. This congruence relation is an equivalence relation. This can be seen as follows:

- Reflexivity: $x \equiv x \pmod{n}$.
- Symmetry: $x \equiv y \pmod{n}$ if $y \equiv x \pmod{n}$ for all x, y, and n.
- Transitivity: If $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n}$, then $x \equiv z \pmod{n}$.

So, congruence modulo n is an equivalence relation, and the equivalence class of the integer a, denoted by $[a]_n$, is the set

$$[a]_n = \{\ldots, a - 2n, a - n, a, a + n, a + 2n, \ldots\}.$$

When the modulus n is known from the context that residue is also denoted $[a]$. This set, consisting of all the integers congruent to a modulo n, is called the congruence class, residue class, or simply residue of the integer a modulo n. ■

Ordering relations

In Table 2.2, we found that the relation "$\leq$" on $\mathbb{N}$ (or on any of the number systems $\mathbb{Z}$ or $\mathbb{R}$) is not an equivalence relation because it is not symmetric. However, the reflexively and transitivity properties of "$\leq$" relation can be used in some applications such as the Big-Oh notation of algorithms which will be studied in Chapter 7. The following definition introduces a property of relations that bridges the gap between symmetric and asymmetric relations.

Definition 2.9 *A relation $\mathcal{R}$ on a set A is called antisymmetric if, for all $x, y \in A$, if $x \,\mathcal{R}\, y$ and $y \,\mathcal{R}\, x$, then $x = y$.*

For example, the relation "$\leq$" on $\mathbb{R}$ has the property that if $x \leq y$ and $y \leq x$, then $x = y$. Therefore, this relation is antisymmetric. Other examples of antisymmetric relations will be presented after the next definition.

Note that if a relation $\mathcal{R}$ is antisymmetric, then $x \,\mathcal{R}\, y$ and $x \neq y$ implies that $y \not\mathcal{R}\, x$. Note also that a relation may be antisymmetric and not symmetric, symmetric and not antisymmetric, both, or neither. Antisymmetry is one of the prerequisites for a partial ordering on a set A. We have the following definition.

Definition 2.10 *A relation $\mathcal{R}$ on a set A is called a partial order (or partial ordering) for A if $\mathcal{R}$ is reflexive on A, antisymmetric and transitive. A set A with partial order R is called a partially ordered set, or poset.*

Besides the above example on the relation "$\leq$" which is a partial order for $\mathbb{R}$, below are two other examples of partial orders taken from Rosen [2002]. The first example shows that the relation "divides" on the set $\mathbb{N}$ of natural numbers is a partial order. Let $\mathcal{D}$ be the relation 'divides" on $\mathbb{N}$, then $1 \,\mathcal{D}\, 7,\ 7 \,\mathcal{D}\, 7,\ 7 \,\mathcal{D}\, 35$ and $13 \,\mathcal{D}\, 78$.

Example 2.13 The relation "divides" on $\mathbb{N}$, is reflexive on $\mathbb{N}$ because every natural number divides itself. Note that if a divides b and b divides c, then a divides c. This implies that the relation "divides" is transitive. Note also that if a divides b and b divides a, then $a = b$, which implies that this relation is antisymmetric. Thus, the relation "divides" is a partial order for $\mathbb{N}$. ■

Example 2.14 Let A be a set. The set inclusion relation "$\subseteq$" on the powerset of A is reflexive on $\mathcal{P}(A)$ and transitive. Note that if $A, B \in \mathcal{P}(A)$ and $A \subseteq B$ and $B \subseteq A$, then $A = B$, which means that this relation is antisymmetric. Thus, the relation "$\subseteq$" is a partial order on the powerset $\mathcal{P}(A)$ for any set A. ■

Let $\mathcal{R}$ be a relation on a set A. Two elements of A are called comparable if they are related by $\mathcal{R}$. For example, in the relation "divides" on $\mathbb{N}$, 7 and 35 are comparable but 10 and 31 are not comparable. Comparability is one of the prerequisites for a total ordering on a set A. We have the following definition.

Definition 2.11 *A partial ordering $\mathcal{R}$ on a set A is called a linear order (or total order) for A if $\mathcal{R}$ if for any two elements x ad y of A, either $x \mathcal{R} y$ or $y \mathcal{R} x$, i.e., x and y are comparable in $\mathcal{R}$. A set A with total order R is called a totally ordered set, linearly ordered set, or loset.*

For example, the relation "$\leq$" is a total order on the set $\mathbb{N}$ of natural numbers, but the "divides" relation is not a total order on $\mathbb{N}$. The total ordering will be used in Chapter 9 to define the topological ordering of graphs.

2.4 Partitions

Partitioning can be used to organize many things around us. The months of a year, for example, are partitioned in several ways: By the four seasons, by the month-lengths, by social/cultural events, etc; see Figure 2.3. A partition of a set is a grouping of its elements into nonempty subsets, in such a way that every element is included in exactly one subset. Formally, we have the following definition followed by some examples of partitions.

Definition 2.12 *Let A be a nonempty set and $\mathcal{A}$ be a set of subsets of A. Then $\mathcal{A}$ is called a partition of A if the following conditions hold:*

(*i*) *If $X \in \mathcal{A}$, then $X \neq \emptyset$.*

(*ii*) $\cup_{X \in \mathcal{A}} X = A$.

(*iii*) *If $X \in \mathcal{A}$ and $Y \in \mathcal{A}$, then $X = Y$ or $X \cap Y = \emptyset$.*

Example 2.15 Here are four different partitions of $\mathbb{N}$:

- $\{\{1\}, \{2\}, \{3\}, \ldots\}$.
- $\{A_0, A_1, A_2, A_3\}$, where $A_i = \{4k + i : k \in \mathbb{N}\}$ for $i = 0, 1, 2, 3$.
- $\{E, O\}$, where E (respectively, O) is the set of even (respectively, odd) numbers.
- $\{\mathbb{N}\}$.

■

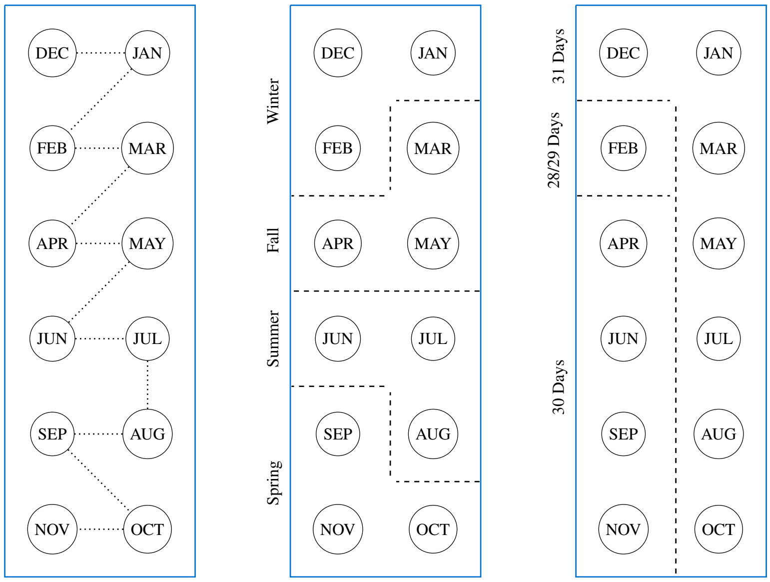


Figure 2.3: The year's months (left), and their partitions by seasons (middle) and by month-lengths (right).

Note that the first and fourth partitions in the above example are the extremes in terms of the number of elements. In fact, for any nonempty set A, $\{\{x\} : x \in A\}$ and $\{A\}$ are always partitions of A.

Example 2.16 Here are three different partitions of the set $\mathbb{R}$ of real numbers:

- $\{[n, n+1) : n \in \mathbb{Z}\}$.
- $\left\{\left[\frac{2n-1}{2}, \frac{2n+1}{2}\right) : n \in \mathbb{Z}\right\}$.
- $\{\mathbb{Q}, \mathbb{Q}'\}$, where $\mathbb{Q}$ (resp., $\mathbb{Q}'$) is the set of rational (resp., irrational) numbers.

A basic theorem of equivalence classes is the following.

Theorem 2.1 *The equivalence classes of any equivalence relation $\mathcal{R}$ in a set A from a partition of A, and any partition of A determines an equivalence relation on A for which the sets in the partition are the equivalence classes.*

Proof For the first part of the proof, we must show that the equivalence classes of $\mathcal{R}$ are nonempty, pairwise-disjoint sets whose union is A. Because $\mathcal{R}$ is reflexive, $x \in [x]$, and so the equivalence classes are nonempty; moreover, since every element $x \in A$ belongs to the equivalence class $[x]$, the union of the equivalence classes is A. It remains to show that the equivalence classes are pairwise disjoint, that is, if two equivalence classes $[x]$ and $[y]$ have an element a in common, then they are in fact the same set. Suppose that $x \,\mathcal{R}\, a$ and $y \,\mathcal{R}\, a$. By symmetry, $a \,\mathcal{R}\, y$, and by transitivity, $x \,\mathcal{R}\, y$. Thus, for any arbitrary element $b \in [x]$, we have $b \,\mathcal{R}\, x$ and, by transitivity, $b \,\mathcal{R}\, y$, and thus $[x] \subseteq [y]$. Similarly, $[y] \subseteq [x]$, and thus $[x] = [y]$.

For the second part of the proof, let $\mathcal{A} = \{A_i\}$ be a partition of A, and define $\mathcal{R} = \{(x, y) :$ there exists i such that $x \in A_i$ and $y \in A_i\}$. We claim that $\mathcal{R}$ is an equivalence relation on A. Reflexivity holds, since $x \in A_i$ implies $x \,\mathcal{R}\, y$. Symmetry holds, because if $x \,\mathcal{R}\, y$, then x and y are in the same set A_i, and hence $y \,\mathcal{R}\, x$. If $x \,\mathcal{R}\, y$ and $y \,\mathcal{R}\, z$, then all three elements are in the same set A_i, and thus $x \,\mathcal{R}\, z$ and transitivity holds. To see that the sets in the partition are the equivalence classes of $\mathcal{R}$, observe that if $x \in A_i$, then $a \in [x]$ implies $a \in A_i$, and $a \in A_i$ implies $a \in [x]$. The proof is complete. ■

Example 2.17 The set $\mathcal{B} = \{B_0, B_1, B_2\}$ is a partition of $\mathbb{Z}$, where

$$B_0 = \{3k : k \in \mathbb{Z}\}, \; B_1 = \{3k + 1 : k \in \mathbb{Z}\}, \; \text{ and } B_2 = \{3k + 2 : k \in \mathbb{Z}\}.$$

The integers x and y are in the same set B_i iff $x = 3n + i$ and $y = 3m + i$ for some integers n and m or, in other words, iff $x - y$ is a multiple of 3. Thus, the equivalence relation associated with the partition $\mathcal{B}$ is the relation of congruence modulo 3 and each B_i is the residue class of i modulo 3, for $i = 0, 1, 2$. ■

2.5 Functions

Functions are extremely important in mathematics and computer science. For example, but not limited to: In Section 3.1, sequences will be defined using functions. In Sections 5.1–5.4, solutions of recurrences will be expressed as functions. In Chapter 7, functions are used to represent how long it takes a computer program to solve problems of a given size.

The concept of a function is very old, but it is relatively recently that it has become standard to define a function as a relation with special properties. We have the following definition.

Definition 2.13 *A function (or mapping) from a set A to a set B is a relation f from A to B such that if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. We write $f : A \to B$ and this is read "f is a function from A to B", or "f maps A to B". The set A is called the domain of f, and B is called the codomain of f. In the case where $A = B$, we say that f is a function on A.*

Note that, in Definition 2.13, no restriction is placed on the sets A and B. They may be sets of numbers, ordered pairs, functions, or even sets of sets of functions.

Example 2.18 The following sets are relations from the set $A = \{1, 2, 3\}$ to the set $B = \{\text{a, b, c}\}$.

$$\begin{aligned} \mathcal{R} &= \{(1, \text{a}), (2, \text{a}), (2, \text{b}), (3, \text{c})\}, & \mathcal{T} &= \{(1, \text{c}), (2, \text{c}), (3, \text{a})\}, \\ \mathcal{S} &= \{(1, \text{b}), (2, \text{a}), (3, \text{c})\}, & \mathcal{U} &= \{(1, \text{a}), (3, \text{c})\}. \end{aligned}$$

The relation $\mathcal{R}$ is not a function from A to B because (2, a) and (2, b) are distinct ordered pairs with the same first coordinates. See Figure 2.4.

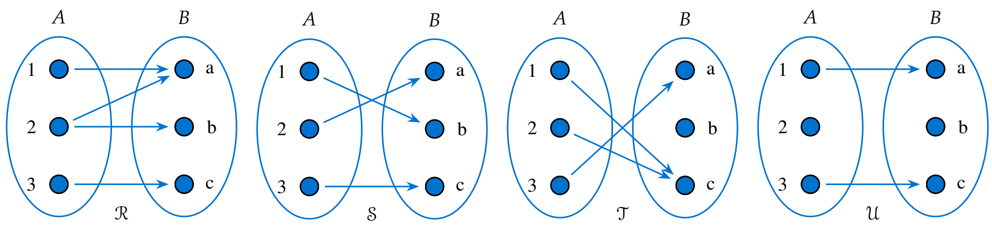


Figure 2.4: Different relations from A to B, some of them are functions from A to B.

The relations $\mathcal{S}$ and $\mathcal{T}$ are functions from A to B. The relation $\mathcal{U}$ is a function from $\{1, 3\}$ to B because the domain of $\mathcal{U}$ is not A, but the set $\{1, 3\}$. ■

The vertical line test is a graphical method for testing whether the graph of a relation represents a graph of a function. The test states that if every vertical line intersects the graph of a relation at most once, then the relation is a function.

Example 2.19 In Figure 2.5, we show two graphs. The graph shown on the left-hand side of Figure 2.5 is for the relation $\mathcal{R}_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ with domain $[-1, 1]$. Since vertical segments intersect the graph of $\mathcal{R}_1$ in two different point, $\mathcal{R}_1$ is not a function from $[-1, 1]$ to $\mathbb{R}$. The graph shown on the right-hand side of Figure 2.5 is for the relation $\mathcal{R}_2 = \{(x, y) \in [-\pi, \pi] \times \mathbb{R} : y = \sin x\}$ with domain $[-\pi, \pi]$. Since every vertical segment intersects the graph of $\mathcal{R}_2$ at at most one point, $\mathcal{R}_2$ is a function from $[-\pi, \pi]$ to $\mathbb{R}$. ■

The function $y = \sin x$ whose graph is shown on the right-hand side of Figure 2.5 is the standard trigonometric sine function. There are different kinds of common functions, such as polynomials, trigonometric, exponential, logarithmic, etc. This can be found in any calculus textbook. In particular, a polynomial of one variable is a function that has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the constant numbers $a_0, a_1, a_2, \ldots, a_n$ are the coefficients.

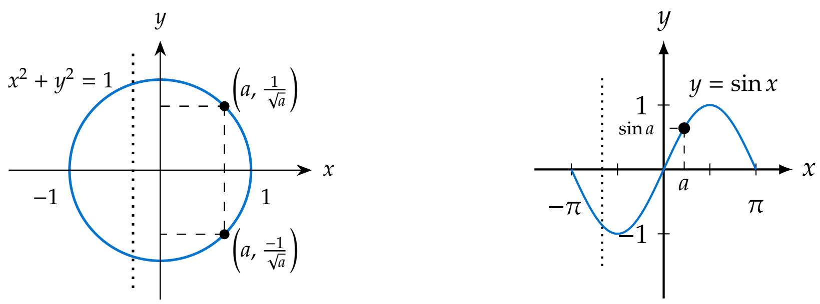


Figure 2.5: The graphs of the circle $x^2 + y^2 = 1$ and the function $y = \sin x$.

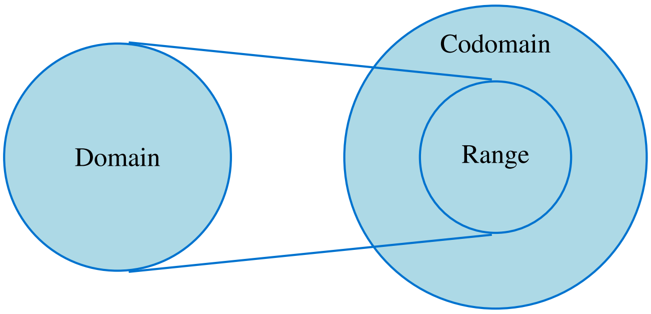


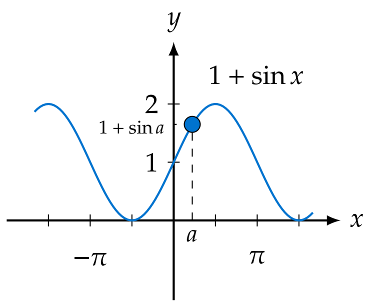
Figure 2.6: Range versus codomain.

Definition 2.14 *Let a and b be elements of sets A and B, respectively. If f is a function from A to B and $f(a) = b$, we say that b is the image of a under f and a is a preimage of b under f. The range, or image, of f is the set of all elements of A.*

Figure 2.6 visually illustrates the difference between the terms range and codomain. When we define a function we specify its domain, its codomain, and the mapping of elements of the domain to elements in the codomain.

Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1 + \sin x$. It can be seen easily by looking at the graph of $f(x)$ which is shown to the right that:
- The domain of f is $\mathbb{R}$.
- The range of f is $[0, 2]$.
- The codomain of f is $\mathbb{R}$.

Since $f(a) = 1 + \sin a$, the value $1 + \sin a$ is the image of a under f, and the value a is a preimage of $1 + \sin a$ under f.



Two functions are equal when they have the same domain, and map each element of their common domain to the same image. A function is called real-valued if its codomain is $\mathbb{R}$, and it is called integer-valued if its codomain is $\mathbb{Z}$. Figure 2.7 shows an example of an integer-valued function.

Another common example of integer-valued functions is the floor function. This function, also known the greatest integer function, has domain $\mathbb{R}$ and codomain $\mathbb{Z}$. It maps each real number x to the greatest integer n such that $n \le x$. We use the notation $\lfloor x \rfloor$ for this function. Specifically, $\lfloor \pi \rfloor = 3$ and $\lfloor -\pi \rfloor = -4$. The ceiling function is also an example of integer-valued functions. This function, also known the smallest integer function, has domain $\mathbb{R}$ and codomain $\mathbb{Z}$. It maps each real number x to the smallest integer n such that $n \ge x$. We use the notation $\lceil x \rceil$ for this function. Specifically, $\lceil \pi \rceil = 4$ and $\lceil -\pi \rceil = -3$.

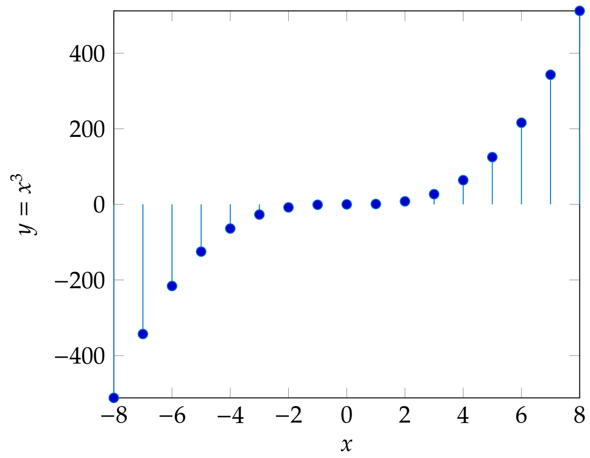


Figure 2.7: The graph of the integer-valued cubic function $y = x^3$ with domain $[-8, 8] \cap \mathbb{Z}$.

Surjections A function is said to be a surjection or an onto function if its range and codomain are the same. For example, the two functions

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R}, && \text{where} \quad f(x) = |x|, \\ g &: \mathbb{R} \to \mathbb{R}_+, && \text{where} \quad g(x) = |x|, \end{aligned}$$

are equal. Here $\mathbb{R}_+$ denotes the set of nonnegative real numbers. The functions f and g have the same range, which is $\mathbb{R}_+$, but f maps to $\mathbb{R}$ while g maps to $\mathbb{R}_+$. Therefore, f is not onto while g is onto. We formally have the following definition.

Definition 2.15 *A function $f : A \to B$ is said to be onto, or a surjection, if every element in B has a preimage, i.e., for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.*

Example 2.20 To show that the function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 2x + 1$, is onto, we must show that for every $t \in \mathbb{R}$, there exists $s \in \mathbb{R}$ such that $f(s) = t$. Let $t \in \mathbb{R}$, and choose $s = (t - 1)/2$. Then $f(s) = 2s + 1 = 2((t - 1)/2) + 1 = t$. Thus, f is onto. ■

In light of Definition 2.15, to show that a function $f : A \to B$ is not a surjection, we need to find a particular $b \in B$ such that $f(a) \neq b$ for all $a \in A$. We have the following example.

Example 2.21 To show that the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = x^2$, is not onto, we must find a value $t \in \mathbb{R}$ that has no preimage in $\mathbb{R}$. Let $t = -1$. Since $x^2 \geq 0$ for every $x \in \mathbb{R}$, there is no $x \in \mathbb{R}$ such that $g(x) = -1$. Thus g is not onto. ■

Injections A function f is said to be an injection or a one-to-one function if no two elements in the domain of f with equal images. For example, the two functions

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R}, && \text{where} \quad f(x) = |x|, \\ g &: \mathbb{R}_+ \to \mathbb{R}, && \text{where} \quad g(x) = |x|, \end{aligned}$$

are different. The functions f and g have the same range, which is $\mathbb{R}_+$, but the domain of f is $\mathbb{R}$ while the domain of g is $\mathbb{R}_+$. Note that $f(1) = f(-1) = 1$, which means that there are two different numbers $-1, 1 \in \mathbb{R}$ with equal images under f. In contrast, no two different numbers in $\mathbb{R}_+$ with equal images under g. Therefore, f is not one-to-one while g is one-to-one. We formally have the following definition.

Definition 2.16 *A function $f : A \to B$ is said to be one-to-one, or an injection, if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.*

Example 2.22 Determine whether each of the following functions is one-to-one on its domain. Justify your answer.

(*i*) $f(x) = x^3 - 8$. (*ii*) $g(x) = \dfrac{x}{x-8}$. (*iii*) $h(x) = x^4 + 3$.

Solution (*i*) Clearly, the domain of f is $\mathbb{R}$. Let $a, b \in \mathbb{R}$, then

$$f(a) = f(b) \implies a^3 - 8 = b^3 - 8 \implies a^3 = b^3.$$

It follows from this that $a = b$. Thus, f is one-to-one.

(*ii*) The domain of g is $\mathbb{R} - \{8\}$. Let $a, b \in \mathbb{R} - \{8\}$, then

$$g(a) = g(b) \implies \frac{a}{a-8} = \frac{b}{b-8} \implies ab - 8a = ab - 8b \implies 8a = 8b.$$

It follows from this that $a = b$. Thus, g is one-to-one.

(*iii*) Clearly, the domain of h is $\mathbb{R}$. Let $a, b \in \mathbb{R}$, then

$$h(a) = h(b) \implies a^4 + 3 = b^4 + 3 \implies a^4 = b^4.$$

It does not follow from this that $a = b$. In fact, this failed "proof" suggests a way to find real numbers with equal images. Indeed, $h(1) = 4 = h(-1)$ while $1 \neq -1$. This shows that h is not one-to-one.

■

In light of Definition 2.16, to show that a function $f : A \to B$ is not an injection, we need to find particular elements $a, b \in A$ such that $a \neq b$ and $f(a) = f(b)$. We have the following example.

Example 2.23 Which one of the following functions is not an injection?

(*i*) $f_1(x) = \sin x$ for $x \in [0, \pi]$.

(*ii*) $f_2(x) = |x + 1|$ for $x \geq 0$.

(*iii*) $f_3(x) = x^2 + 1$ for $x \geq 0$.

(*iv*) $f_4(x) = \cos x$ for $x \in [-\pi, 0]$.

Solution The correct answer is (*i*). In fact, the function $f_1(x) = \sin x$, $x \in [0, \pi]$, is not an injection because $f_1(\pi/3) = f_1(2\pi/3) = \sqrt{3}/2$ while $\pi/3 \neq 2\pi/3$. ■

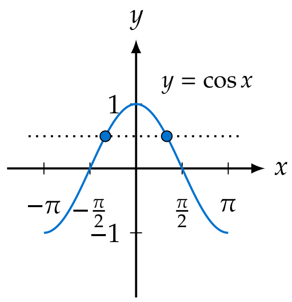


Figure 2.8: The graphs of the function $y = \cos x$ with two different domains.

The horizontal line test is a graphical method for testing whether the graph of a function represents a graph of an injection. The test states that if every horizontal line intersects the graph of a function at most once, then the function is an injection.

Example 2.24 In Figure 2.8, we show two graphs. The graph shown on the left-hand side of Figure 2.8 is for the trigonometric function $f(x) = \cos x$ with domain $[-\pi, \pi]$. Since vertical segments intersect this graph in two different point, $f(x)$ with domain $[-\pi, \pi]$ is not an injection. The graph shown on the right-hand side of Figure 2.8 is for the same function, $f(x) = \cos x$, but with domain $[0, \pi]$. Since every vertical segment intersects this graph in at most one point, $f(x)$ with domain $[0, \pi]$ is an injection. ■

Bijections Bijective functions are essential to many areas of mathematics. For example, in Section 4.2, we will see that bijections arise in the definition of graph isomorphism.

Definition 2.17 *A function is said to be a one-to-one correspondence, or a bijection, if it is both injection and surjection.*

The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $g(x) = |x|$ is an example of a bijection. See also Figure 2.9 which shows examples of different types of functions. Only the function shown on the right-hand side of Figure 2.9 is bijection.

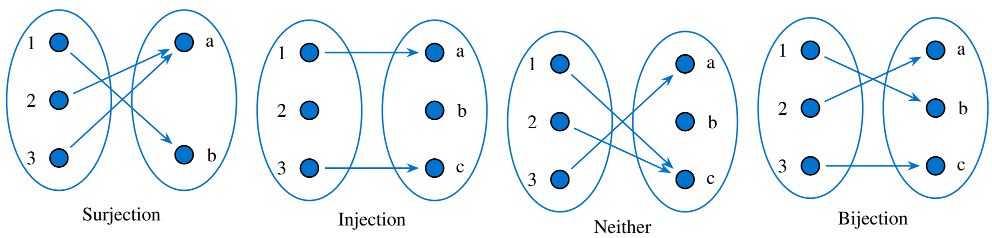


Figure 2.9: Examples of different types of functions.

Exercises

2.1 Choose the correct answer for each of the following multiple-choice questions/items.

(a) One of the following statements is true about mathematical induction proofs, which is

(i) Every mathematical induction proof is a two-part proof, and only the first part is necessary.

(ii) Every mathematical induction proof is a two-part proof, and only the second part is necessary.

(iii) Every mathematical induction proof is a two-part proof, and both parts are absolutely necessary.

(iv) None of the above is true.

(b) If induction on n is used to prove that

$$\sum_{i=1}^{n} i^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1), \quad \text{for } n \in \mathbb{N},$$

then the base case (basis step) is

(i) $\sum_{i=1}^{n} 1^4 = \frac{1}{30} 1(1+1)(2+1)(3+3-1),\ n \in \mathbb{N}.$

(ii) $\sum_{i=1}^{n} i^1 = \frac{1}{30} 1(1+1)(2+1)(3+3-1),\ n \in \mathbb{N}.$

(iii) $\sum_{i=1}^{1} i^4 = \frac{1}{30} 1(1+1)(2+1)(3+3-1).$

(iv) none of the above.

(c) If induction on n is used to prove that

$$\sum_{i=1}^{n} i^3 = \frac{1}{4} n^2(n+1)^2, \quad \text{for } n \in \mathbb{N},$$

then the inductive step shows that

(i) $\sum_{i=1}^{k+1} i^3 = \frac{1}{4}(k+1)^2(k+2)^2.$

(ii) $\sum_{i=1}^{k+1} i^3 = \frac{1}{4}(k+1)^2(k+2)^2 \longrightarrow \sum_{i=1}^{k} i^3 = \frac{1}{4}k^2(k+1)^2.$

(iii) $\sum_{i=1}^{k} i^3 = \frac{1}{4}k^2(k+1)^2 \longrightarrow \sum_{i=1}^{k+1} i^3 = \frac{1}{4}k^2(k+1)^2.$

(iv) $\sum_{i=1}^{k} i^3 = \frac{1}{4}k^2(k+1)^2 \longrightarrow \sum_{i=1}^{k+1} i^3 = \frac{1}{4}(k+1)^2(k+2)^2.$

(d) If we list all of the proper subsets of the set $\{\{\emptyset\}\}$, we get

(i) $\emptyset$.

(ii) $\emptyset, \{\emptyset\}$.

(iii) $\{\emptyset\}$.

(iv) no proper subsets.

(e) Let D be a domain and $X = \{x \in D : P(x)\}$. Only one of the following statements is false, which is

(i) $a \in X \longrightarrow P(a)$.
(ii) $P(a) \longrightarrow a \in X$.
(iii) $\neg P(a) \longrightarrow a \notin X$.
(iv) $\neg P(a) \longrightarrow a \notin D$.

(f) The binary relation $\{(a,a),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b)\}$ on the set $\{a,b,c\}$ is

(i) reflective, symmetric and transitive.
(ii) irreflexive, symmetric and transitive.
(iii) irreflexive and antisymmetric.
(iv) neither reflective, nor irreflexive but transitive.

(g) The relation $x \mathcal{R} y$ if $|x| = |y|$ is

(i) transitive and symmetric.
(ii) reflexive, symmetric and transitive.
(iii) reflexive and asymmetric.
(iv) irreflexive, antisymmetric and transitive.

(h) Let $A = \{1,2,3,4,5,6,7\}$. Which one of the following is not a partition of A?

(i) $\{\{1,2,5\},\{3,6\},\{4,7\}\}$.
(ii) $\{\{1,2,5,7\},\{3\},\{4,6\}\}$.
(iii) $\{\{1,5,7\},\{3,4\},\{2,5,6\}\}$.
(iv) $\{\{1,2,3,4,5,6,7\}\}$.

(i) Which one of the following relations is not a function?

(i) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = |x|\}$.
(ii) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : x = |y|\}$.
(iii) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x^3\}$.
(iv) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 = y^3\}$.

(j) Which one of the following functions is not a bijection?

(i) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x\}$.
(ii) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x^3\}$.
(iii) $\{(x,y) \in \mathbb{R}_+ \times \mathbb{R} : y = |x|\}$.
(iv) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x^2\}$.

2.2 Archimedean principle states that for all natural numbers n and m, there exists a natural number s such that $m < sn$. This principle can be proven by induction on n. To prove this for $n = 1$, choose s to be $m + 1$, then $m < m + 1 = sn$. So the base step is true. Assuming the principle is true when $n = k$ for some $k \in \mathbb{N}$, complete the induction proof by establishing the inductive step.

2.3 Use mathematical induction to prove that, for all $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

2.4 Use mathematical induction to prove that the cardinality of the powerset of a finite set A is equal to 2^n if the cardinality of A is n. Show all the steps in the proof.

2.5 Use the induction method to show that the recurrence

$$T(n) = \begin{cases} 1, & \text{if } n = 1; \\ 3T(n-1) + 4, & \text{if } n = 2, 3, 4, \ldots, \end{cases}$$

has the solution given in (2.5).

2.6 Let A, B and C be any three sets. Decide whether each of the following implications true or false? And if it is false, give an example that shows that.

(*a*) $(A \in B) \wedge (B \in C) \longrightarrow (A \in C)$. (*b*) $A \cup C \subseteq B \cup C \longrightarrow A \subseteq B$.

2.7 Find the number of subsets of the given set. Justify your answer.

(*a*) $S = \{$fundamentals, discrete, structures, combinatorics, optimization$\}$.

(*b*) $T = \{n \in \mathbb{N} : n$ is an even number between 21 and 41$\}$.

2.8 Let $S = \{$q, r, s, t, u, v, w, x, y, z$\}$ be a space or universe of three sets A, B and C, where $A = \{$q, s, u, w, y$\}, B = \{$q, s, y, z$\}$, and $C = \{$v, w, x, y, z$\}$. Identify the set $(A \cup C') \cap B'$ by listing its members in set braces.

2.9 Let $\mathcal{R}$ be the relation on a set of all people defined as $x \mathcal{R} y$ if and only if x and y have the same birthday (out of 365 possible birthdays). Show that $\mathcal{R}$ is an equivalence relation. How many equivalence classes does $\mathcal{R}$ have?

2.10 Show that the exponential function $f : \mathbb{R} \to [0, \infty)$ defined by $f(x) = e^x$ is a bijection. Is the exponential function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = e^x$ a bijection? Justify your answer.

Notes and sources

The history of set theory can be traced to the late 19th century, with significant contributions from mathematicians like Georg Cantor, who is often regarded as the founder of modern set theory. In the 1870s, he introduced the concept of a set as a collection of distinct objects and developed a formal theory of infinite sets. His work on the theory of transfinite numbers and the notion of different "infinities" revolutionized the field of mathematics. Cantor's book Cantor [1883] formed the basis for the growth and formalization of set theory in the 20th century.

This chapter introduced several set-theoretic concepts that serve as the foundation for various mathematical structures and algorithms in computer science. To be more precise, the chapter covered mathematical induction, sets along with their properties, relations, equivalence relations, ordering relations, partition, and functions along with their characteristics. The content included in this chapter not only provided readers with essential mathematical tools, but also prepared them for more advanced studies in mathematics and computer science.

As we conclude this chapter, it is worth noting that the cited references and others, such as Cusack and Santos [2021], Pinter [2014], Mott et al. [1986], Joshi [1989], Lipschutz [c1981.], Lin and Lin [1985], Kunen [1983], Fraenkel et al. [1973], Jech [1997], Schwartz et al. [2011], Scheurer [1994], Moschovakis [2006], Knuth [1997], also serve as valuable sources of information pertaining to the subject matter covered in this chapter. The code that created Figure 2.1 is due to StackExchange [2013]. We used and modified a code due to StackExchange [2020] to draw Venn diagrams in Figure 2.2. Exercises 2.2 and 2.6 are due to Smith et al. [2014].

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CHAPTER 3

ANALYTIC AND ALGEBRAIC STRUCTURES

Chapter overview: This chapter provides core mathematical concepts that underpin diverse areas, including combinatroics and optimization. It provides essential insights into some foundational analytic and algebraic topics, including sequences, series, matrices, subspaces, bases, polydedra, and convex cones. We also present Farkas' lemma, which is a crucial result in linear programming. The chapter serves as an indispensable resource for those seeking an understanding of the analytic and algebraic structures that form a bedrock of various disciplines. This chapter concludes with a set of exercises to encourage readers to apply their knowledge and enrich their understanding.

Keywords: Sequences and series, Subspaces, Polyhedra, Farkas' lemma

Contents

B. Alzalg, *Combinatorial and Algorithmic Mathematics: From Foundation to Optimization*

The first two sections of this chapter study analytic concepts, such as sequences, summations and series, which are used extensively in solving recurrences in Chapter 5. The last three sections of this chapter study algebraic and geometric structures, such as subspaces, bases, convex sets, convex cones, convex hulls, and polyhedra, which are, together with Farkas' lemma of Section 3.5, used extensively in solving linear programming in Chapter 10.

3.1 Sequences

A sequence of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$. The values $a(1), a(2), \ldots$, which for simplicity are usually written as $a_1, a_2, \ldots$, are called the terms of the sequence. In particular, a_n is called the nth term of a. The sequence is often written as $\{a_1, a_2, a_3, \ldots\}$, $\{a_n\}_{n=1}^{\infty}$, or even more simply $\{a_n\}$.

Example 3.1 The first five terms of the sequence $\{a_n\} = \{\frac{n}{n+2}\}$ are

$$a_1 = 1/3, a_2 = 2/4, a_3 = 3/5, a_4 = 4/6, \text{ and } a_5 = 5/7.$$

■

Example 3.2 It is not hard to see that the nth term of the sequence that has the five given terms $b_1 = -1/4, b_2 = 2/9, b_3 = -3/16, b_4 = 4/25$ and $b_5 = -5/36$ is

$$b_n = (-1)^n \frac{n}{(n+1)^2}.$$

■

A sequence can be infinite, as in Examples 3.1 and 3.2, or finite if it has a limited number of terms.

A sequence can also defined by a recurrence relation, which expresses each term as a combination of the previous terms. For example, the Fibonacci sequence[1], $f_0, f_1, f_2, \ldots$, is defined by the recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \text{ for } n = 2, 3, 4, \ldots, \quad \text{where } f_0 = 0 \text{ and } f_1 = 1.$$

Using the recurrence formula we have

$$\begin{aligned} f_2 &= f_1 + f_0 = 1 + 0 = 1, & f_5 &= f_4 + f_3 = 3 + 2 = 5, \\ f_3 &= f_2 + f_1 = 1 + 1 = 2, & f_6 &= f_5 + f_4 = 5 + 3 = 8, \\ f_4 &= f_3 + f_2 = 2 + 1 = 3, & f_7 &= f_6 + f_5 = 8 + 5 = 13. \end{aligned}$$

Solving recurrences is an important subject that will be studied exclusively in this chapter.

A sequence $\{a_n\}$ is called bounded above if there is a number M such that $a_n \le M$ for all n. Here, M is called an upper bound for $\{a_n\}$. A sequence $\{a_n\}$ is called bounded below if there is a number m such that $a_n \ge m$ for all n. Here, m is called a lower bound for $\{a_n\}$. The sequence is called bounded if it is both bounded above and bounded below. For example,

[1]The Fibonacci sequence is named after the Italian mathematician who was born in 1170.

since $0 \leq \frac{1}{n} \leq 2$ for every n, the sequence $\{\frac{1}{n}\}$ is bounded below by 0 and is bounded above by 1.

Now, we study the limits of sequences. Because we are only concerned with limits involving infinity, we introduce the following definition.

Definition 3.1 (Limit) *Let $a_n : \mathbb{N} \to \mathbb{R}$ be a sequence and $L \in \mathbb{R}$. It is said that the limit of a_n, as n approaches infinity, is L, and written $\lim_{n\to\infty} a_n = L$, if for every real number $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n \geq n_0$.*

The following example illustrates how Definition 3.1 is applied.

Example 3.3 Prove that $\lim_{n\to\infty} 1/n = 0$.

Solution Let $\epsilon > 0$ and choose an integer $n_0 > 1/\epsilon$. Then $1/n_0 < \epsilon$. Now, for any $n \geq n_0$, we have

$$0 < \frac{1}{n} < \frac{1}{n_0},$$

and so

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{n_0} < \epsilon.$$

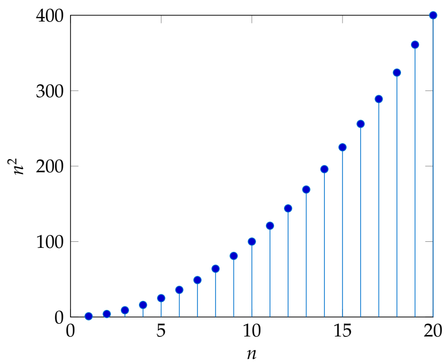
By Definition 3.1, we conclude that $\lim_{n\to\infty} 1/n = 0$. ■

Example 3.4

As n grows larger and larger without limit, the output of the mathematical function expressed as n^2 also increases without bounds. In simpler terms, the values of the quadratic function n^2 become progressively larger, without any upper bound or limit, as $n \to \infty$. See the discrete graph shown to the right. Therefore, the limit

$$\lim_{n\to\infty} n^2$$

does not exist.



■

A sequence that has a limit is said to be convergent. A sequence that has no limit is said to be divergent. For instance, the sequence $\{1/n\}$ in Example 3.3 is convergent to 0, and the sequence $\{n^2 + 4\}$ in Example 3.4 is divergent.

Theorem 3.1 *Every convergent sequence is bounded.*

Proof Assume that $a_n \to L$. Choose any positive number, say 1, and use it as ϵ. Then we can see that there exists a positive integer n_0 such that

$$|a_n - L| \leq 1 \text{ for all } n \geq n_0.$$

Note that

$$|a_n| - |L| \le ||a_n| - |L|| \le |a_n - L|.$$

It follows that

$$|a_n| \le |L| + 1 \quad \text{for all} \quad n \ge n_0.$$

Consequently, we have

$$a_n \le \max\{|a_1|, |a_2|, \ldots, |a_{n_0-1}|, L+1\} \quad \text{for all} \quad n.$$

This proves that $\{a_n\}$ is bounded, and hence proves the theorem. ■

The truth of the following corollary follows from the contrapositive law (see Table 1.8) and Theorem 3.1.

Corollary 3.1 *Every unbounded sequence is divergent.*

For example, the sequences

$$\{a_n\} = n^2 \quad \text{and} \quad \{b_n\} = n\, e^n$$

are unbounded. Therefore, each of these sequences is divergent.

Using Definition 3.1, we can prove the following theorems.

Theorem 3.2 *Let* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$, *where* L *and* M *are finite real numbers. Then*

(*i*) $\lim_{n\to\infty} k\, a_n = kL$ *for any* $k \in \mathbb{R}$.

(*ii*) $\lim_{n\to\infty}(a_n \pm b_n) = L \pm M$.

(*iii*) $\lim_{n\to\infty} (a_n b_n) = LM$.

(*iv*) *If* $M \neq 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$.

Theorem 3.3 *Let* k *be a constant. The following limits hold.*

(*i*) $\lim_{n\to\infty} k = k$.

(*ii*) $\lim_{n\to\infty} n^k = \begin{cases} \infty, & \text{if } k > 0; \\ 0, & \text{if } k < 0. \end{cases}$

(*iii*) *If* $k > 0$, $\lim_{n\to\infty} \log_k n = \infty$.

(*iv*) $\lim_{n\to\infty} k^n = \begin{cases} \infty, & \text{if } k > 1; \\ 0, & \text{if } 0 < k < 1. \end{cases}$

Theorem 3.4 *Let* $\{a_n\}$ *be a sequence. If* $\lim_{n\to\infty} a_n = \infty$, *then* $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

A well-known and very useful theorem for computing limits is the following.

Theorem 3.5 (L'Hospital's rule at infinity) *Let* $f(n)$ *and* $g(n)$ *be two real-valued differentiable functions on* $\mathbb{N}$. *If*

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = 0 \quad \text{or} \quad \lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty,$$

then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

This version of L'Hospital's rule is restricted to the limits where the variable n approaches infinity since these are only limits of interest in our context.

Example 3.5 Test the following sequences for convergence.

(*i*) $\{a_n\} = \left\{\frac{\ln n}{n^2}\right\}$. (*ii*) $\{b_n\} = \left\{\frac{n^2}{e^n}\right\}$. (*iii*) $\{c_n\} = \left\{\frac{n^2}{e^n}\right\}$.

Solution (*i*) Both numerator and denominator tend to ∞ as $x \to \infty$. Using L'Hospital's rule, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n^2} = \lim_{n\to\infty} \frac{1/n}{2n} = \lim_{n\to\infty} \frac{1}{2n^2} = 0.$$

Therefore, $\{a_n\}$ converges to 0.

(*ii*) Both numerator and denominator tend to ∞ as $x \to \infty$. Applying L'Hospital's rule twice, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{e^n} = \lim_{n\to\infty} \frac{2n}{e^n} = \lim_{n\to\infty} \frac{2}{e^n} = 0.$$

Therefore, $\{b_n\}$ converges to 0.

(*iii*) Both the numerator and denominator tend to 0 as $x \to \infty$. Using L'Hospital's rule, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{\pi/n} - 1}{1/n} = \lim_{n\to\infty} \frac{e^{\pi/n}(-\pi/n^2)}{(-1/n^2)} = \pi \lim_{n\to\infty} e^{\pi/n} = \pi.$$

Therefore, $\{c_n\}$ converges to π.

3.2 Summations and series

Summations and series are used to describe algebraic patterns. Series are of great interest because computers and smartphones use them internally to calculate many functions.

If we add the terms of an infinite sequence $\{a_k\}_{k=0}^{\infty}$ we get

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots . \tag{3.1}$$

We write

$$\sum_{k=0}^{\infty} a_k \tag{3.2}$$

(read "the sum of a sub k from k equals 0 to k equals infinity") to indicate the sum (3.1). The infinite sum (3.2) is called an infinite series. The corresponding sequence $\{s_n\}$ defined by the finite sum

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{k=0}^{n} a_k$$

is called the sequence of partial sums of the series (3.2).

Definition 3.2 *Given the infinite series* $\{a_k\}_{k=0}^{\infty}$. *If the sequence of partial sums* $\{s_n\}$, *with* $s_n = \sum_{k=0}^{n} a_k$, *converges to a finite limit* L, *then the series* $\sum_{k=0}^{\infty} a_k$ *is said to converge to* L, *written as*

$$\sum_{k=0}^{\infty} a_k = L.$$

The number L *is called the sum of the series. If the sequence of partial sums diverges, then the series* $\sum_{k=0}^{\infty} a_k$ *diverges.*

Using Definition 3.2, we can prove the following theorem.

Theorem 3.6 *If* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{k=0}^{\infty} b_k$ *are convergent series, then their sum, difference and scalar multiplication by any constant are convergent series. Moreover,*

$$(i)\ \sum_{k=0}^{\infty} a_k \pm \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} (a_k \pm b_k). \qquad (ii)\ \sum_{k=0}^{\infty} \alpha a_k = \alpha \sum_{k=0}^{\infty} a_k,\ \forall \alpha \in \mathbb{R}.$$

It is important to point out that the convergence or divergence of an infinite series is not affected by where you start the summation because the limit of the sequence of partial sums does not depend on where you begin the summation. Therefore $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=m}^{\infty} a_k$ converges, for any positive integer m.

Example 3.6 Test the following series for convergence. If it converges, find its sum.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

Solution Note that

$$\frac{1}{k(k+1)} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

It follows that the sequence of partial sums is

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}. \end{aligned}$$

Now, as $n \to \infty$, $s_n \to 1$. This means that the series converges to 1. Therefore

$$\sum_{k=0}^{\infty} \frac{1}{k(k+1)} = 1.$$

This answers the question raised in the example. ■

Infinite series with the property that its terms can be arranged in pairs with opposite signs, except for the first and last term, are called telescoping series. For instance, the infinite series given in Example 3.6 is a telescoping series.

The following theorem states that kth term of a convergent series tends to 0.

Theorem 3.7

$$\textit{If } \sum_{k=m}^{\infty} a_k \textit{ converges, then } a_k \to 0 \textit{ as } k \to \infty.$$

Proof Let

$$s_n = \sum_{k=m}^{n} a_k.$$

Since the series $\sum_{k=m}^{\infty} a_k$ converges, its sequence of partial sums, $\{s_n\}$, converges to some number L. That is, $s_n \to L$. Hence, $s_{n-1} \to L$ as well. Since $a_n = s_n - s_{n-1}$, we have $a_n \to L - L = 0$. A change in notation gives $a_k \to 0$. The proof is complete. ■

The truth of the following corollary follows from the contrapositive law (see Table 1.8) and Theorem 3.7.

Corollary 3.2 (The divergence test)

$$\textit{If } a_k \nrightarrow 0 \textit{ as } k \to \infty, \textit{ then } \sum_{k=m}^{\infty} a_k \textit{ diverges.}$$

For example, since $\lim_{k\to\infty}(k/(k+1)) = 1 \neq 0$, using the divergence test we conclude that the series $\sum_{k=0}^{\infty} \frac{k}{k+1}$ diverges.

We now introduce the geometric series, which is the most famous convergent series. The series is first presented in its finite version.

Theorem 3.8 (The finite geometric series) *Let $x \in \mathbb{R} - \{0\}$, and n and m be integers such that $0 \leq m \leq n$. Then*

$$\sum_{k=m}^{n} x^k = \begin{cases} \dfrac{x^m - x^{n+1}}{1-x} & \text{if } x \neq 1, \\ n - m + 1 & \text{if } x = 1. \end{cases} \tag{3.3}$$

Proof Let

$$s_n = \sum_{k=m}^{n} x^k.$$

To compute s_n, first multiply both sides of the equality by r and then manipulate the resulting sum as follows.

$$x s_n = x \sum_{k=m}^{n} x^k = \sum_{k=m}^{n} x^{k+1} = \sum_{j=m+1}^{n+1} x^j = \left(\sum_{j=m}^{n} x^j \right) + (x^{n+1} - x^m).$$

It follows that

$$x s_n = s_n + (x^{n+1} - x^m).$$

Solving for s_n shows that if $x \neq 1$, then

$$s_n = \frac{x^m - x^{n+1}}{1-x}.$$

If $x = 1$, then $s_n = \sum_{k=m}^{n} 1 = n - m + 1$ as desired. This proves the theorem. ■

Corollary 3.3 *Let $x \in \mathbb{R}$, and n and m be integers such that $0 \leq m \leq n$. Then*

$$\sum_{k=m}^{n} f(x) = (n - m + 1) f(x), \tag{3.4}$$

where $f(\cdot)$ is a real-valued function that does not depend on k.

The following example is a direct application of (3.3) and (3.4).

Example 3.7 Find

(*i*) $\displaystyle\sum_{k=1}^{9} \left(\frac{3}{4}\right)^k.$ (*ii*) $\displaystyle\sum_{k=1}^{9}\sum_{j=0}^{9} \left(\frac{3}{4}\right)^k.$

Solution (*i*) Using (3.3), we have

$$\sum_{k=1}^{9} \frac{3}{4}^k = \frac{\left(\frac{3}{4}\right)^{10} - \frac{3}{4}}{\frac{3}{4} - 1} = 3\left(1 - \left(\frac{3}{4}\right)^9\right) \approx 2.775.$$

(*ii*) This is an example of a double summation. To evaluate this, first compute the inner summation and then continue by computing the outer summation as follows.

$$\sum_{k=1}^{9}\sum_{j=0}^{9} \left(\frac{3}{4}\right)^k = \sum_{k=1}^{9} \left((9 - 0 + 1)\left(\frac{3}{4}\right)^k\right) = 10 \sum_{k=1}^{9} \left(\frac{3}{4}\right)^k \approx (10)(2.775) = 27.75,$$

where we used (3.4) to obtain the first equality, and used item (*i*) to obtain the approximate equality.

■

Now, we state and prove the infinite version of the geometric series.

Theorem 3.9 (The infinite geometric series) *If $x \in \mathbb{R}$, then*

$$\sum_{k=m}^{\infty} x^k = \begin{cases} \dfrac{x^m}{1-x} & \text{if } |x| < 1, \\ \text{diverges} & \text{if } |x| \geq 1. \end{cases} \tag{3.5}$$

Proof Let

$$s_n = \sum_{k=m}^{n} x^k.$$

We break the proof into two cases:
Case 1: When $|x| = 1$. If $x = 1$, then $s_n = n - m + 1$, and hence $\{s_n\}$ diverges as $n \to \infty$. Therefore $\sum_{k=m}^{\infty} x^k$ diverges. If $x = -1$, then $x^k = (-1)^k \nrightarrow 0$ as $k \to \infty$, and hence by the divergence test, the series $\sum_{k=m}^{\infty} x^k$ diverges.
Case 2: When $|x| \neq 1$, the same steps followed in the proof of Theorem 3.8 can be used to prove that

$$s_n = \frac{x^{n+1} - x^m}{x - 1}.$$

If $|x| > 1$, then $\{x^{n+1}\}$ is unbounded, and hence $\{s_n\}$ diverges as $n \to \infty$. Therefore, $\sum_{k=m}^{\infty} x^k$ diverges. If $|x| < 1$, then $\{x^{n+1}\} \to 0$ as $n \to \infty$, and hence

$$s_n \to \frac{x^m}{1 - x}.$$

Therefore

$$\sum_{k=m}^{\infty} x^k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{x^m}{1 - x} = \frac{x^m}{1 - x}.$$

This proves the theorem. ■

The following example is a direct application of (3.5).

Example 3.8 Test the series for convergence. If it converges, find its sum.

(i) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{5}{4}\right)^k.$ (ii) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{5}{4}\right)^{3-k}$

Solution (i) The series diverges as $5/4 > 1$.

(ii) We have

$$\sum_{k=1}^{\infty} \left(\frac{5}{4}\right)^{3-k} = \left(\frac{5}{4}\right)^3 \sum_{k=1}^{\infty} \left(\frac{4}{5}\right)^k = \left(\frac{5}{4}\right)^3 \frac{(4/5)^3}{1 - 4/5} = \frac{1}{1/5} = 5.$$

Thus the series converges to 5.

■

We can use the geometric series to derive an explicit formula, also called a closed form, for some series.

Example 3.9 Use the geometric series to derive a closed form for the following series.

$$\sum_{k=0}^{\infty} (-1)^k x^k, \ |x| < 1.$$

Solution Since $|-x| = |x| < 1$, using (3.5), we have

$$\sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=0}^{\infty} (-x)^k = \frac{1}{1 - (-x)} = \frac{1}{1 + x}.$$

■

The proof of the following theorem is left as an exercise for the reader.

Theorem 3.10 *Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$. Then*

$$f(x)g(x) = \sum_{k=0}^{m+n} \left(\sum_{r=0}^{k} a_r b_{k-r} \right) x^k, \tag{3.6}$$

where we use the convention that $a_i = 0$ for all integers $i > m$ and $b_j = 0$ for all integers $j > n$.

The following is the infinite version of Theorem 3.10.

Theorem 3.11 *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then*

$$f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{r=0}^{k} a_r b_{k-r} \right) x^k. \tag{3.7}$$

The following example shows how Theorem 3.11 is applied.

Example 3.10 Find a series expansion for the expression $1/(1-x)^2$, $|x| < 1$.

Solution Since $|x| < 1$, we have

$$\frac{1}{1-x} = \sum_{k=0}^{n} x^k.$$

Using Theorem 3.11, we have

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} \left(\sum_{r=0}^{k} 1 \right) x^k = \sum_{k=0}^{\infty} (k+1) x^k.$$

This answers the question raised in the example. ■

Another way to approach the series expansion for the expressions, such as that in Example 3.10, is to use Theorem 3.12. The proof of the following theorem can be found in any standard calculus textbook, see for example [Salas et al., 2006, Theorem 12.9.1].

Theorem 3.12 *If $\sum_{k=0}^{\infty} a_k x^k$ converges on $(-c, c)$, then $\sum_{k=0}^{\infty} \frac{d}{dx}(a_k x^k)$ also converges on $(-c, c)$. Moreover, if*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \textit{for all} \quad x \in (-c, c),$$

then f is differentiable on $(-c, c)$ and

$$f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}(a_k x^k) \quad \textit{for all} \quad x \in (-c, c).$$

Sum	Closed form	Sum	Closed form
$\sum_{k=1}^{n} 1$	n	$\sum_{k=0}^{n} x^k,\ x \neq 1$	$\frac{1-x^{n+1}}{1-x}$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$	$\sum_{k=0}^{\infty} x^k,\ \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$	$\sum_{k=0}^{\infty} (-1)^k x^k,\ \lvert x\rvert < 1$	$\frac{1}{1+x}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$	$\sum_{k=0}^{\infty} (k+1) x^k,\ \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Table 3.1: Some useful summation formulas.

The result in Example 3.10 can be derived from Theorem 3.12 by differentiation as follows.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{k=0}^{n} x^k\right) = \sum_{k=0}^{n} \frac{d}{dx}(x^k) \sum_{k=1}^{n} kx^{k-1} = \sum_{k=0}^{n} (k+1)x^k.$$

It is important to point out that the above differentiation is valid provided that $|x| < 1$.

Table 3.1 provides closed forms for commonly occurring summations. Some summations given in the left-hand column of Table 3.1 were proven in Section 2.1 by induction (see Corollary 2.1 and Exercises 2.1 and 2.3).

3.3 Matrices, subspaces, and bases

In this section, we review some notions and concepts from elementary linear algebra.

Matrices

Throughout this and other subsequent chapters, we will use vectors and matrices. A matrix of dimension $n \times m$ is an array of numbers a_{ij}:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}.$$

We assume that the entries of A are real. A row vector is a matrix with $n = 1$, and a column vector is a matrix with $m = 1$. The word *vector* will always mean *column vector* unless the contrary is explicitly stated. So, a vector of dimension n is an array of numbers x_i:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}.$$

Scalars will be always denoted by lower case characters, vectors will be always denoted by lower case boldface characters, and matrices will be always denoted by upper case characters.

We use $\mathbb{R}^n$ to denote the set of all nth-dimensional vectors. The standard inner product of $\mathbb{R}^n$ is defined as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \triangleq \boldsymbol{x}^{\mathsf{T}} \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i,$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Here $\boldsymbol{x}^{\mathsf{T}}$ denotes the transpose of the vector $\boldsymbol{x}$. The Euclidean norm (also called the 2-norm) of $\boldsymbol{x} \in \mathbb{R}^n$ is denoted as $\|\cdot\|$, and is defined to be the square root of the inner product of a vector with itself. That is

$$\|\boldsymbol{x}\| \triangleq \sqrt{\boldsymbol{x}^{\mathsf{T}} \boldsymbol{x}} = \sqrt{\sum_{i=1}^{n} x_i^2},$$

for $\boldsymbol{x} \in \mathbb{R}^n$.

The transpose A^{T} of an $n \times m$ matrix A is the $m \times n$ matrix

$$A^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

A square matrix is a matrix with the same number of rows and columns. A square matrix A is called symmetric if it is equal to its own transpose matrix, i.e., $A = A^{\mathsf{T}}$. We use I to denote the identity matrix, which is a square matrix whose diagonal entries are ones and its off-diagonal entries are zeros.

A square matrix is called a diagonal matrix all of the entries off the main diagonal are zero. That is, a square matrix D is diagonal if $d_{ij} = 0$ whenever $i \neq j$. A square matrix is called upper triangular if all the entries above the main diagonal are zero. Therefore, an upper triangular matrix has the form:

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & 0 & & \times & \times \\ & & & & \times \end{bmatrix}$$

Likewise, a square matrix is called lower triangular if all the entries above the main diagonal are zero.

Let A be a square matrix. If there exists a square matrix B of the same dimensions satisfying $AB = BA = I$, we say that A is invertible or nonsingular. Such a matrix B, called the inverse of A, is unique and is denoted by A^{-1}. The matrix A is called an orthogonal matrix if $A^{\mathsf{T}}A = I$. In other words, a square matrix A is orthogonal if its transpose is equal to its inverse matrix, i.e., $A^{\mathsf{T}} = A^{-1}$.

The determinant is a function whose input is a square matrix A and whose output is a number. We use the notation $\det(A)$ to denote the determinant of a square matrix A. The reader can consult any linear algebra textbook, such as Anton and Rorres [2014], to see how to compute $\det(\cdot)$.

If a system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution, it is said to be consistent. A consistent system has either exactly one solution or an infinite number of solutions. Therefore, for a system of linear equations, we have three possibilities: The system has a unique solution, it has infinitely many solutions, or it is inconsistent. We give, without proof, the following standard result in this context. For a proof, see for example [Anton and Rorres, 2014, Theorem 4.8.8].

Theorem 3.13 *Let A be a square matrix. Then, the following statements are equivalent:*

(a) *The matrix A is invertible.*

(b) *The matrix A^{T} is invertible.*

(c) *The determinant of A is nonzero.*

(d) *The rows of A are linearly independent.*

(e) *The columns of A are linearly independent.*

(f) *For every vector $\boldsymbol{b}$, the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.*

(g) *There exists some vector $\boldsymbol{b}$ such that the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.*

(h) $\det(A) \neq 0$.

In this section (and throughout the book), we use $\mathbb{R}^{n\times m}$ to denote the set of all real matrices of dimension $n \times m$.

Definition 3.3 *A matrix $A \in \mathbb{R}^{n\times n}$ is called positive semidefinite (respectively, positive definite) if it is symmetric and $x^{\mathsf{T}}A\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$ (respectively, $x^{\mathsf{T}}A\boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n - \{\boldsymbol{0}\}$).*

The eigenvalues of a matrix are the roots of its characteristic polynomial (for more detail, consult any linear algebra text; for example refer to Renardy [1996]). As an alternative to Definition 3.3, we say that a square matrix is positive semidefinite (respectively, positive definite) if it is symmetric and all its eigenvalues are nonnegative (respectively, positive). Note that every positive definite matrix is invertible.

Subspaces and bases

Subspaces and bases are introduced in this part. We have the following definition.

Definition 3.4 *A set $S \subset \mathbb{R}^n$ is a subspace if $a\boldsymbol{x} + b\boldsymbol{y} \in S$ for every $\boldsymbol{x}, \boldsymbol{y} \in S$ and every $a, b \in \mathbb{R}$. If, in addition, $S \neq \mathbb{R}^n$, we say that S is a proper subspace.*

Note that the zero vector, $\mathbf{0}$, must belong to every subspace (take $a = b = 0$). As an example, every line passing the origin is a subspace of $\mathbb{R}^2$. As another example, every plane passing the origin is a subspace of $\mathbb{R}^3$.

Definition 3.5 *We say the vectors $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(m)} \in \mathbb{R}^n$ are linearly dependent if there exists $a_1, a_2, \ldots, a_m \in \mathbb{R}$, not all of them zero, such that $\sum_{k=1}^{m} a_k \boldsymbol{x}^{(k)} = \mathbf{0}$; otherwise, they are called linearly independent.*

For example, clearly the two vectors $(1, 0)$ and $(0, 1)$ are linearly independent in $\mathbb{R}^2$. For another example, it is also clear that the three vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are linearly independent in $\mathbb{R}^3$. Note that the maximum number of linearly independent points in $\mathbb{R}^n$ is n.

As a basic fact in linear algebra, the maximum number of linearly independent rows of a matrix A is equal to the maximum number of linearly independent columns of A.

Definition 3.6 *The rank of a matrix A, denoted by rank(A), is the maximum number of linearly independent rows (columns) of A.*

Definition 3.7 *The span of a finite number of vectors $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(m)}$ in $\mathbb{R}^n$ is the subspace of $\mathbb{R}^n$ defined as the set of all vectors $\boldsymbol{y}$ of the form $\boldsymbol{y} = \sum_{k=1}^{m} a_k \boldsymbol{x}^{(k)}$, where $a_1, a_2, \ldots, a_m \in \mathbb{R}$. Any such vector $\boldsymbol{y}$ is called a linear combination of $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(m)}$.*

Definition 3.8 *A basis of a nonzero subspace $S \subset \mathbb{R}^n$ is a collection of vectors that are linearly independent and whose span is S. The dimension of a subspace is the number of vectors in any basis for the subspace.*[a]

[a] Every basis of a given subspace has the same number of vectors and this number is its dimension.

For example, the rank of the 3×3 identity matrix is 3. The space $\mathbb{R}^3$ has $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ as a span. This spanning set is also a basis. Therefore, the dimension of $\mathbb{R}^3$ is 3. In general, the dimension of $\mathbb{R}^n$ is n.

Definition 3.9 *Let S be a subspace of $\mathbb{R}^n$ and $\boldsymbol{x}_0 \notin S$ be a vector. The set $S_0 = \boldsymbol{x}_0 + S = \{\boldsymbol{x}_0 + \boldsymbol{x} : \boldsymbol{x} \in S\}$ is called an affine subspace parallel to S.*

It is not hard to see that the dimension of S_0 is equal to the dimension of the subspace S. If S is an m-dimensional subspace of $\mathbb{R}^n$ with $m < n$, there will be $n - m$ linearly independent vectors orthogonal to S. The set of such orthogonal vectors is indeed a subspace of $\mathbb{R}^n$.

Definition 3.10 *If $S \subseteq \mathbb{R}^n$ is a subspace, then the subspace*

$$S^{\perp} \triangleq \left\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^{\mathsf{T}} \boldsymbol{y} = 0 \text{ for } \boldsymbol{y} \in S\right\}$$

is called the orthogonal subspace of S.

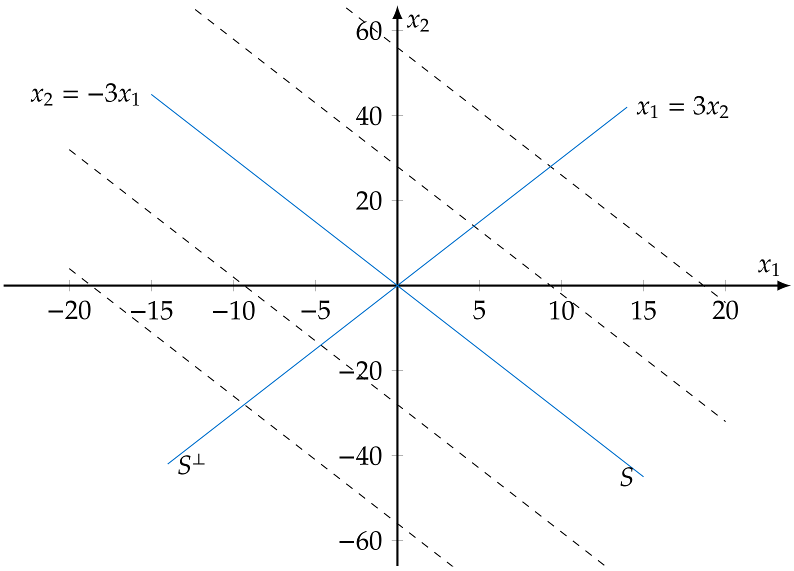


Figure 3.1: A graphical illustration for the subspace in Example 3.11 and affine subspaces parallel to it.

We give, without proof, the following standard result in this context.

Proposition 3.1 *Let A be an $m \times n$ matrix. The orthogonal of the subspace*

$$S = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{0}\}$$

is the subspace

$$S^{\perp} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = A^{\mathsf{T}}\boldsymbol{u}, \boldsymbol{u} \in \mathbb{R}^m\}.$$

Example 3.11 The line $x_2 = -3x_1$, which passes through the origin, is a subspace of $\mathbb{R}^2$; see Figure 3.1. The equation of the line can be written as

$$A\boldsymbol{x} = 0, \text{ where } A = \begin{bmatrix} 3 & 1 \end{bmatrix} \text{ and } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

All points on this line can be described by

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix} u, \text{ for } u \in \mathbb{R}.$$

So, the set $\{(-1,3)^{\mathsf{T}}\}$ is a basis for the given subspace, and hence its dimension equals 1. Note that the vector $(-1,3)^{\mathsf{T}}$ is orthogonal to this subspace and every affine subspace parallel to it. To see this, note that by Proposition 3.1 we have

$$
\begin{aligned}
\left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : 3x_1 + x_2 = 0\right\}^{\perp} &= \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : \begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0\right\}^{\perp} \\
&= \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} u, u \in \mathbb{R}\right\} \\
&= \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 = 3x_2\right\} \\
&= \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : \begin{bmatrix} -1 \\ 3 \end{bmatrix}^{\mathsf{T}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0\right\}.
\end{aligned}
$$

■

In essence, matrices, subspaces, and bases are important concepts in linear algebra, and define the foundational framework for understanding and manipulating multidimensional spaces.

3.4 Convexity, polyhedra, and cones

This section introduces the definitions of convex functions, convex sets, convex hulls, polyhedra, cones, and related notions.

Definition 3.11 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex if for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and every $\lambda \in [0, 1]$, we have*

$$f(\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) + (1 - \lambda) f(\boldsymbol{y}).$$

To visually illustrate the above definition, see Figure 3.2. Examples of convex functions on their domains are x^2, e^x, and $-\log x$.

Definition 3.12 *A set $T \subseteq \mathbb{R}^n$ is said to be convex if $\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in T$ for any $\boldsymbol{x}, \boldsymbol{y} \in T$ and $\lambda \in [0, 1]$.*

See Figure 3.3, which shows convex and nonconvex sets in $\mathbb{R}^2$. We now introduce the convex hull of points in $\mathbb{R}^n$.

Definition 3.13 *A point $\boldsymbol{x} \in \mathbb{R}^n$ is a convex combination of points of $S \subseteq \mathbb{R}^n$ if there exist a finite set of points $\{\boldsymbol{x}^{(i)}\}_{i=1}^t$ in S and $\boldsymbol{\lambda} \in \mathbb{R}^t_+$ with $\sum_{i=1}^t \lambda_i = 1$ such that $\boldsymbol{x} = \sum_{i=1}^t \lambda_i \boldsymbol{x}^{(i)}$. The convex hull of S, denoted by conv(S), is the set all points that are convex combinations of points in S.*

Figure 3.4 shows a one-dimensional convex hull and three two-dimensional convex hulls.

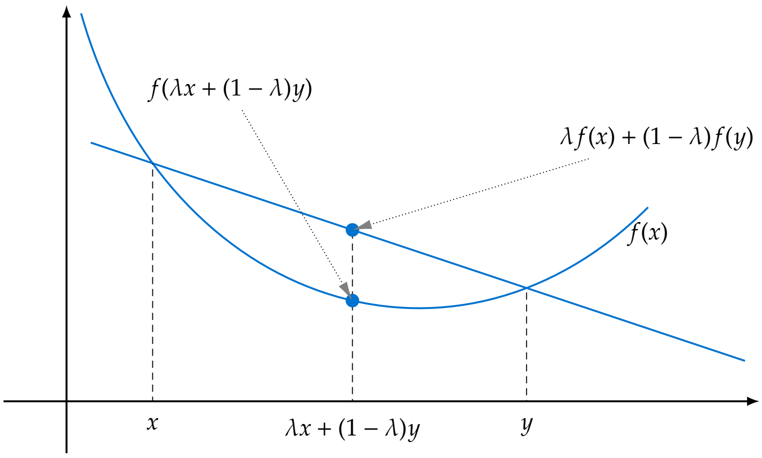


Figure 3.2: Illustration of the definition of a convex function on $\mathbb{R}$.

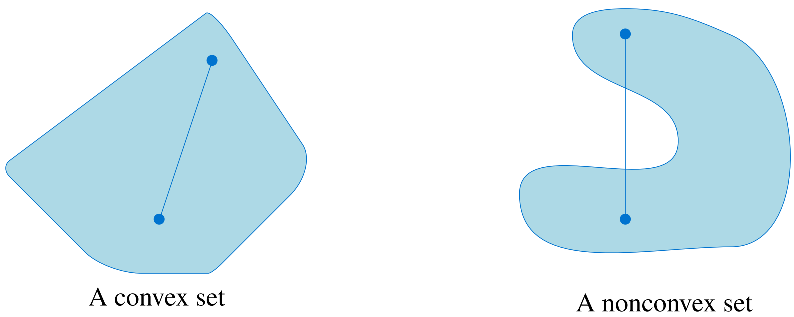


Figure 3.3: A convex set versus a nonconvex set in $\mathbb{R}^2$.

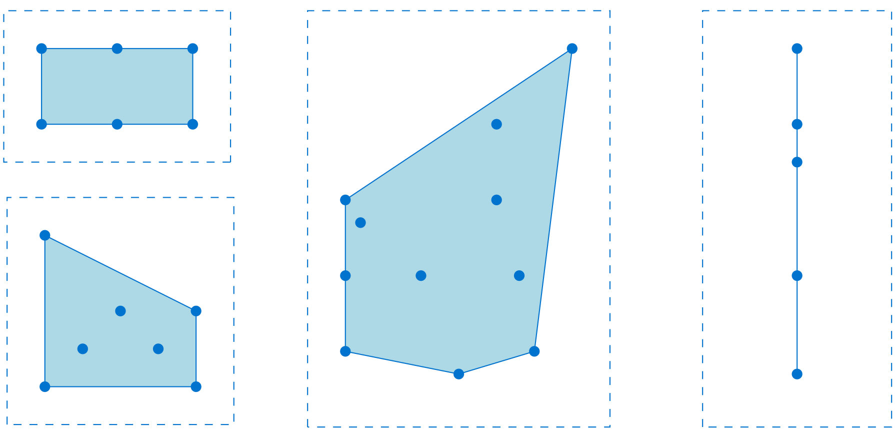


Figure 3.4: Four convex hulls; three in $\mathbb{R}^2$ and one in $\mathbb{R}$.

Lemma 3.1 *The convex hull of a finite number of vectors is a convex set.*

Proof Let H be the convex hull of a finite number of vectors, say $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(k)}$. Let also $\boldsymbol{y} = \sum_{i=1}^{t} \alpha_i \boldsymbol{x}^{(i)} \in H$, $\boldsymbol{z} = \sum_{i=1}^{t} \beta_i \boldsymbol{x}^{(i)} \in H$, where $\alpha_i, \beta_i \geq 0$, and $\sum_{i=1}^{t} \alpha_i = \sum_{i=1}^{t} \beta_i = 1$. Then, for any $\lambda \in [0, 1]$, we have

$$\lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{z} = \lambda \left(\sum_{i=1}^{t} \alpha_i \boldsymbol{x}^{(i)} \right) + (1-\lambda) \left(\sum_{i=1}^{t} \beta_i \boldsymbol{x}^{(i)} \right) = \sum_{i=1}^{t} (\lambda \alpha_i + (1-\lambda)\beta_i) \boldsymbol{x}^{(i)}.$$

Note that $\sum_{i=1}^{t} (\lambda \alpha_i + (1-\lambda)\beta_i) = \lambda \sum_{i=1}^{t} \alpha_i + (1-\lambda) \sum_{i=1}^{t} \beta_i = \lambda + (1-\lambda) = 1$. Hence, $\alpha \boldsymbol{y} + \beta \boldsymbol{z} \in H$. Thus, H is a convex set. The proof is complete. ■

A hyperplane is a subspace whose dimension is one less than that of its surrounding space. More formally, we have the following definition.

Definition 3.14 *A hyperplane is the set of points that satisfy a linear equation. That is, if $\boldsymbol{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, then the subspace $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}^{\mathsf{T}} \boldsymbol{x} = b\}$ is a hyperplane in $\mathbb{R}^n$.*

A half-space is either of the two parts into which a hyperplane divides an affine space. More formally, we have the following definition.

Definition 3.15 *A half-space is the set of points that satisfy a linear inequality. That is, if $\boldsymbol{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, then the subspace*

$$H = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}^{\mathsf{T}} \boldsymbol{x} \geq b\}$$

is a half-space in $\mathbb{R}^n$.

A polyhedron is the intersection of a finite number of half-spaces. More formally, we have the following definition.

Definition 3.16 *A polyhedron is the set of points that satisfy a finite number of linear inequalities. That is, if A is an $m \times n$ matrix and $\boldsymbol{b} \in \mathbb{R}^m$, then the subspace*

$$P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \geq \boldsymbol{b}\}$$

is a polyhedron in $\mathbb{R}^n$.

Definition 3.17 *A polyhedron $P \subseteq \mathbb{R}^n$ is said to be bounded if there exists a positive constant K such that*

$$P \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : -K \leq x_i \leq K \text{ for } i = 1, 2, \ldots, n\}.$$

Definition 3.18 *A polytope is a bounded polyhedron. That is, if A is an $m \times n$ matrix, $\boldsymbol{b} \in \mathbb{R}^m$ and K is a positive constant, then the subspace $\{\boldsymbol{x} \in [-K, K]^n : A\boldsymbol{x} \geq \boldsymbol{b}\}$ is a polytope in $\mathbb{R}^n$.*

Lemma 3.2 *A polyhedron is a convex set.*

Proof Let $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \geq \boldsymbol{0}\}$ be a given polyhedron. Let also $\boldsymbol{x}, \boldsymbol{y} \in P$ and $\lambda \in [0, 1]$. Then $A\boldsymbol{x} \geq \boldsymbol{b}$ and $A\boldsymbol{y} \geq \boldsymbol{b}$. It follows that

$$A(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) = \lambda A\boldsymbol{x} + (1-\lambda)A\boldsymbol{y} \geq \lambda\boldsymbol{b} + (1-\lambda)\boldsymbol{b} = \boldsymbol{b}.$$

Hence, $\alpha\boldsymbol{x} + \beta\boldsymbol{y} \in P$. Thus, P is a convex set. The proof is complete. ∎

Definition 3.19 *A set $\mathcal{C} \subseteq \mathbb{R}^n$ is called a cone if $\lambda\boldsymbol{x} \in \mathcal{C}$ for any $\boldsymbol{x} \in \mathcal{C}$ and $\lambda > 0$.*

Lemma 3.3 *The polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \geq \boldsymbol{0}\}$ is a cone.*

Proof Let $\boldsymbol{x}, \boldsymbol{y} \in P$ and $\alpha > 0$. Then $A\boldsymbol{x} \geq \boldsymbol{0}$. It follows that $A(\alpha\boldsymbol{x}) = \alpha A\boldsymbol{x} \geq \alpha\boldsymbol{0} = \boldsymbol{0}$. Hence, $\alpha\boldsymbol{x} \in P$. Thus, P is a cone. The proof is complete. ∎

A convex cone is a cone that is also convex, i.e., a cone that is closed under addition. So, a cone $\mathcal{C}$ is convex if $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$. The following combines Definitions 3.12 and 3.19.

Definition 3.20 *A set $\mathcal{C} \subseteq \mathbb{R}^n$ is a convex cone if $\alpha\boldsymbol{x} + \beta\boldsymbol{y} \in \mathcal{C}$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$ and $\alpha, \beta > 0$.*

The following is a corollary to Lemmas 3.2 and 3.3. It can be also proven directly using Definition 3.20.

Corollary 3.4 *The polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \geq \boldsymbol{0}\}$ is a convex cone.*

We now introduce the conic hull of points in $\mathbb{R}^n$.

Definition 3.21 *A point $\boldsymbol{x} \in \mathbb{R}^n$ is a conic combination of points of $S \subseteq \mathbb{R}^n$ if there exist a finite set of points $\{\boldsymbol{x}^{(i)}\}_{i=1}^t$ in S and $\boldsymbol{\lambda} \in \mathbb{R}^t_+$ such that $\boldsymbol{x} = \sum_{i=1}^t \lambda_i \boldsymbol{x}^{(i)}$. The conic hull of S, denoted by $\text{cone}(S)$, is the set all points that are conic combinations of points in S.*

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_t \in \mathbb{R}^n$. According to Definitions 3.13 and 3.21, we have

$$\begin{aligned} \text{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_t) &\triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = \textstyle\sum_{i=1}^t \lambda_i \boldsymbol{v}_i, \sum_{i=1}^t \lambda_i = 1, \lambda_i \geq 0, i = 1, \ldots, t\}, \\ \text{cone}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_t) &\triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = \textstyle\sum_{i=1}^t \lambda_i \boldsymbol{v}_i, \lambda_i \geq 0, i = 1, \ldots, t\}. \end{aligned}$$

Note that a convex combination differs from a conic combination in the sense that we do not require that $\sum_{i=1}^t \lambda_i = 1$ in the latter. Figure 3.5 shows conic and convex hulls of points in $\mathbb{R}^2$. A cone is said to be regular if it is a closed, convex, pointed, solid cone.

Definition 3.22 *Let $\mathcal{V}$ be a finite-dimensional Euclidean space over $\mathbb{R}$ with an inner product "$\langle \cdot, \cdot \rangle$". The dual cone of a regular cone $\mathcal{K} \subset \mathcal{V}$ is defined as*

$$\mathcal{K}^\star \triangleq \{\boldsymbol{s} \in \mathcal{V} : \langle \boldsymbol{x}, \boldsymbol{s} \rangle \geq 0, \ \forall \boldsymbol{x} \in \mathcal{K}\}.$$

A regular cone $\mathcal{K}$ is said to be self-dual if $\mathcal{K} = \mathcal{K}^\star$.

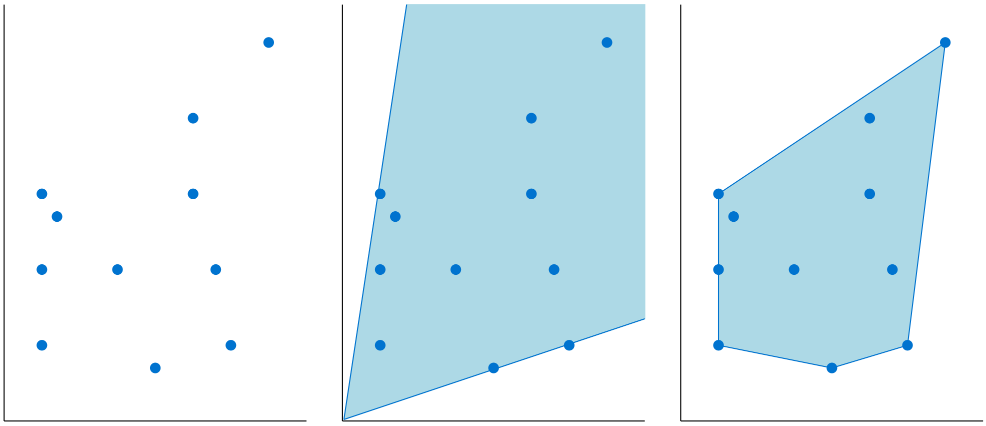


Figure 3.5: The conic and convex hulls of some points in $\mathbb{R}^2$.

Examples of self-dual cones include:

- The cone of nonnegative orthant of $\mathbb{R}^n$ (i.e., the polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq \mathbf{0}\}$ with A equals the identity matrix);
- The second-order cone (see Lemma 11.3);
- The cone of real symmetric positive semidefinite matrices;
- The cone of complex Hermitian positive semidefinite matrices;
- The cone of quaternion Hermitian positive semidefinite matrices.

Examples of non-self-dual cones include the pth-order cones, the hyperbolic cones (when $p \neq 2$), the cone of copositive matrices, the doubaly nonnegative cone, the power cone, and the exponential cone.

3.5 Farkas' lemma and its variants

Farkas' lemma plays a central role in the development of the field of optimization, and more specifically, linear programming duality (see Chapter 10). In this section, we state and prove two versions of Farkas' lemma. First, we give, without proof, the following result which will be used in the proof of Farkas' lemma.

Theorem 3.14 (Separating hyperplane) *Let $Q \subseteq \mathbb{R}^n$ be a closed, nonempty and convex set. Let $\boldsymbol{b} \in \mathbb{R}^n, \boldsymbol{b} \notin Q$. Then there exist $\mathbf{0} \neq \boldsymbol{v} \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ such that $\boldsymbol{v}^\mathsf{T}\boldsymbol{b} > \beta$ and $\boldsymbol{v}^\mathsf{T}\boldsymbol{q} < \beta$ for all $\boldsymbol{q} \in Q$.*

For a proof of Theorem 3.14, see for example Bertsimas and Tsitsiklis [1997].

Theorem 3.15 (Farkas' lemma (Version I)) *Let $A \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Then exactly one of the following two conditions holds for given A and $\boldsymbol{b}$:*

(1) $\exists \boldsymbol{x} \in \mathbb{R}^n$ *such that* $A\boldsymbol{x} = \boldsymbol{b}$ *and* $\boldsymbol{x} \geq \boldsymbol{0}$*;*

(2) $\exists \boldsymbol{y} \in \mathbb{R}^m$ *such that* $A^\top \boldsymbol{y} \geq \boldsymbol{0}$ *and* $\boldsymbol{b}^\top \boldsymbol{y} < 0$.

Proof The proof consists of two steps. At first, we prove that we cannot have both (1) and (2) simultaneously, then we prove that if (1) does not hold then (2) does.

Suppose, in the contrary, that we can have both (1) and (2) simultaneously. That is, there are $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^m$ satisfying $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{y}^\top \boldsymbol{b} < 0$, with $\boldsymbol{0} \leq \boldsymbol{x}$ and $\boldsymbol{0} \leq A^\top \boldsymbol{y}$. Then

$$0 \leq \left(\boldsymbol{y}^\top A\right)\boldsymbol{x} = \boldsymbol{y}^\top (A\boldsymbol{x}) = \boldsymbol{y}^\top \boldsymbol{b} < 0,$$

giving the desired contradiction. Thus, we cannot have both (1) and (2) together.

Now we prove that (2) holds if (1) does not hold. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ be the columns of A. Then $A\boldsymbol{x} = \sum_{j=1}^n x_j \boldsymbol{a}_j$. Note that the vector $\boldsymbol{b}$ satisfies $A\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{x} \geq \boldsymbol{0}$ if and only if $\boldsymbol{b} \notin Q$, where Q is the nonempty (as it contains $\boldsymbol{0}$), closed, convex set:

$$Q \triangleq \operatorname{cone}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = \left\{ \sum_{j=1}^n x_j \boldsymbol{a}_j : x_j \geq 0, j = 1, \ldots, n \right\}.$$

Therefore, if Condition (1) does not hold then $\boldsymbol{b} \notin Q$. The separating hyperplane theorem (Theorem 3.14) implies that there exists $\boldsymbol{0} \neq \boldsymbol{v} \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ such that $\boldsymbol{v}^\top \boldsymbol{b} > \beta$ and $\boldsymbol{v}^\top \boldsymbol{q} < \beta$ for all $\boldsymbol{q} \in Q$. In particular, because $\boldsymbol{0}, x_1\boldsymbol{a}_1, \ldots, x_n\boldsymbol{a}_n \in Q$, we have $0 = \boldsymbol{v}^\top \boldsymbol{0} < \beta$ and $\boldsymbol{v}^\top (x_j \boldsymbol{a}_j) < \beta$ for all $j = 1, 2, \ldots, n$. Letting $x_j \longrightarrow \infty$, we have $\boldsymbol{v}^\top \boldsymbol{a}_j < \beta / x_j \longrightarrow 0$, which implies that $\boldsymbol{v}^\top \boldsymbol{a}_j \leq 0$ for all $j = 1, 2, \ldots, n$. Picking $\boldsymbol{y} = -\boldsymbol{v}$, we get

$$A^\top \boldsymbol{y} = -A^\top \boldsymbol{v} = -\begin{bmatrix} \boldsymbol{v}^\top \boldsymbol{a}_1 \\ \boldsymbol{v}^\top \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{v}^\top \boldsymbol{a}_n \end{bmatrix} \geq \boldsymbol{0}, \quad \text{and} \quad \boldsymbol{b}^\top \boldsymbol{y} = -\boldsymbol{b}^\top \boldsymbol{v} < -\beta < 0.$$

Thus Condition (2) holds. The proof is complete. ∎

Theorem 3.16 (Farkas' lemma (Version II)) *Let $A \in \mathbb{R}^{m \times n}$ and $\boldsymbol{c} \in \mathbb{R}^n$. Then exactly one of the following two conditions holds for given A and $\boldsymbol{c}$:*

(1) $\exists \boldsymbol{y} \in \mathbb{R}^m$ *such that* $A^\top \boldsymbol{y} \leq \boldsymbol{c}$*;*

(2) $\exists \boldsymbol{x} \in \mathbb{R}^n$ *such that* $A\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{c}^\top \boldsymbol{x} < 0$ *and* $\boldsymbol{x} \geq \boldsymbol{0}$.

The following condition is equivalent to (2):

(2′) $\exists \boldsymbol{x} \in \mathbb{R}^n$ *such that* $A\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{c}^\top \boldsymbol{x} = -1$ *and* $\boldsymbol{x} \geq \boldsymbol{0}$.

Proof The proof consists of three steps or parts. At first, we prove that (2) and (2′) are equivalent. Then, we prove that we cannot have both (1) and (2) simultaneously. And finally, we prove that if (2) does not hold then (1) does.

The proof of the first part is left as an exercise for the reader (see Exercise 3.11). Now, we prove that we cannot have both (1) and (2) simultaneously. Suppose, on the contrary, that we can have both (1) and (2) simultaneously. That is, there are $\boldsymbol{y} \in \mathbb{R}^m$ and $\boldsymbol{x} \in \mathbb{R}^n$ satisfying $A^\mathsf{T}\boldsymbol{y} \le \boldsymbol{c}$ and $A\boldsymbol{x} = \boldsymbol{0}$, with $\boldsymbol{c}^\mathsf{T}\boldsymbol{x} < 0$ and $\boldsymbol{x} \ge \boldsymbol{0}$. Then

$$0 = (A\boldsymbol{x})^\mathsf{T}\,\boldsymbol{y} = \boldsymbol{x}^\mathsf{T}(A^\mathsf{T}\boldsymbol{y}) \le \boldsymbol{x}^\mathsf{T}\boldsymbol{c} < 0,$$

giving the desired contradiction. Thus, we cannot have both (1) and (2) together.

Finally, we prove if (2) is not true (and hence (2′) is not true as well), then (1) is true. Assume that (2′) does not hold, i.e., there is no $\boldsymbol{x}$ satisfying $A\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{c}^\mathsf{T}\boldsymbol{x} = -1$, and $\boldsymbol{x} \ge \boldsymbol{0}$, or equivalently, satisfying $\hat{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \ge \boldsymbol{0}$, where

$$\hat{A} = \begin{bmatrix} A \\ \boldsymbol{c}^\mathsf{T} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} \boldsymbol{0} \\ -1 \end{bmatrix}.$$

This means that Condition (1) of Farkas' lemma (Version I) does not hold. It follows that Condition (2) of Farkas' lemma (Version I) holds, i.e., there is $\hat{\boldsymbol{y}}$ satisfying $\hat{A}^\mathsf{T}\hat{\boldsymbol{y}} \ge \boldsymbol{0}$ and $\boldsymbol{b}^\mathsf{T}\hat{\boldsymbol{y}} < 0$. Let α be the last component of $\hat{\boldsymbol{y}}$, and $\boldsymbol{p}$ be the subvector of its remaining components. That is, $\hat{\boldsymbol{y}} = (\boldsymbol{p}^\mathsf{T}, \alpha)^\mathsf{T}$. Then $\boldsymbol{b}^\mathsf{T}\hat{\boldsymbol{y}} < 0$ implies that $\alpha > 0$. Also $\hat{A}^\mathsf{T}\hat{\boldsymbol{y}} \ge \boldsymbol{0}$ implies that

$$\begin{bmatrix} A \\ \boldsymbol{c}^\mathsf{T} \end{bmatrix}^\mathsf{T} \begin{bmatrix} \boldsymbol{p} \\ \alpha \end{bmatrix} = \begin{bmatrix} A^\mathsf{T} \vdots\, \boldsymbol{c} \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ \alpha \end{bmatrix} \ge \boldsymbol{0}. \tag{3.8}$$

Note that (3.8) can be written as $A^\mathsf{T}\boldsymbol{p} + \alpha\boldsymbol{c} \ge \boldsymbol{0}$ or $A^\mathsf{T}(\frac{-\boldsymbol{p}}{\alpha}) \le \boldsymbol{c}$. Thus, Condition (1) holds with $\boldsymbol{y} = \frac{-1}{\alpha}\boldsymbol{p}$. The proof is complete. ■

In conclusion, Farkas' lemma represents a fundamental theorem in linear programming and optimization. By considering both versions I and II of this lemma, we gain a more comprehensive and versatile understanding of the relationships between feasible and optimal solutions in the world of linear programming.

Exercises

3.1 Choose the correct answer for each of the following multiple-choice questions/items.

(*a*) Which of the following sequences is bounded above?

(*i*) $\left\{\dfrac{n^2}{n+2}\right\}$.

(*ii*) $\left\{\dfrac{3^n}{2^n+32}\right\}$.

(*iii*) $\left\{(-1)^{2n+1}\sqrt{n}\right\}$.

(*iv*) $\left\{\dfrac{2^n}{(n+1)^2}\right\}$.

(*b*) Which one of the following sequences diverges?

(*i*) $\left\{\frac{n^3}{n^3+1}\right\}$.

(*ii*) $\left\{\frac{1}{n}\ln\frac{1}{n}\right\}$.

(*iii*) $\left\{\frac{n^2\ln n}{e^n}\right\}$.

(*iv*) $\left\{\frac{n^2+1}{n+1000}\right\}$.

(*c*) Which one of the following sequences converges to a number that is not 4/9?

(*i*) $\left\{\frac{4n-1}{9n}\right\}$.

(*ii*) $\left\{\frac{4n-1}{9n^2}\right\}$.

(*iii*) $\left\{\frac{(2n+1)^2}{(3n-1)^2}\right\}$.

(*iv*) $\left\{\frac{4n^2-1}{9n^2}\right\}$.

(*d*) $\sum_{k=0}^{\log_{3/2} n} n$ equals

(*i*) $(1+\log_{3/2} n)n$.

(*ii*) $n\log_{3/2} n$.

(*iii*) $(1+\log n)n$.

(*iv*) $n\log n$.

(*e*) Which of the following is a subspace of $\mathbb{R}^3$?

(*i*) All vectors of the form $(a,0,0)$.

(*ii*) All vectors of the form $(a,1,1)$.

(*iii*) All vectors of the form (a,b,c) where $b=a+c+1$.

(*iv*) None of the above.

(*f*) The value of α such that the vector $(\alpha,7,-4)$ is a linear combination of vectors $(-2,2,1)$ and $(2,1,-2)$ is

(*i*) 2. (*ii*) -2. (*iii*) 0. (*iv*) -1.

(*g*) The composition of two convex functions is

(*i*) convex. (*ii*) nonconvex.

(*h*) The intersection of two convex sets is convex.

(*i*) convex. (*ii*) nonconvex.

3.2 Use induction and L'Hospital's rule to show that, for each positive k,

$$\lim_{n\to\infty}\frac{(\ln n)^k}{n}=0.$$

3.3 Find $\sum_{k=30}^{60} k^2$.

3.4 Test the following series for convergence. If it converges, find its sum.

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)(2k-1)}.$$

3.5 Test the following series for convergence.

$$(i)\ \sum_{k=1}^{\infty} \frac{1}{2+3^{-k}}. \qquad (ii)\ \sum_{k=1}^{\infty} \frac{1}{2+(0.3)^{k}}.$$

3.6 Test the following series for convergence. If it converges, find its sum.

$$(i)\ \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k}. \qquad (iii)\ \sum_{k=2}^{\infty} (-1)^k \left(\frac{2}{5}\right)^{k-2}.$$

$$(ii)\ \sum_{k=1}^{\infty} \left(\frac{5}{4}\right)^{3-k}. \qquad (iv)\ \sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^{k}.$$

3.7 Prove that the inverse of a positive definite matrix is also positive definite.

3.8 Find the orthogonal of the subspace $H = \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = 2x_2\}$.

3.9 Let $f_1, f_2, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ be convex functions. Show that the function f defined by $f(x) = \max_{i=1,\ldots,m} f_i(\boldsymbol{x})$ is also convex.

3.10 Prove that the intersection of a finite number of convex sets is convex. Argue if the intersection of an infinite number of convex sets is convex.

3.11 Prove that Conditions (2) and (2′) in Farkas' lemma (Version II) are equivalent.

Notes and sources

The histories of mathematical analysis and linear algebra date back to ancient civilizations, with roots in Babylon, ancient Greece, Arabia, and India. Looking at the analysis timeline, Eudoxus and Archimedes made significant contributions to the understanding of calculus-related concepts. However, the formalization and development of mathematical analysis as we know it today began in the 17th century with the work of European mathematicians like Isaac Newton. His book Newton [1687] marked a pivotal moment in the history of mathematical analysis. Over the subsequent centuries, mathematical analysis evolved and expanded, with contributions from other European mathematicians like Cauchy, Weierstrass, and Riemann; see for example Cauchy [1821].

Looking at the algebra timeline, the term "algebra" itself is derived from the Arabic word "Al-Jabr", introduced by Al-Khwarizmi in his work during the 9th century. Al-Khwarizmi's book entitled "Kitab al-Mukhtasar fi Hisab al-Jabr wal-Muqabala" (see, for instance, Surhone et al. [2010]) laid the foundation for solving linear and quadratic equations, marking a significant turning point in the development of algebra. Over the centuries, algebra evolved and expanded, with contributions from European mathematicians including Arthur Cayley. His work Cayley [1889] on matrices, in particular, laid the groundwork for the modern study of linear algebra. Mathematical analysis and linear algebra continued to flourish as essential branches of mathematics, with their modern forms being shaped in the 19th century.

This chapter offered some fundamental concepts from mathematical analysis and linear algebra that serve as the basis for various fields, including combinatorics and optimization. It provided essential insights into foundational analytic and algebraic topics, such as sequences, series, matrices, subspaces, bases, polyhedra, and convex cones. Furthermore, we presented Farkas' lemma, which is a pivotal result in linear programming. The chapter acts as an invaluable resource for individuals seeking an understanding of the analytic and algebraic structures that form the foundation of the disciplines under study in this book.

As we conclude this chapter, it is worth noting that the cited references and others, such as Cusack and Santos [2021], Stewart [2015], Anton et al. [2021], Minton and Robert T Smith [2011], Thomas et al. [2005], also serve as valuable sources of information pertaining to analytic structures covered in this chapter. Also, the cited references and others, such as Moh [2020], Neri [2016], Datta [2010], Meyer [2000], Bartl and Dubey [2017], Roos [2009], Jeyakumar [2009], Bartl [2008] also serve as valuable sources of information pertaining to algebraic and geometric structures covered in this chapter. We used and modified a code due to StackExchange [2019] to create Figure 3.3.

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Part II

COMBINATORICS

CHAPTER 4

GRAPHS

Chapter overview: This chapter serves as a comprehensive introduction to essential concepts in graph theory, providing a foundation for both theoretical understanding and practical problem-solving in diverse fields. More precisely, it delves into the world of graphs and covers key topics such as graph properties, graph coloring, and directed graphs. Readers will also gain an understanding of how graphs can be analyzed and compared for structural similarity through isomorphism. They will also study special graphs including Eulerian and Hamiltonian graphs and explore the art of graph coloring and its applications in various areas. This chapter concludes with a set of exercises to encourage readers to apply their knowledge and enrich their understanding.

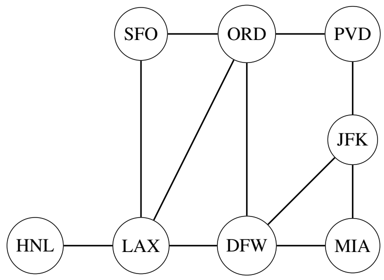
Keywords: Graphs, Graph isomorphism, Graph coloring, Digraphs

Contents

B. Alzalg, *Combinatorial and Algorithmic Mathematics: From Foundation to Optimization*

Suppose that we want to visualize flight routes in North America. A natural representation is with a graph as follows.

- Create a set of nodes (also-called vertices), with each node representing a city. Each node is labeled by the city's three-letter airport code.
- Connect any 2 cities (nodes) which have a flight route between them with a line (called an edge). We can also label this edge with the mileage of the route.



Some flight routes in North America.

A way to think about graphs is:

- Vertices (nodes) specify some entities we are interested in.
- Edges specify the relationships between the entities (vertices).

After we represent our data (flight) with a graph, we can answer a lot of interesting questions such as:

- Can we reach one city from another city?
- What is the route with the minimum number of connections between two cities?
- What is the minimum mileage route between two cities?

Many interesting questions can be answered efficiently, but for some, there is no hope of an efficient answer. This chapter is devoted to introducing graph concepts and terminology. Note that the graphs in this chapter are a completely different concept from the graphs used for plotting functions in calculus.

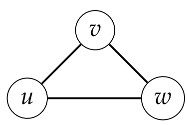
4.1 Basic graph definitions

In this section, we introduce some basic graph definitions. This includes directed and undirected graphs, simple and multi-graphs, subgraphs, paths, cycles, connected components, trees, spanning trees, complete graphs and bipartite graphs.

Definition 4.1 *A graph, $G = (V, E)$, is a finite set of vertices, V, and a finite set of edges, E, where each edge (u, v) connects two vertices, u and v.*

Example 4.1

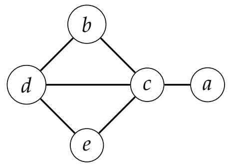
In the graph shown to the right, we have $G = (V, E)$ where $V = \{u, v, w\}$ is the set of the vertices and $E = \{(u, w), (u, v), (w, v)\}$ is the set of edges. These vertices, namely u, v, and w, form the fundamental building blocks of the graph. These edges, namely (u, w), (u, v) and (w, v), explicitly define the links between the respective vertices.



Graph of Example 4.1.

Example 4.2

In the graph provided on the right, we have $G = (V, E)$ where $V = \{a, b, c, d, e\}$ is the set of the vertices and $E = \{(a, c), (b, c), (c, e), (b, d), (d, e), (c, d)\}$ is the set of edges. Overall, this representation of a graph, characterized by its vertices and edges, plays a pivotal role in visualizing the relationships and connections that exist within the given graphical representation.



Graph of Example 4.2.

Directed and undirected graphs We classify the edges into two distinct categories: Directed edges and undirected edges. The directed edge has an origin vertex and a destination vertex, whereas for the undirected edge, there is no designated beginning or end to the edges. An easy way to think about a directed edge is a flight going from one vertex to another. If there is a flight going from San Francisco international airport (SFO) to Chicago O'Hare international airport (ORD), it does not necessarily mean that there is a flight going from ORD to SFO. On the other hand, an undirected edge can be thought of as a network of "friends". For example, if Noah is a friend of Zaid, then this necessarily means that Zaid is also a friend of Noah. See Figure 4.1.

This leads us to classify the graphs into two distinct categories: Directed graphs and undirected graphs. In directed graphs, all edges are directed. In undirected graphs, all edges are undirected (see Figure 4.1). Note that (u, v) and (v, u) are two different edges if the graph is directed, but they are the same edge if the graph is undirected.

In this chapter, we first study the undirected graphs, then we study the directed graphs. Applications of directed and undirected graphs include, but not limited to:

- Transportation networks: City map, highway network, flight network, etc.
- Computer networks: Local area network, internet, web, etc.
- Linguistics, physics, chemistry, biology, etc.

Two vertices u and v of a graph G are adjacent if there exists an edge (u, v) in G. In Figure 4.2, left-hand and middle side pictures do not represent graphs, while the right-hand picture represents a graph because looking at it in three-dimensional space one finds that the vertices u and v are adjacent and the vertices w and x are adjacent.

Simple and multi-graphs A self-loop is an edge that connects a vertex to itself. Multi-edges (or parallel edges) are edges that have the same endpoints in undirected graphs, or the same origin and destination in directed graphs.

Figure 4.1: Directed edges forming a directed graph (left) and undirected edges forming a undirected graph (right).

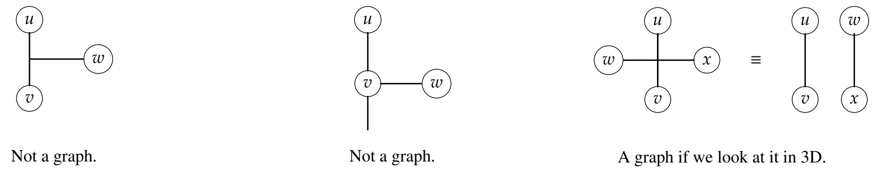
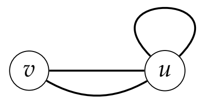


Figure 4.2: Examples of graphs and non-graphs.

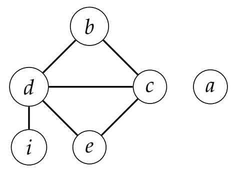
In the graph shown to the right, there is a self-loop at vertex u and there are two parallel edges between the vertices u and v. If a graph does not have parallel edges and self-loops, then it is said to be simple. A multi-graph can have multiple edges between the same two vertices and self-loops.



A non-simple graph.

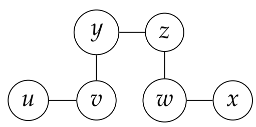
In this chapter, we will deal almost exclusively with simple graphs. In other words, unless otherwise specified, when we say "graph" we mean "simple graph".

The vertex degree Let $G = (V, E)$ be a graph. If $u, v \in V$ and $(u, v) \in E$, we say that the edge (u, v) is incident to vertices u and v. The degree of a vertex $v \in V$, denoted as $\deg(v)$, is the number of edges incident on v. A vertex whose degree is zero is called isolated. For example, in the graph shown to the right, we have $\deg(i) = 1, \deg(b) = \deg(e) = 2, \deg(c) = 3, \deg(d) = 4$ and $\deg(a) = 0$. So, a is an isolated vertex. The degree list of an undirected graph is the non-decreasing sequence of its vertex degrees. For instance, the degree list of the graph shown to the right is $0, 1, 2, 2, 3, 4$.

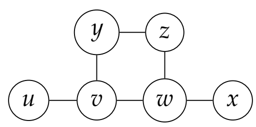


A graph with a degree sequence $0, 1, 2, 2, 3, 4$.

Paths and cycles A path P_n from a vertex u to a vertex u' is a graph written as a sequence $(v_0, v_1, v_2, \ldots, v_n)$ of $n + 1$ vertices, such that $u = v_0$ and $u' = v_n$, and n edges (v_{i-1}, v_i) for $i = 1, 2, \ldots, n$. The length of a path is the number of its edges. So, the path P_n has length n. A subpath of a path is a contiguous subsequence of its vertices. A path is simple if all vertices in the path are distinct. Revisiting vertices and/or edges are allowed in non-simple paths. Figure 4.3 shows simple and non-simple paths. The simple path shown to the left has length 5 and has, for instance, the path (v, y, z, w) as a subpath.



A simple path: (u, v, y, z, w, x).



A non-simple path: (u, v, w, z, y, v, w, x).

Figure 4.3: Simple and non-simple paths.

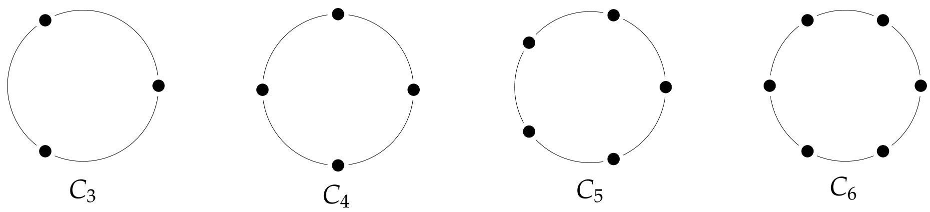
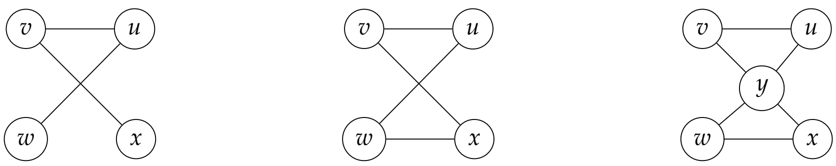


Figure 4.4: Cycle graphs.



An acyclic graph.

A simple cycle C_4: (v, u, w, x, v).

A non-simple cycle: (v, u, y, w, x, y, v).

Figure 4.5: Acyclic and (simple and non-simple) cycle graphs.

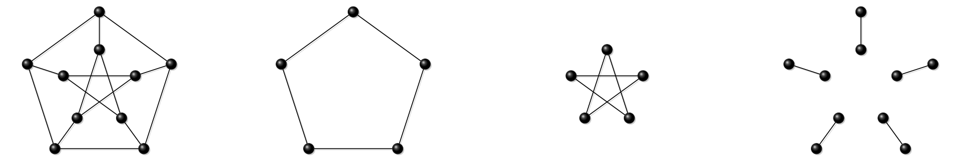
A cycle is a path on which the first vertex is equal to the last vertex and all edges are distinct. So, a cycle C_n is a path P_n with $v_0 = v_n$. In Figure 4.4, we show some cycles. A cycle is simple if all its vertices, except the first and the last one, are distinct. A graph with no simple cycles is called acyclic. In Figure 4.5, we show an acyclic graph, a simple cycle graph, and a non-simple cycle graph.

Subgraphs and connected components A graph $H = (V', E')$ is called a subgraph of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A spanning subgraph of a graph G is a subgraph of G that contains all the vertices of G. That is, $H = (V', E')$ is a spanning subgraph of a graph $G = (V, E)$ if $V' = V$ and $E' \subseteq E$. The Petersen graph is well-known graph in graph theory.[1] Figure 4.6 shows three subgraphs of the Petersen graph, P, and only one of them is a spanning subgraph.

A graph is called connected if every vertex is reachable from all other vertices, and disconnected otherwise. That is, there is a path between every pair of distinct vertices of a connected graph. In Figure 4.7, we show four graphs; each one has 20 vertices; we see that the two left-hand side ones are connected, whereas the two right-hand side ones are disconnected because the five external vertices are not reachable from any of other internal vertices.

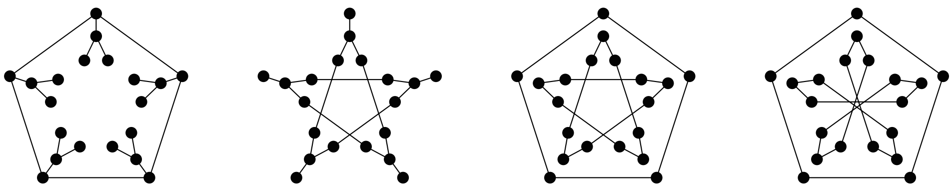
A connected component G' of a graph G is a maximal connected subgraph of G. By maximality we mean that there is no way to add into G' any vertices and/or edges of G which are not currently in G' in such a way that the resulting subgraph is connected.

[1]The Petersen graph serves as a useful example and counterexample for many problems in graph theory.



The Petersen graph P. A subgraph of P. A subgraph of P. A span. subgraph of P.

Figure 4.6: The Petersen graph P and some subgraphs of P.



A connected graph. A connected graph. A disconnected graph. A disconnected graph.

Figure 4.7: Connected and disconnected graphs.

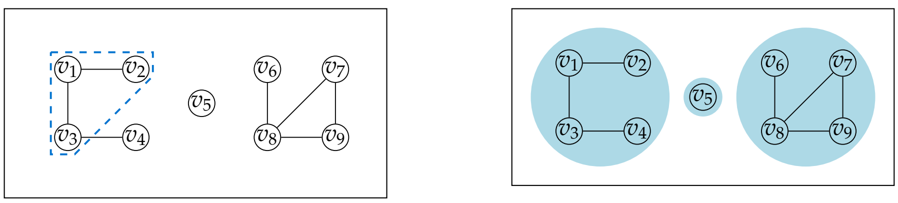


Figure 4.8: A graph with three connected components.

Example 4.3 The graph, say G, shown to the right in Figure 4.8 has 9 vertices and 7 edges. Note that the subgraph surrounded by the blue dashed lines is not a connected component of G. The reason for that is if we add a new vertex v_4 and a new edge (v_3, v_4), the resulting subgraph is still connected.

The graph G has 3 connected components, where each corrected component is framed by a blue disk background shown to the left in Figure 4.8. Note that, for instance, if we add the vertex v_5 to the right- or left-hand side connected component, the resulting subgraph is not connected anymore. ■

Example 4.4 The graph shown in Figure 4.9 has 18 vertices, 12 edges, and 7 connected components. These components are framed by blue ellipsoid backgrounds. ■

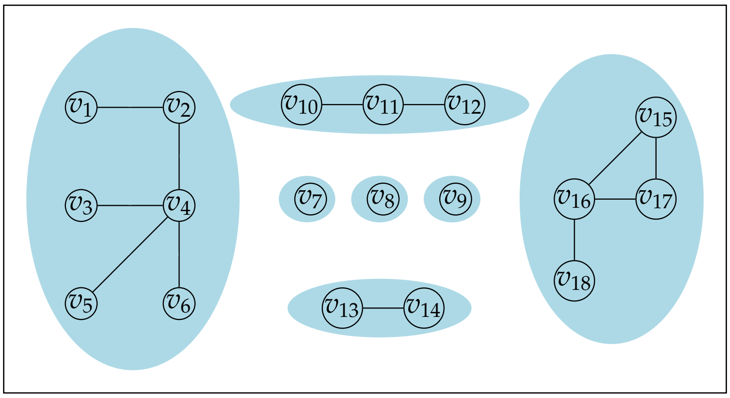


Figure 4.9: A graph with seven connected components.

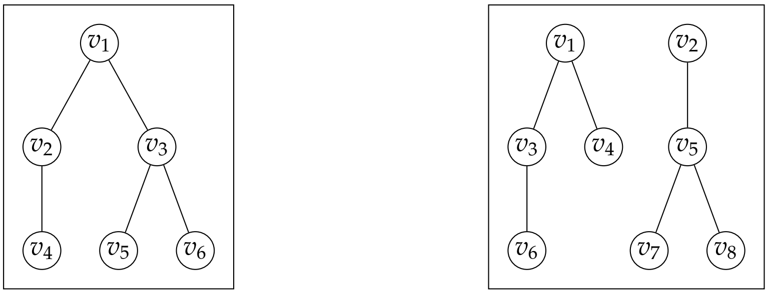


Figure 4.10: A tree graph (left) and a forest graph (right).

The following remark can be directly proven on the basis of Definitions 2.7 and 2.8.

Remark 4.1 *The connected components of a graph define equivalence classes over its vertices under the "is reachable from" relation.*

From Theorem 2.1, the "is reachable from" relation defines a partition on the vertices, with two vertices being in the same partition (or the same equivalence class) if and only if they are in the same connected component.

Trees and spanning trees One of the important classes of graphs is the trees. The importance of trees is evident from their applications in various areas, especially theoretical computer science and molecular evolution. A tree is a connected undirected graph with no cycles (i.e., connected, acyclic, undirected graph). A forest is any undirected graph without cycles. Figure 4.10 shows a tree and a forest. Note that the connected components of a forest are trees.

A spanning tree of a connected graph is a spanning subgraph that is a tree. Note that a spanning tree is not unique unless the graph is a tree. Figure 4.11 shows a graph and two of its spanning trees. Spanning trees have applications to the design of communication networks.

A spanning forest of a graph is a spanning subgraph that is a forest. Note that disconnected graphs have spanning forests and do not have spanning trees, and that only connected graphs have spanning trees.

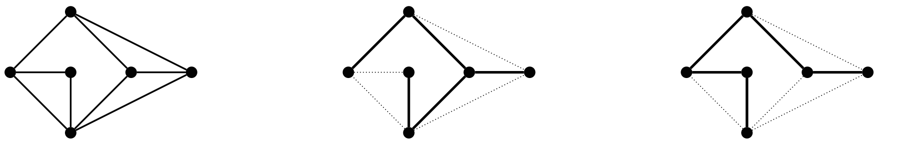


Figure 4.11: A graph with spanning trees.

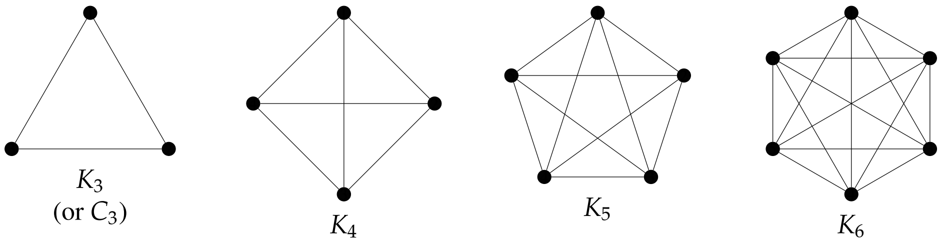


Figure 4.12: Complete graphs.

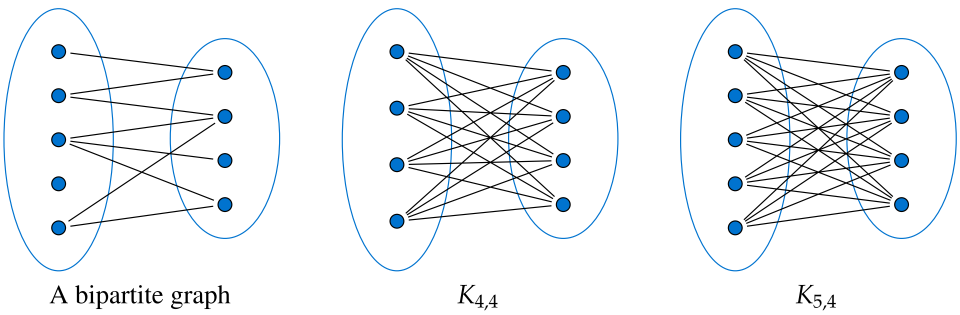


Figure 4.13: Bipartite and complete bipartite graphs.

Complete and bipartite graphs A complete graph of n vertices, denoted as K_n, is a graph in which every pair of vertices is adjacent. In Figure 4.12, we show some complete graphs. An empty graph is a graph with no edges, i.e., $E = \emptyset$.

A bipartite graph is a graph in which its vertices can be partitioned into two sets such that, for every edge (u, v) in the graph, u is in one of the sets and v is in the other. A complete bipartite graph, denoted as $K_{m,n}$, is a bipartite graph in which each vertex in one of the sets is joined to each vertex in the other. Figure 4.13 shows bipartite and complete bipartite graphs.

A planar graph is a graph that can be embedded in the plane. In other words, it can be drawn on the plane in such a way that its edges intersect only at their endpoints (i.e., no edges cross each other). See Figure 4.14.

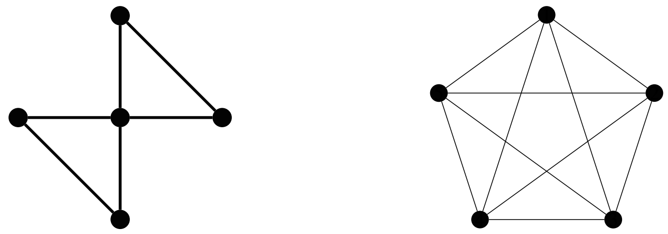


Figure 4.14: A planar graph (left) and a nonplanar graph (right).

Figure 4.15: A graph G (left) and its complement $\bar{G}$ (right).

The complement of a graph $G = (V, E)$ is the graph $\bar{G} \triangleq (V, \bar{E})$ on the same set of vertices as of G such that there will be an edge between two vertices u and v in V, if and only if there is no edge in between u and v in V. See Figure 4.15.

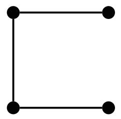
A dense graph is a graph in which the number of edges is close to the maximal number of edges. The opposite, a graph with only a few edges, is a sparse graph. The distinction between sparse and dense graphs is rather vague, and depends on the context. Formally speaking, a graph $G = (V, E)$ is dense if $|E|$ is close to $|V^2|$ (i.e., $|E| \approx |V|^2$), and G is sparse if $|E|$ is much less than $|V^2|$ (i.e., $|E| \approx |V|$).

The matching (or assignment) problem is the problem of choosing an optimal assignment of a number of applicants to a number of jobs. We can model this with a bipartite graph. Here we assume that there are n_1 job applicants and n_2 job openings. Each applicant has a subset of the job openings s/he is interested in. Conversely, each job opening can only accept one applicant out of some subset of the applicants. In other words, there are certain allowed or "compatible" pairings of applicants to jobs. We can find an assignment of jobs for applicants in such a way all the pairings are allowed and as many applicants as possible get jobs. This can be modeled with a bipartite graph as follows: On the left side we have the applicants and on the right side we have the jobs. We draw an edge between applicant u and job v if they are compatible. The problem of finding a maximum matching in a given graph is an NP-hard problem in general (the class of NP-hard problems will be introduced briefly in Section 7.7), but if the graph is bipartite, the problem can be solved efficiently.

A bridge or cut-edge is an edge of a graph whose deletion increases its number of connected components. In Figure 4.16, we show two graphs, one has three bridges while the other has no bridges.

The first characterization of trees is that a graph is a tree if and only if it is connected and every edge is a bridge. Further characterizations of trees will be given in the next section.

A graph with three edges and three bridge edges.



A graph with three edges and no bridge edges.

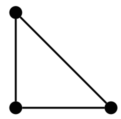
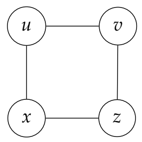
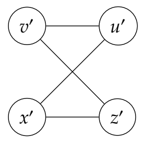


Figure 4.16: Graphs with/without bridges.



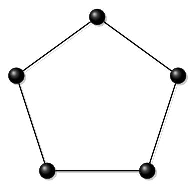
Graph G



Graph H

A bijection between
$V(G)$ and $V(H)$:
$f(u) = u'$, $f(v) = v'$,
$f(x) = x'$, $f(z) = z'$.

Figure 4.17: An isomorphism between two graphs.



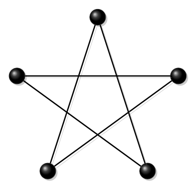


Figure 4.18: A pair of isomorphic graphs.

4.2 Isomorphism and properties of graphs

In this section, we discuss graph isomorphism and present some graph properties.

Graph isomorphism

What does it mean for two graphs to be "the same"? We say that two graphs are isomorphic when the vertices of one can be relabeled to match the vertices of the other in a way that preserves adjacency. Formally, we have the following definition.

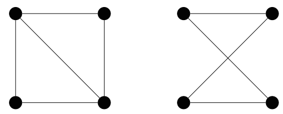
Definition 4.2 *We say that two graphs G and H are isomorphic if there is a bijection $f : V(G) \to V(H)$ such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H.*

See Figure 4.17 which shows an example of two isomorphic graphs. It is also not hard to see that the two graphs with solid vertices in Figure 4.18 are another example of an isomorphic pair of graphs.

In fact, given two graphs, it is often really hard to tell if they are isomorphic, but it is usually easier to see if they are not isomorphic. We have the following remark which shows that isomorphic graphs have the same number of vertices and the same number of edges.

Remark 4.2 *If two graphs have different numbers of vertices or different numbers of edges, then they are not isomorphic.*

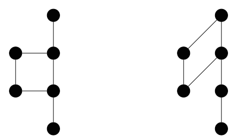
For example, the two connected graphs shown to the right have 4 vertices, but they are not isomorphic because they have different numbers of edges.



Having the same number of vertices and the same number of edges is not a guarantee that two graphs will be isomorphic. We have the following remark which shows that isomorphic graphs have the same degree lists.

Remark 4.3 *If two graphs have different degree lists, then they are not isomorphic.*

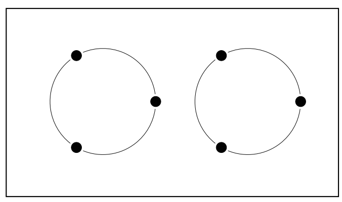
As an example, the two connected graphs shown to the right have 6 vertices and 6 edges, but they are not isomorphic because they have different degree lists.



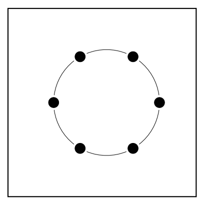
Having the same number of vertices, the same number of edges, and the same degree lists is not a guarantee that two graphs will be isomorphic. We have the following remark which shows that isomorphic graphs have the same number of connected components.

Remark 4.4 *If two graphs have different number of connected components (for instance, one graph is connected, and the other is not), then they are not isomorphic.*

As an example, the two graphs shown to the right have 6 vertices, 6 edges, and the degree list is the same, but they are not isomorphic because one of them is connected while the other is not.



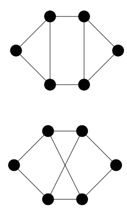
C_3 duplicated.
(Disconnected graph).



The cycle C_6.
(Connected graph).

Finally, we also point out that having the same number of vertices, the same number of edges, the same degree lists, and the same number of connected components is not a guarantee that two graphs will be isomorphic.

As an example, both the two connected graphs shown to the right have 6 vertices, 8 edges, 1 connected component, and the degree list is the same, but they are not isomorphic. However, the graph on the top has a cycle of length 3, and the minimum length of any cycle in the graph on the bottom is 4. So these two graphs are not isomorphic.



Graph properties

In this part, we present some graph properties which are stated in the following theorems.

Theorem 4.1 *In an undirected graph $G = (V, E)$, we have*

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Proof Every edge (w, u) has 2 endpoints (w and u). The first endpoint w contributes exactly 1 to $\deg(u)$. The second endpoint u contributes exactly 1 to $\deg(w)$. Thus, each edge contributes exactly 2 to the sum on the left. This proves the theorem. ■

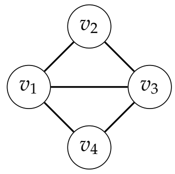
The following example is a direct application of Theorem 4.1.

Example 4.5

In the graph, $G = (V, E)$, shown to the right, we have

$$\sum_{v \in V} \deg(v) = \sum_{i=1}^{4} \deg(v_i) = 3 + 2 + 3 + 2 = 10 = 2|E|.$$

This matches the result in Theorem 4.1 above.



Graph of Example 4.5.

■

Throughout the rest of this section, for a given undirected graph $G = (V, E)$, we let $n = |V|$ and $m = |E|$. The following is the bipartite version of Theorem 4.1.

Theorem 4.2 *In a bipartite graph $G = (A \cup B, E)$, we have*

$$\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v) = m.$$

Proof Since A and B are partition sets of the vertex set, V, of the bipartite graph G (i.e., $A \cup B = V$ and $A \cap B = \emptyset$), each vertex in V is either in A or B, but not both. Then, from Theorem 4.1, we have

$$\sum_{v \in V} \deg(v) = \sum_{v \in A} \deg(v) + \sum_{v \in B} \deg(v) = 2m.$$

Note that each edge has one endpoint in A and one endpoint in B. So, we have $\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v)$. Thus,

$$2\sum_{v \in A} \deg(v) = 2\sum_{v \in B} \deg(v) = 2m.$$

The result follows by dividing each term by 2. The proof is complete. ■

For example, in the bipartite graph shown on the left-hand side of Figure 4.13, we have

$$\sum_{v \text{ in the first partite set}} \deg(v) = \sum_{v \text{ in the second partite set}} \deg(v) = 8 = |E|.$$

Theorem 4.3 *In an undirected graph, we have* $m \leq n(n-1)/2$.

Proof Form Theorem 4.1, we have $2m = \sum_{v \in V} \deg(v)$. Note that $\deg(v) \leq n-1$ because each vertex has degree at most $n-1$. It follows that

$$m = \frac{1}{2}\sum_{v \in V} \deg(v) \leq \frac{1}{2}\sum_{v \in V}(n-1) = \frac{1}{2}\sum_{i=1}^{n}(n-1) = \frac{1}{2}n(n-1).$$

The proof is complete. ■

From Theorem 4.3, we can conclude the following corollary.

Corollary 4.1 *In an undirected graph* $G = (V, E)$, *we have* $|E| = O(V^2)$[a].

[a] To be consistent with the notations in the literature, throughout this book, we use $O(V)$, $O(V^2)$ and $O(V+E)$ to mean that $O(|V|)$, $O(|V|^2)$ and $O(|V|+|E|)$, respectively.

The following theorem tells us when the inequality in Theorem 4.3 is actually an equality.

Theorem 4.4 *An* n*-vertex complete graph has* $n(n-1)/2$ *edges.*

Proof We prove that $|E(K_n)| = n(n-1)/2$ by induction on n. The base case is trivial: K_1 has one vertex and hence no edges and $n(n-1)/2 = (1)(0)/2 = 0$. Assume that the statement is true for $n = \ell$, i.e., $|E(K_\ell)| = \ell(\ell-1)/2$, for some $\ell \in \mathbb{N}$. Now, we prove that $|E(K_{\ell+1})| = (\ell+1)\ell/2$. Let H be a subgraph of $K_{\ell+1}$ created by removing any vertex v and all edges incident to v. Then $H = K_\ell$ and

$$|E(K_{\ell+1})| = |E(K_\ell)| + \deg(v) = \ell(\ell-1)/2 + \ell = (\ell+1)\ell/2.$$

Thus, any n-vertex complete graph has $n(n-1)/2$ edges. The proof is complete. ■

For example, in the complete graph K_6 (see K_6 at the right-hand side of Figure 4.12), we have $6(6-1)/2 = 15 = |E|$.

Theorem 4.5 *The number of odd-degree vertices is always even.*

Proof By Theorem 4.1, the sum of all the degrees is twice the number of edges. Because the sum of the degrees is even and the sum of the degrees of vertices with even degrees is also even, the sum of the degrees of vertices with odd degrees must be even. Since the sum of the degrees of vertices with odd degrees is even, it must be an even number of those vertices. ■

For instance, in Example 4.5, the number of odd-degree vertices is two, which is even. We have also the following example.

Example 4.6 Determine whether each of the following sequences is a degree sequence of a graph? Answer by 'yes' or 'no' and justify your answer.

(a) $(1,1,2,2,2,3,3,3)$. (b) $(0,0,1,1,2,2,3,3)$. (c) $(0,0,1,1,1,1,6)$.

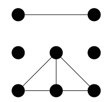


Figure 4.19: Graph of Example 4.6 (b).

Solution (a) The answer is no because we have an odd number of odd-degree vertices.

(b) Yes. To justify this answer, a graph that represents this degree sequence is given in Figure 4.19.

(c) No. The justification for this answer is that there is a vertex of degree 6. So, the minimum number of vertices that this vertex must be adjacent to is 6. Therefore, all 7 vertices must be connected. However, the degree sequence has two vertices of degree 0, contradicting this.

■

The most important formula for studying planar graphs is Euler's formula.[2]

Theorem 4.6 (Euler's formula) *Let $G = (V, E)$ be a finite, connected, planar graph that is drawn in the plane without any edge intersections. Let also F be the set of faces (regions bounded by edges, including the outer, infinitely large region) of G, then*

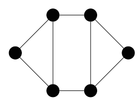
$$|V| - |E| + |F| = 2.$$

Proof We prove the theorem by induction on m where $m = |E|$. If $m = 0$, then it consists of a single vertex with a single region surrounding it, and hence $|V| - |E| + |F| = 1 - 0 + 1 = 2$. This proves the base case.

For the inductive step, we assume that for some graph, the formula is true for any $m = k$ for some $k \in \mathbb{N}$. To prove that the formula is true for $m = k + 1$, choose any edge $e \in E$. If e connects two vertices, then contracting it would reduce $|V|$ and $|E|$ by one. Otherwise, the edge e separates two faces, then removing it would reduce $|F|$ and $|E|$ by one. In either case, the result follows the inductive hypothesis. This proves the theorem. ■

As an illustration, in the graph shown to the right, we have

$$|V| - |E| + |F| = 6 - 8 + 4 = 2.$$



We end this section with the following theorem which characterizes the tree graph.

Theorem 4.7 *Let G be an undirected graph on n vertices and m edges. The following statements are equivalent.*

(a) *G is a tree.*

(b) *Any two vertices in G are connected by a unique simple path.*

[2]Euler's formula was first proved by Leonhard Euler (1707 - 1783), a Swiss mathematician who made important and influential discoveries in many branches of mathematics.

(c) *G is connected but each edge is a bridge.*

(d) *G is connected and* $m = n - 1$.

(e) *G is acyclic and* $m = n - 1$.

(f) *G is acyclic but if any edge is added to G it creates a cycle.*

Proof We need to show that $(a) \to (b) \to (c) \to (d) \to (e) \to (f) \to (a)$.

$(a) \to (b)$: Given that every tree is connected, it guarantees the existence of at least one path connecting any pair of vertices in the graph G. To establish the uniqueness of this path, we employ a proof by contradiction. Let us assume there are at least two distinct paths connecting a particular pair of vertices in G. The union of these paths would inevitably form a cycle within the graph, implying that G contains a cycle. However, this directly contradicts the established fact that G is a tree. Therefore, our initial assumption of multiple paths between vertices in G must be false, ultimately confirming the presence of a unique path between any two vertices in the tree.

$(b) \to (c)$: Due to the existence of a distinct path connecting any pair of vertices in graph G, it can be unequivocally declared as connected. Now, consider an arbitrary edge (x, y) within G, denoting it as path $P = (x, y)$ from vertex x to vertex y. Since this path is unique, its removal from G results in the absence of any alternative path between x and y, rendering $G - (x, y)$ disconnected. This observation holds true for any edge (x, y) within G, demonstrating that the removal of any edge e from G results in a disconnected graph $G - e$. Thus, we conclude that while G itself is connected, the removal of any edge e from G leads to a disconnected graph.

$(c) \to (d)$: By assumption, G is connected. So we need only to show that $m = n - 1$. We use proof by induction. It is clear that a connected graph with $n = 1$ or $n = 2$ vertices has $n - 1$ edges. Assume that every graph with fewer than n vertices satisfying (c) also satisfies (d). Let G be an n-vertex connected graph but $G - e$ is disconnected for every edge e of G. Let e' be any edge of G. Now, $G - e'$ is disconnected. Hence $G - e'$ has two connected components. Let G_1 and G_2 be the connected components of G. Let n_i and m_i be the number of vertices and edges in G_i, for $i = 1, 2$. Now, each component satisfies (c) because $n_i < n$ for $i = 1, 2$. By induction hypothesis, we have $m_i = n_i - 1$, for $i = 1, 2$. So, $m = m_1 + m_2 + 1 = (n_1 - 1) + (n2 - 1) + 1 = n - 1$. Thus, by mathematical induction, G has exactly $n - 1$ edges.

$(d) \to (e)$: We have to show that every connected graph G with n vertices and $n - 1$ edges is acyclic. We use proof by induction. For $n = 1, 2$, it is clear that all connected graphs with n vertices and $n - 1$ edges are acyclic. Assume that every connected graph with fewer than n vertices satisfying (d) is acyclic. Let G be an n-vertex connected graph with $n - 1$ edges. Because G is connected and has $n - 1$ edges, G has a vertex, say x, of degree 1. Let $G' = G - \{x\}$. Then, G' is connected and has $n - 1$ vertices and $n - 2$ edges. By induction hypothesis, G' is acyclic. Because x is a 1-degree vertex in G, x can not be in any cycle of G. Since $G' = G - \{x\}$ is acyclic, G must be acyclic. Thus, by mathematical induction, every n-vertex connected graph with $n - 1$ edges is acyclic.

$(e) \to (f)$: Assume that G is acyclic and that $m = n - 1$. Let G_i, $1 \le i \le k$, be the connected components of G. Because G is acyclic, G_i is acyclic for $i = 1, 2, \ldots, k$. Hence, each G_i, $1 \le i \le k$, is a tree. Let n_i and m_i be the number of vertices and edges in G_i for each

$i = 1, 2, \ldots, k$. As (a) implies (e), we have $m = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k}(n_i - 1) = n - k$. Therefore, $k = 1$. This means that G is connected. Hence, G must be a tree. Since (a) implies (b), any two vertices in G are connected by a unique path. Thus, the addition of any new edge to G would inevitably create a cycle.

$(f) \to (a)$: The proof here is left to the reader as an exercise (see Exercise 4.5). ■

4.3 Eulerian and Hamiltonian graphs

This section introduces Eulerian and Hamiltonian paths and cycles. Eulerian graphs are motivated by the 18th century problem about the seven bridges of Königsberg.

Königsberg bridge problem Graph theory began in 1736 when Leonhard Euler solved the well-known Königsberg bridge problem. The city of Königsberg in Prussia (now Kaliningrad in Russia) is set on both sides of the Pregel river, which has seven bridges across it (see Figure 4.20). In 1735, citizens of the city were wondering if it is possible to:

- Walk over all the bridges exactly once returning to where you started.
- Walk over all the bridges exactly once ending at a different place than where you started.

The negative resolution by Euler in 1736 laid the foundations of graph theory.

Eulerian paths and cycles A graph is called Eulerian when it contains an Eulerian cycle. We have the following definition.

Definition 4.3 *An Eulerian path in a graph is a path that visits every edge exactly once (allowing for revisiting vertices). An Eulerian cycle in a graph is an Eulerian path that starts and ends on the same vertex.*

Note that, from Definition 4.3, every Eulerian cycle is an Eulerian path. Figure 4.21 shows an Eulerian path and cycle. Note that, in the Eulerian path shown to the left in Figure 4.21, all vertices have even degrees. Note also that, in the Eulerian cycle shown to the right in Figure 4.21, there are exactly two vertices of odd degrees. The following two theorems, which are due to Euler, are regarded as an excellent characterization for Eulerian paths and cycles.

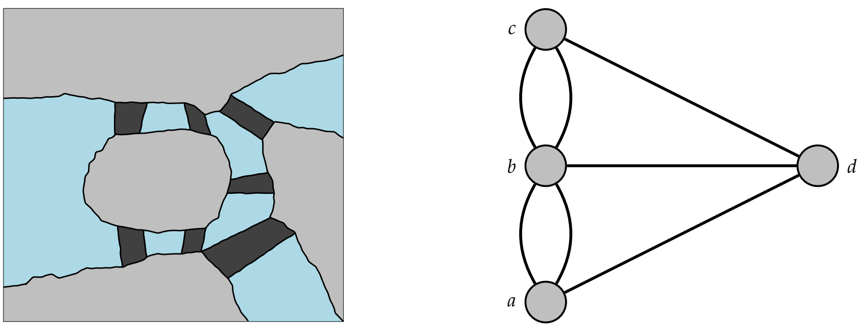


Figure 4.20: The graph of Königsberg bridge problem has four vertices a, b, c and d (representing land masses) and seven edges (representing the bridges).

An Eulerian path: (u, v, w, x, y, w). An Eulerian cycle: (u, v, w, x, y, w, u).

Figure 4.21: An Eulerian path and cycle.

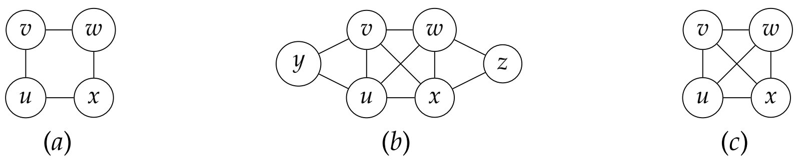
Theorem 4.8 *A connected undirected graph has an Eulerian cycle if and only if the degrees of all the vertices are even.*

Proof An Eulerian cycle is a traversal of all the edges of a simple graph once and only once, staring at one vertex and ending at the same vertex. We can repeat vertices as many times as we want, but you can never repeat an edge once it is traversed. We prove that if a connected undirected graph has an Eulerian cycle, then the degrees of all the vertices are even. Let G be a graph that has an Eulerian cycle. Every time we arrive at a vertex during our traversal of G, we enter through one edge and exit through another. Thus, there must be an even number of edges incident to every vertex. Therefore, every vertex of G has an even degree. This proves one direction of the theorem. The proof of the other direction is left to the reader as an exercise. ∎

Theorem 4.9 *A connected undirected graph has an Eulerian path that is not a cycle if and only if there are exactly two vertices of odd degrees.*

Proof We prove that if a connected undirected graph has exactly two vertices of odd degrees, then it has an Eulerian path. Let G be a graph with exactly two vertices, say u and v, of odd degrees. Adding the edge (u, v) to G forms a graph having all even degrees. By Theorem 4.8, the resulting graph contains an Eulerian cycle. This cycle uses the edge (u, v). Thus, deleting the edge (u, v) from the resulting graph yields an Eulerian path in the graph G. This proves one direction of the theorem. The proof of the other direction is left as an exercise. ∎

Example 4.7 Determine if the following graphs have an Eulerian cycle, Eulerian path, or neither? If there is one, give it. Otherwise, justify why there is none.



Solution (*a*) Note that all vertices in the graph are of even degrees. From Theorem 4.8, the graph has an Eulerian cycle. Also, from Theorem 4.9, the graph does not have an Eulerian path. The Eulerian cycle is (u, v, w, x, u).

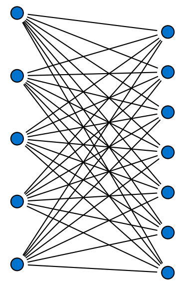
(*b*) Note that all vertices in the graph are of even degrees. From Theorem 4.8. the graph has an Eulerian cycle, Also, from Theorem 4.9, the graph does not have an Eulerian path. The Eulerian cycle is $(y, v, w, z, x, u, v, x, w, u, y)$.

(c) Note that all vertices in the graph are of odd degrees. From Theorem 4.8, the graph does not have an Eulerian cycle. Also, from Theorem 4.9, the graph does not have an Eulerian path.

To illustrate the usefulness Theorems 4.8 and 4.9, we present some examples.

Example 4.8 Determine if $K_{5,7}$ has an Eulerian cycle, Eulerian path, or neither. Justify your answer.

Soltion. The complete bipartite graph $K_{5,7}$ has 12 vertices: 7 vertices of degree 5, and 5 vertices of degree 7. By Theorem 4.8, because not all vertices of $K_{5,7}$ are of even degrees, the graph $K_{5,7}$ does not have an Eulerian cycle. In addition, by Theorem 4.8, because $K_{5,7}$ does not have exactly two vertices of odd degrees, the graph $K_{5,7}$ does not have an Eulerian path.



$K_{5,7}$

Finally, revisiting the Königsberg bridge problem, note that the degrees of all 4 vertices of the graph shown in Figure 4.20 are odd. So, applying Theorems 4.8 and 4.9 to the Königsberg bridge graph, we understand now why Euler was led to a negative resolution of the problem.

Hamiltonian paths and cycles A graph is called Hamiltonian when it contains a Hamiltonian cycle. We have the following definition.

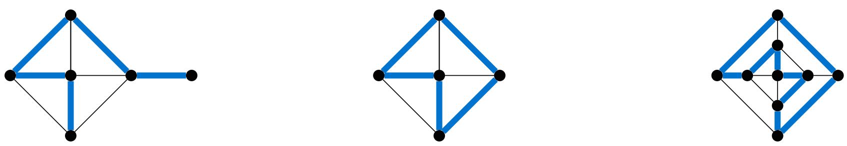
Definition 4.4 *A Hamiltonian path in a graph is a path that visits every vertex in the graph exactly once. A Hamiltonian cycle in a graph is a Hamiltonian path that starts and ends on the same vertex.*

See Figure 4.22, which shows a Hamiltonian path and Hamiltonian cycles.

Note that if a graph has a Hamiltonian cycle, then it automatically has a Hamiltonian path by deleting any edge of the cycle.

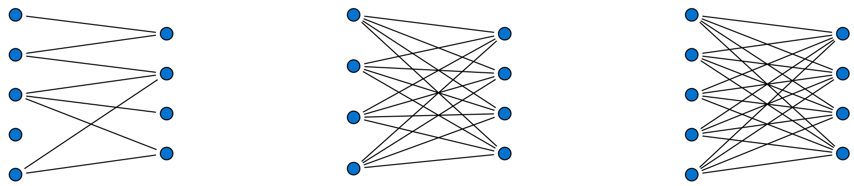
The following theorem is due to Dirac. The proof of this theorem is beyond the scope of this book and is omitted. The proof can be found in any standard textbook on graph theory.

Theorem 4.10 *If each vertex of a connected graph with $n \geq 3$ vertices is adjacent to at least $n/2$ vertices, then the graph has a Hamiltonian cycle (hence, also has a Hamiltonian path).*



A Hamiltonian path. A Hamiltonian cycle. A Hamiltonian cycle.

Figure 4.22: A graph with a Hamiltonian path and graphs with Hamiltonian cycles.

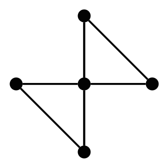


A bipartite graph $K_{4,4}$ $K_{5,4}$

Figure 4.23: Some bipartite graphs. Among them, only $K_{4,4}$ satisfies the hypothesis of Dirac's theorem, hence it is Hamiltonian.

Note that the converse of Theorem 4.10 is not necessarily true, i.e., if a connected graph with $n \geq 3$ vertices has a Hamiltonian cycle, then it is not necessarily that each vertex of the graph is adjacent to at least $n/2$ vertices. The graph shown on the right-hand side of Figure 4.22 is an example, and, indeed, many other graphs can also be taken to verify that the converse of Theorem 4.10 is not true in general. For example, any circle C_n, where $n \geq 5$, can be considered.

Note also that, if n is odd in Theorem 4.10, we take the ceiling of $n/2$. In the graph shown to the right, we give a 5-vertex graph, in which the degree of each vertex is greater than or equal $\lfloor 5/2 \rfloor = 2$, but the graph is non-Hamiltonian. (Why?).



A non-Hamiltonian graph.

Figure 4.23 shows some bipartite graphs. Among them, only the complete bipartite graph $K_{4,4}$ satisfies the condition that each vertex has a degree at least half the number of vertices. Based on Theorem 4.10, $K_{4,4}$ has a Hamiltonian cycle. It is also not hard to see that the complete bipartite graph $K_{5,4}$ has a Hamiltonian path but does not have a Hamiltonian cycle.

Generally, from Theorem 4.10, any complete bipartite graph $K_{n,n}$, where $n \geq 2$, has a Hamiltonian cycle (hence, has also a Hamiltonian path). Additionally, the complete bipartite graphs $K_{n,n+1}$ and $K_{n+1,n}$, where $n \geq 1$, have Hamiltonian paths but have no Hamiltonian cycles. Finally, from Theorem 4.10, any complete graph K_n, where $n \geq 2$, has a Hamiltonian cycle (hence, also has a Hamiltonian path).

4.4 Graph coloring

We start with the following example inspired by Hussin et al. [2011]. It introduces graph coloring and motivates the reader to some of its applications.

Example 4.9 Every semester, the registrar ensures that no student faces the inconvenience of having to take two exams simultaneously. To address this challenge, we employ a graph modeling approach, creating a graph denoted as $G = (V, E)$. Here, the vertex set V directly corresponds to all classes, assigning one vertex per class. The set E is defined based on student enrollments: an edge connects vertices u and v if there exists at least one student who is enrolled in both class u and class v. Our objective is to allocate each vertex to a specific time slot while ensuring that vertices connected by an edge are assigned distinct slots.

The most straightforward approach is to assign each final exam its dedicated time slot, resulting in n time slots if there are n classes. However, this can lead to an overly long final exam period. Therefore, we are looking for a solution that not only satisfies the scheduling requirements but also minimizes the total number of time slots.

One way to conceptualize the concept of "assigning a time slot" is to view each time slot as a unique color. In essence, given graph G, our task is to color each vertex in such a way that connected vertices do not share the same color.

In Figure 4.24, we illustrate this process by coloring a graph with 12 vertices, showcasing how to schedule final exams for these 12 classes. Each class corresponds to a vertex labeled as c_i for $i = 1, 2, \ldots, 12$. It is assumed that every class c_i contains exactly three students, each enrolled in three different classes aside from c_i. The left side of Figure 4.24 demonstrates a coloring approach utilizing four distinct colors, while the right side employs only two colors, presenting alternative scheduling possibilities. ■

Now we are ready to introduce the properly colored graphs. Formally, we have the following definition.

Definition 4.5 *A proper coloring is an assignment of colors to the vertices of a graph so that no two adjacent vertices have the same color.*

Figure 4.25 shows a properly colored graph and an improperly colored graph. Note that the two colorings used for the 12-vertex graph shown in Figure 4.24 are both proper.

Definition 4.6 *A (proper) k-coloring of a graph is a proper coloring using at most k colors. A graph that has a k-coloring is said to be k-colorable.*

Clearly, every graph $G = (V, E)$ is $|V|$-colorable because we can assign a different color to each vertex. We are usually interested in the minimum number of colors we can get away with and still properly color a graph. We have the following theorems.

Figure 4.24: Scheduling final exams for twelve classes by coloring a graph of 12 vertices using four colors (left) and coloring the same graph using only two colors (right).

Figure 4.25: Proper graph coloring (left) versus improper graph coloring (right).

Theorem 4.11 *A graph is 2-colorable if and only if it is bipartite.*

Proof Consider a graph G that is 2-colorable, meaning we can assign either red or blue to each vertex in such a way that no edge connects two vertices of the same color. We denote the subset of red-colored vertices as A and the subset of blue-colored vertices as B. Since all vertices within set A are red, there are no edges connecting them, and the same holds for set B. Consequently, every edge in the graph connects one endpoint in set A and the other in set B, which defines G as bipartite.

Conversely, let us assume that G is a bipartite graph, implying we can partition its vertices into two distinct subsets, A and B, where every edge connects a vertex in A to one in B. By coloring every vertex in A as red and every vertex in B as blue, we get a proper 2-coloring of G. Therefore, G is indeed 2-colorable. ■

Theorem 4.12 *The complete graph K_n is not $(n-1)$-colorable.*

Proof Consider any vertex coloring of the complete graph K_n that employs a maximum of $n-1$ colors. Given that there are precisely n vertices, it follows that there must be at least one pair of vertices, say u and v, which are assigned the same color. However, (u, v) forms an edge within K_n. This particular edge possesses two endpoints sharing the identical color. Consequently, the coloring is improper, leading to the conclusion that K_n cannot be colored with just $n-1$ colors. ■

Definition 4.7 *The chromatic number of a graph G is denoted by $\chi(G)$ and is defined to be the minimum positive integer k such that G is k-colorable. A graph G is called k-chromatic if $\chi(G) = k$.*

From Definition 4.7, a graph is called k-chromatic if it is k-colorable, but not $(k-1)$-colorable. For example, by Theorem 4.12, the complete graph K_n is n-chromatic (hence $\chi(K_n) = n$). Note that, from Theorem 4.11, a graph G is bipartite iff $\chi(G) \leq 2$, and the equality holds iff G has at least one edge. In particular, the complete bipartite graph is 2-chromatic (hence $\chi(K_{m,n}) = 2$). See Figure 4.23. We also have the following theorem.

Theorem 4.13 *If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.*

Proof Let G be a graph that has H as a subgraph. Let also $\chi(G) = k$, then G is k-colorable, hence so is H. This implies that $\chi(H) \leq k = \chi(G)$. ■

Finding the chromatic number of a graph is an optimization problem. In general, when $\chi(G) \geq 3$, computing the chromatic number of G is an NP-hard problem, which belongs to a class of problems that will be introduced briefly in Section 7.7.

4.5 Directed graphs

All graphs that we have looked at so far are undirected graphs. Their edges are also said to be undirected. Sometimes it is necessary to associate directions with edges and get a directed graph instead.

Figure 4.26: A simple digraph (left) versus a non-simple digraph (right).

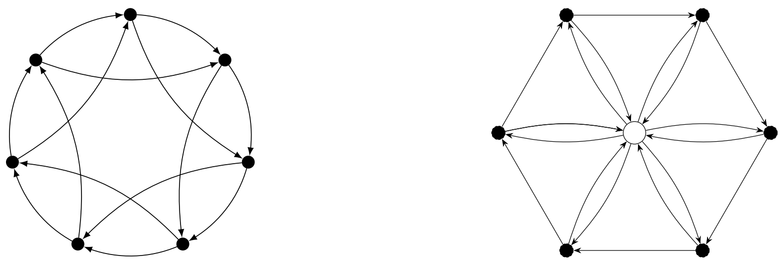


Figure 4.27: Balanced digraphs.

Before formally introducing the definition of a directed graph, it is worth mentioning that we can use relations in this definition. The representation of a relation $\mathcal{R}$ on a set A can be introduced with a directed graph as follows: A collection of vertices, one for each element in A, and directed edges, where an edge exists from vertex u to vertex v if and only if $u \mathcal{R} v$. It is also standard to treat a directed graph as we define it below.

Definition 4.8 *A directed graph, or digraph, is a graph whose edges have a defined direction, usually edges are represented as ordered pairs where the ordered pair* (u, v) *indicates that there is a directed edge from vertex u to vertex v.*

A slightly different set of definitions are used for directed graphs. A simple digraph has no loops and no multiple edges. See Figure 4.26. All directed graphs that we consider in the book are simple.

Vertex in-degree and out-degree If $e = (u, v)$ is a directed edge in a digraph, then e is called incident from u and incident to v. The in-degree of a vertex v, denoted as $\deg_{\text{in}}(v)$, is the number of edges incident to v (i.e., those entering the vertex v). The out-degree of a vertex v, denoted as $\deg_{\text{out}}(v)$, is the number of edges incident from v (i.e., those leaving the vertex v). For example, in the graph shown on the left-hand side of Figure4.26, we have $\deg_{\text{in}}(v) = \deg_{\text{out}}(u) = 2$ and $\deg_{\text{out}}(v) = \deg_{\text{in}}(u) = \deg_{\text{out}}(w) = \deg_{\text{in}}(w) = 1$.

A vertex v is called balanced if its in-degree is equal to its out-degree. A directed graph is balanced if every vertex is balanced. In Figure 4.27, we show two balanced directed graphs in which each vertex has 2 incoming and 2 outgoing edges, except the non-solid vertex which has 6 incoming and 6 outgoing edges.

Figure 4.28: A valid directed path from u to z (left) versus a "non-valid path" from u to z (right).

The following is the directed version of Theorem 4.1.

Theorem 4.14 *In a directed graph $G = (V, E)$, we have*

$$\sum_{v \in V} \deg_{in}(v) = \sum_{v \in V} \deg_{out}(v) = |E|.$$

Proof Every directed edge (w, u) has 2 endpoints (w and u). The first endpoint w contributes exactly 1 to $\deg_{\text{out}}(w)$. The second endpoint u contributes exactly 1 to $\deg_{\text{in}}(u)$. Thus, each directed edge contributes exactly 1 to exactly one of the sums on the left. The proof is complete. ■

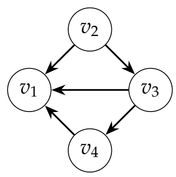
The simple calculations in the following example match the result in Theorem 4.14.

Example 4.10
In the digraph, $G = (V, E)$, shown to the right, we have

$$\sum_{v \in V} \deg_{\text{in}}(v) = \sum_{i=1}^{4} \deg_{\text{in}}(v_i) = 3 + 0 + 1 + 1 = 5 = |E|$$

and

$$\sum_{v \in V} \deg_{\text{out}}(v) = \sum_{i=1}^{4} \deg_{\text{out}}(v_i) = 0 + 2 + 2 + 1 = 5 = |E|.$$



Digraph of Example 4.10.

■

Directed paths, cycles and trees A directed path, or dipath, in a directed graph is a sequence of edges joining a sequence of distinct vertices where the edges are all directed in the same direction. See Figure 4.28.

A directed cycle, or dicycle, is a directed path (with at least one edge) whose first and last vertices are the same. A directed tree, or ditree, is a directed graph whose underlying graph is a tree. A directed forest is a family of disjoint directed trees. See Figure 4.29.

A semicycle is a directed cycle in which some of its edges have been reversed. In other words, a semicycle constitutes an (undirected) cycle when neglecting its directionality. A cycle in a digraph is either a directed cycle or a semicycle. For example, the circle given by the sequence (v, w, x, y, v) in the digraph shown to the right in Figure 4.30 is a directed cycle, whereas that given by the same sequence in the digraph shown in the middle of Figure 4.30 is a semicircle.

After introducing the weakly connectedness in this section, we can see that a directed tree can now be redefined to be a weakly connected digraph that has no cycles.

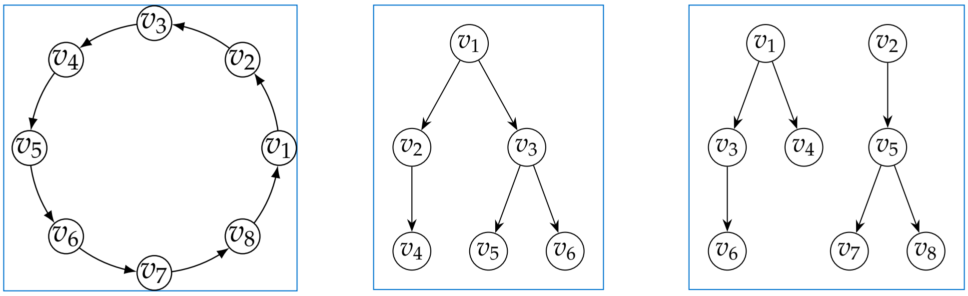


Figure 4.29: A directed cycle (left), a directed tree (middle), and a directed forest (right).

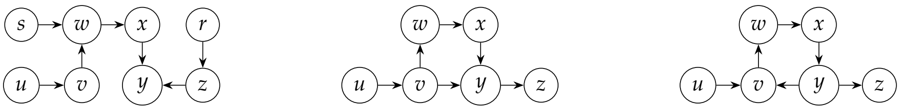
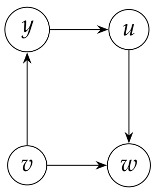


Figure 4.30: A directed tree (left), a DAG (middle), and a non-DAG (right).

The vertex y is not reachable from the vertex w, but the underlying undirected graph is connected.



In the graph shown on the right, every two vertices are reachable from each other.

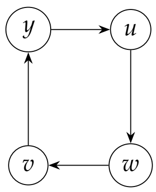


Figure 4.31: A weakly connected digraph (left) versus a strongly connected digraph (right).

We also have the following definition.

Definition 4.9 *A directed acyclic graph (DAG) is a digraph that has no directed cycles.*

Figure 4.30 shows a directed tree, a DAG, and a non-DAG. In the next chapter, we will see that a DAG can be redefined to be a digraph that has the so-called topological ordering.

Connectedness A directed graph is called weakly connected if replacing all of its directed edges with undirected edges produces a connected (undirected) graph. A directed graph is called strongly connected if every two vertices are reachable from each other. In other words, a digraph strongly connected if there is a path in each direction between each pair of vertices of the digraph. See Figure 4.31.

A strongly connected component G' of a digraph G is a maximal strongly connected directed subgraph of G. By maximality, we mean that there is no way to add into G' any vertices and/or directed edges of G which are not currently in G' in such a way that the resulting directed subgraph is strongly connected.

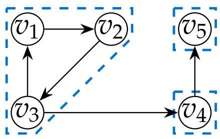


Figure 4.32: A digraph with three strongly connected components.

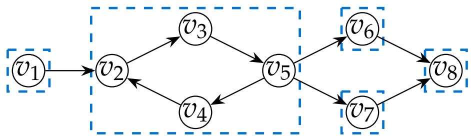


Figure 4.33: A digraph with five strongly connected components.

The digraph shown in Figure 4.32, for example, has three strongly connected components which are surrounded by blue dashed polygons. For another example, the digraph shown in Figure 4.33 has five strongly connected components which are surrounded by blue dashed polygons. Note that a strongly connected digraph has only one strongly connected component.

Remark 4.5 *The strongly connected components of a digraph define equivalence classes over its vertices under the "are mutually reachable from" relation.*

Remark 4.5 can be directly proven on the basis of Definitions 2.7 and 2.8.

Exercises

4.1 Choose the correct answer for each of the following multiple-choice questions/items.

(a) If a simple graph has 3 vertices, then the number of vertices with odd degrees can only be:

(i) 0. (ii) 1. (iii) 2. (iv) 0 or 2.

(b) What is the number of edges present in a graph having 244 degrees total?

(i) 61. (ii) 122. (iii) 244. (iv) 488.

(c) Let $G = (V, E)$ be an undirected graph. The following statements are equivalent, except for one that is not. Which one?

(i) Any two vertices in G are connected by a simple path.
(ii) G is connected but each edge is a bridge.
(iii) G is connected and $|V|^2 = |E|^2 + 2|E| + 1$.
(iv) G is acyclic but if any edge is added to G it creates a cycle.

(d) Let G be a simple connected graph with n vertices, where n is an odd number greater than 2. Which one of the following statements is the contrapositive of Dirac's theorem for graphs?

(i) If every vertex of G is adjacent to at least $(n+1)/2$ vertices, then G has a Hamiltonian cycle.

(ii) If G has no Hamiltonian cycle, then every vertex of G is adjacent to at most $(n+1)/2$ vertices.

(iii) If G has no Eulerian cycle, then each vertex of G is adjacent to less than $(n+1)/2$ vertices.

(iv) Every vertex of G has degree at most $n/2$ if G has no Hamiltonian cycle.

(e) Let p and q be even numbers greater than two. Which one of the following statements is false?

(i) The complete bipartite graph $K_{2,q}$ has an Eulerian cycle.

(ii) The complete bipartite graph $K_{2,q}$ has an Eulerian path.

(iii) The complete graph K_p has no Eulerian cycle.

(iv) The complete bipartite graph $K_{p,q}$ has an Eulerian cycle.

(f) Let p be an integer number greater than one. Which one of the following graphs has no Hamiltonian cycle?

(i) The complete graph K_p.

(ii) The complete bipartite $K_{p,p}$.

(iii) The complete bipartite $K_{p,p+1}$.

(iv) The cycle C_p.

(g) The chromatic number of an even cycle (i.e., a cycle with even length) equals:

(i) 2. (ii) 3. (iii) 2 or 3. (iv) its length.

(h) Let $G = (V, E)$ be a digraph, where $V = \{a, b, c, d, e, f, g, h, i\}$ and $E = \{(a, d), (d, a), (b, c), (c, f), (f, b), (g, h), h, g), (b, e), (e, d), (e, i), (h, e)\}$. What is the number of strongly connected components in G?

(i) 3. (ii) 4. (iii) 5. (iv) 6.

(i) Which one of the following statements is true?

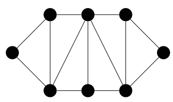
(i) There are two isomorphic graphs with different degree sequences.

(ii) There are no two non-isomorphic graphs with the same degree sequence.

(iii) There are no two isomorphic graphs with the same degree sequences.

(iv) There are two non-isomorphic graphs with the same degree sequence.

4.2 Find a spanning tree for the following graph so that no vertex has a degree of 4.



4.3 Are the graphs G_1 and G_2 shown below isomorphic? Why or why not?

4.4 Determine whether each of the following sequences is a degree sequence of a graph? Answer by 'yes' or 'no' and justify your answer.

(a) $(1, 2, 3, 3, 4, 5, 5)$. (b) $(0, 0, 1, 1, 1, 1, 4, 4)$. (c) $(0, 2, 2, 2, 2, 4, 4, 4)$.

4.5 Prove that the direction "$(f) \to (a)$" in Theorem 4.7.

4.6 Give an undirected graph with 5 vertices, all of which must be at least degree 3. It must have an Eulerian cycle and a Hamiltonian cycle. Can you give another undirected graph?

4.7 Does the bipartite graph $K_{5,7}$ have a Hamiltonian cycle or path? Justify your answer.

4.8 Let G be a k-colorable simple graph. Provide a tight upper bound[3] on the maximum number of colors that one could need to properly color the graph if

(a) one adds an edge between two vertices in G.

(b) one removes an edge from G.

4.9 Is the statement that "Any graph with a vertex of degree d is $d + 1$-colorable" true or false? If it is false, explain why.

4.10 Find the strongly connected components of the digraph shown in Figure 4.34.

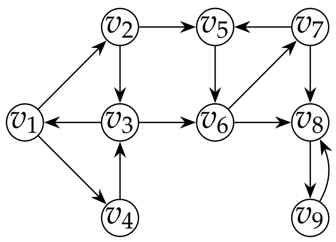


Figure 4.34: Digraph of Exercise 4.10.

[3]By a tight upper bound, we look for an "interesting answer". For example, the upper bound $|V|$ is not an interesting answer in terms of credit.

Notes and sources

Graph theory has its roots in the 18th century. The Swiss mathematician Leonhard Euler is often credited with pioneering graph theory through his work on the "Seven Bridges of Königsberg" problem in 1735; see Euler [1736]. Euler's solution to this problem laid the foundation for the study of graphs and led to the development of graph theory as a distinct mathematical discipline. The theory continued to evolve in the 19th and 20th centuries, with contributions from mathematicians like Gustav Kirchhoff and Arthur Cayley, who made significant advancements in the study of graphs and their applications in various fields, see for instance Cayley [1889].

This chapter served as a comprehensive introduction to essential concepts in graph theory, providing a foundation for both theoretical understanding and practical problem-solving in diverse fields. It covered key topics such as graph properties, graph coloring, and directed graphs. Readers also gained an understanding of how graphs could be analyzed and compared for structural similarity through isomorphism. We studied special graphs, including Eulerian and Hamiltonian graphs, and explored the art of graph coloring with applications.

As we conclude this chapter, it is worth noting that the cited references and others, such as Rosen [2002], Cusack and Santos [2021], Mott et al. [1986], Diestel [2012], Harris et al. [2008], Bondy and Murty [2008], Bollobás [2002], Godsil and Royle [2001], Erciyes [2021], Rahman [2017], Kagaris and Tragoudas [2008], Gabow et al. [2003], also serve as valuable sources of information pertaining to the subject matter covered in this chapter. The code that created Figure 4.13 is due to StackExchange [2011]. The code that created the left-hand side picture in Figure 4.20 is due to StackExchange [2014]. We modified a code due to StackExchange [2015] to create Figure 4.24. We modified a code due to StackExchange [2020] to create Figure 4.27.

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CHAPTER 5

RECURRENCES

Chapter overview: Recurrences are equations or inequalities that are used to describe function in terms of their value on smaller inputs. In this chapter, we learn some recurrence-solving techniques, namely guess-and-confirm, recursion-iteration, generating functions, and recursion-tree. This chapter concludes with a set of exercises to encourage readers to apply their knowledge and enrich their understanding.

Keywords: Recurrences, Recurrence-solving techniques, Generating functions, Recursion-tree

Contents

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Within this chapter, we acquire knowledge of various methods for solving recurrences. A formal definition of a recurrence looks like:

Definition 5.1 *A recurrence is an equation or inequality that describes a function in terms of its value on smaller inputs.*

There are five efficient methods for solving recurrences, namely the guess-and-confirm method, the iteration method, the recursion-tree method, the generating functions method, and the master method. In this chapter, we present the first four methods. The fifth method (master method) is beyond the scope of this book. However, there are a number of good references for solving recurrences using the master method, see for example [Cormen et al., 2001, Section 4.5].

5.1 Guess-and-confirm

In this section, we learn what we mean by solving recurrences and then present the guess-and-confirm method for solving them.

As a matter of example, it can be shown (see Exercise 5.2 (b)) that the following recurrence

$$T(1) = 1, \quad T(n) = 3T(n-1) + 4, \quad n = 1, 2, 3, \ldots, \tag{5.1}$$

describes the function

$$T(n) = 3^n - 2, \quad n = 0, 1, 2, \ldots \,. \tag{5.2}$$

An explicit formula (also called a closed formula) is said to be a solution of a recurrence relation if its terms satisfy the recurrence relation. For example, the formula (5.2) is the solution of the recurrence relation (5.1). The initial conditions for a recurrence relation specify the terms that precede the first term where the recurrence relation takes effect. For example, $T(1) = 1$ is the initial condition for the recurrence relation (5.1).

The following are some instances of recurrence formulas.

- A recurrence formula that loops through the input to eliminate one item. For example:

$$T(n) = T(n-1) + 1, \; T(n) = T(n-1) + n, \; \text{etc.}$$

- A recurrence formula that halves the input. For example:

$$T(n) = T(n/2) + 1, \; T(n) = T(n/2) + n, \; \text{etc.}$$

- A recurrence formula that splits the input into two halves. For example:

$$T(n) = 2T(n/2) + 1, \; \text{etc.}$$

Each remaining section of this chapter describes a method for solving recurrences.

As we mentioned earlier, the technique under discussion in this section is the guess-and-confirm method. This method follows a systematic approach in which we initially employ repeated substitutions to make an educated guess or hypothesis regarding the explicit form of a recurrence formula. Once we have this proposed expression, we proceed to utilize mathematical induction, a fundamental proof technique of Section 2.1, to rigorously establish and validate the accuracy of our conjecture. Due to this dual-step process, the guess-and-confirm method is also recognized as the substitution/induction method, as it distinctly involves both of these fundamental components.

To illustrate the practical application of this method, we will provide examples that showcase its effectiveness and how it is systematically employed to derive and validate recurrence formulas. Through these examples, readers will gain a deeper understanding of the guess-and-confirm method and its value as a tool in solving recurrence relations.

Example 5.1 Use the guess-and-confirm method to solve the following recurrences.

(a) $T(n)$[1] $= \begin{cases} c_1, & \text{if } n = 1; \\ T(n-1) + c_2, & \text{if } n > 1. \text{ Here, } c_1 \text{ and } c_2 \text{ are constants.} \end{cases}$

(b) $T(n)$[2] $= \begin{cases} 0, & \text{if } n = 1; \\ T(n/2) + 1, & \text{if } n > 1, \text{ and } n \text{ is a power of 2.} \end{cases}$

Solution (a) Using repeated substitutions, we have

$$\begin{array}{rcl} T(1) & = & c_1, \\ T(2) & = & T(1) + c_2 = c_1 + c_2, \\ T(3) & = & T(2) + c_2 = c_1 + 2c_2, \\ T(4) & = & T(3) + c_2 = c_1 + 3c_2, \\ & \vdots & \\ T(n) & = & T(n-1) + c_2 = c_1 + (n-1)c_2, \text{ for all } n \geq 1. \end{array}$$

We prove the nth-term guess by mathematical induction. The base case is trivial: $T(1) = c_1$. For the inductive hypothesis: we assume that $T(k) = c_1 + (k-1)c_2$, for $k < n$. We now use the inductive hypothesis to prove the statement for $k + 1$. Note that

$$T(k+1) = c_2 + T(k) = c_2 + c_1 + (k-1)c_2 = c_1 + kc_2,$$

which completes the inductive step. Thus $T(n) = c_1 + (n-1)c_2$ for $n \geq 1$.

(b) Using repeated substitutions, we have

$$\begin{array}{rcl} T(1) & = & 0 = \log 1, \\ T(2) & = & T(2/2) + 1 = T(1) + 1 = 0 + 1 = 1 = \log 2, \\ T(4) & = & T(4/2) + 1 = T(2) + 1 = 1 + 1 = 2 = \log 4, \\ T(8) & = & T(8/2) + 1 = T(4) + 1 = 2 + 1 = 3 = \log 8, \\ & \vdots & \\ T(n) & = & \log n, \text{ where } n > 1 \text{ and } n \text{ is a power of 2.} \end{array}$$

[1]This recurrence formula is the one that arises from the factorial function algorithm (Algorithm 7.28).
[2]This recurrence formula is the one that arises from the binary search algorithm (Algorithm 8.6).

We prove the nth-term guess by mathematical induction. The base case is trivial: $T(1) = 0 = \log 1$. For the inductive hypothesis: we assume that $T(k) = \log k$, for $k < n$. We now use the inductive hypothesis to prove the statement for n. Note that

$$T(n) = T(n/2) + 1 = \log(n/2) + 1 = \log n - \log 2 + 1 = \log n.$$

This completes the inductive step and hence completes the proof. ■

Example 5.2 Use the guess-and-confirm method to determine a good upper bound on the following recurrence.

$$T(n) = \begin{cases} 1, & \text{if } n = 0, 1; \\ T(n-1) + T(n-2) + 1, & \text{if } n > 1. \end{cases}$$

Solution Using repeated substitutions, we have

$$\begin{array}{rcl} T(0) & = & 1 \leq 2^0, \\ T(1) & = & 1 \leq 2^1, \\ T(2) & = & T(1) + T(0) + 1 = 3 \leq 2^2, \\ T(3) & = & T(2) + T(1) + 1 = 5 \leq 2^3, \\ T(4) & = & T(3) + T(2) + 1 = 9 \leq 2^4, \\ T(5) & = & T(4) + T(3) + 1 = 15 \leq 2^5, \\ & \vdots & \\ T(n) & = & T(n-1) + T(n-2) + 1 \leq 2^n, \text{ for all } n \geq 1. \end{array}$$

We prove the nth-term guess, which is $T(n) \leq 2^n$, by induction on n. The base case is trivial: we have $T(0) \leq 2^0 = 1$ (same for $T(1)$). For the inductive hypothesis: we assume that $T(k) \leq 2^k$. We now use the inductive hypothesis to prove the statement for $k + 1$. Note that

$$T(k+1) = T(k) + T(k-1) + 1 \leq 2^k + 2^{k-1} + 1 \leq 2^k + 2^k = 2(2^k) = 2^{k+1},$$

where the last inequality follows from the fact that $2^{k-1} + 1 \leq 2^k$ for all $k > 0$. This completes the inductive step and hence completes the proof. In conclusion, we have $T(n) \leq 2^n$ for $n = 0, 1, 2, \ldots$. ■

5.2 Recursion-iteration

Within this section, we introduce the iteration method, a valuable technique for resolving recurrence relations. This method is characterized by a systematic approach where we dissect the given recurrence relation into a sequence of individual terms, each of which helps derive the expression for the nth term based on the preceding terms. This process continues iteratively until we reach the desired solution for the nth term of the recurrence.

The iteration method is a powerful tool for solving recurrence relations, as it allows for a step-by-step and cumulative approach to understanding and deriving the expressions for the terms in the sequence. Through the examples presented in this section, we aim to provide practical illustrations of how the iteration method is effectively applied, allowing readers to grasp its significance and applicability in solving various types of recurrence relations.

Example 5.3 Use the iteration method to solve the following recurrence formula.

$$\begin{aligned} T(1) &= c_1, \\ T(n) &= c_2 + T(n-1), \quad n > 1, \end{aligned} \tag{5.3}$$

where c_1 and c_2 are constants.

Solution From (5.3), we have

$$\begin{aligned} T(n) &= c_2 + T(n-1) \\ &= c_2 + (c_2 + T(n-2)) \\ &= c_2 + (c_2 + (c_2 + T(n-3))) \\ &= 3c_2 + T(n-3) \\ &\vdots \\ &= kc_2 + T(n-k). \end{aligned}$$

Let $k = n-1$, then $n-k = 1$ and we have

$$T(n) = (n-1)c_2 + T(1) = (n-1)c_2 + c_1.$$

■

Example 5.4 Use the iteration method to solve the following recurrence formula.

$$\begin{aligned} T(1) &= c, \\ T(n) &= c + T(n/2), \end{aligned} \tag{5.4}$$

where c is a constant.

Solution From (5.4), we have

$$\begin{aligned} T(n) &= c + T(n/2) \\ &= c + (c + T(n/4)) \\ &= c + (c + (c + T(n/8))) \\ &= 4c + T(n/2^4) \\ &\vdots \\ &= ck + T(n/2^k). \end{aligned}$$

Let $n = 2^k$, then $k = \log n$. It follows that $T(n) = ck + T(n/2^k) = c \log n + c$, where $c = T(n/2^k) = T(1)$. ■

Example 5.5 Use the iteration method to solve the following recurrence formula.

$$T(n) = n + 2T(n/2). \tag{5.5}$$

Solution From (5.5), we have

$$\begin{aligned} T(n) &= n + 2T(n/2) \\ &= n + 2(n/2 + 2T(n/4)) \\ &= n + 2(n/2 + 2(n/4 + 2T(n/8))) \\ &= 3n + 2^3 T\left(n/2^3\right) \\ &\vdots \\ &= kn + 2^k T\left(n/2^k\right). \end{aligned}$$

Let $n = 2^k$, then $k = \log n$. It follows that $T(n) = nk + 2^k T(n/2^k) = n\log n + nT(1)$ as desired. ■

Change of variables Sometimes we need to transform the recurrence to one that we solved before. We have the following example.

Example 5.6 Solve the recurrence $T(n) = 2T(\sqrt{n}) + \log n$.

Solution Set $m = \log n$, then $n = 2^m$ and

$$T(2^m) = 2T(2^{m/2}) + m. \tag{5.6}$$

After renaming m to k and letting $S(k) = T(2^k)$, the formula (5.6) becomes $S(k) = 2S(k/2) + k$, which is the same recurrence formula (5.6). In Algorithm 8.9, we used the iteration method to solve this recurrence and found that $S(k) = k\log k + kS(1)$. It immediately follows that

$$T(n) = T(2^m) = S(m) = m\log m + mS(1) = \log n \log(\log n) + c\log n,$$

where $c = T(2)$. ■

5.3 Generating functions

Recurrence relations can be solved by finding a closed form for the associated generating function.

Definition 5.2 *Let $\{a_n\}$ be a sequence of real numbers. The generating function for $\{a_k\}$ is the infinite series*

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{k=1}^{\infty} a_k x^k.$$

Below are some examples of generating functions.

Example 5.7 The generating function for the sequence $1, 1, 1, \ldots$ is

$$1 + x + x^2 + \cdots = \sum_{k=1}^{\infty} x^k = \frac{1}{1-x} \quad \text{for } |x| < 1,$$

where the last equality was obtained by the geometric series (3.5). ■

Example 5.8 The generating function for the sequence $1, a, a^2, \ldots$ is

$$1 + ax + a^2x^2 + \cdots = \sum_{k=1}^{\infty} a^k x^k = \sum_{k=1}^{\infty} (ax)^k = \frac{1}{1-ax} \quad \text{for } |ax| < 1^3, \tag{5.7}$$

where the last equality was obtained by the geometric series (3.5). ■

We can define generating functions for finite sequences of real numbers by extending a finite sequence $a_1, a_2, \ldots, a_n$ into an infinite sequence by setting $a_j = 0$ for $j \geq n+1$. We have the following example.

Example 5.9 The generating function for the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{3}, \ldots, \binom{n}{n}$ is the finite series

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{k=1}^{n} \binom{n}{k} x^k = (1+x)^n,$$

where the last equality was obtained by the binomial theorem (Theorem 6.1). ■

In the generating function method for solving recurrences, we transform the recurrence relation for the terms of a sequence into an equation involving a generating function. This equation can then be solved to find a closed form for the generating function. The recurrences that we solve in the following example are taken from Rosen [2002] with little modification in their form.

Example 5.10 Use the generating function method to solve the following recurrences.

$$(a)\ T(n) = \begin{cases} 4, & \text{if } n = 0; \\ 4T(n-1), & \text{if } n \geq 1. \end{cases} \qquad (b)\ T(n) = \begin{cases} 1, & \text{if } n = 0; \\ 8T(n-1) + 10^{n-1}, & \text{if } n \geq 1. \end{cases}$$

Solution (*a*) Let $g(x)$ be the generating function for the sequence $\{a_n\} = \{T(n)\}$. Then $g(x) = \sum_{k=0}^{\infty} T(k)\, x^k$. Using the recurrence relation, we have

$$g(x) = T(0) + \sum_{k=1}^{\infty} T(k)x^k = 4 + \sum_{k=1}^{\infty} 4T(k-1)x^k = 4 + 4x \sum_{k=1}^{\infty} T(k-1)x^{k-1} = 4 + 4x \sum_{k=0}^{\infty} T(k)x^k.$$

It follows that

$$g(x) = 4 + 4xg(x).$$

[3] Equivalently, $|x| < 1/|a|$ for $a \neq 0$.

Solving for $g(x)$, we get

$$g(x) = \frac{4}{1-4x} = 4\sum_{k=0}^{\infty}(4x)^k = \sum_{k=0}^{\infty}\underbrace{4^{k+1}}_{T(k)}x^k,$$

where the second equality was obtained by the geometric series (5.7). Therefore,

$$T(n) = 4^{n+1}$$

for any $n \geq 0$.

(*b*) Let $g(x)$ be the generating function for the sequence $\{a_n\} = \{T(n)\}$. Then

$$g(x) = \sum_{k=0}^{\infty} T(k)\,x^k.$$

Using the recurrence relation, we have

$$\begin{aligned}
g(x) &= T(0) + \sum_{k=1}^{\infty} T(k)x^k \\
&= 1 + \sum_{k=1}^{\infty}(8T(k-1) + 10^{k-1})x^k \\
&= 1 + 8x\sum_{k=1}^{\infty} T(k-1)x^{k-1} + x\sum_{k=0}^{\infty} 10^{k-1}x^{k-1} \\
&= 1 + 8x\sum_{k=0}^{\infty} T(k)x^k + x\sum_{k=0}^{\infty} 10^k x^k \;=\; 1 + 8xg(x) + \frac{x}{1-10x},
\end{aligned}$$

where the last equality was obtained by the geometric series (5.7). It follows that

$$g(x) = 1 + 8xg(x) + \frac{x}{1-10x}.$$

Solving for $g(x)$, we get

$$g(x) = \frac{1}{(1-8x)}\left(1 + \frac{x}{1-10x}\right) = \frac{1-9x}{(1-10x)(1-8x)}.$$

Note that

$$1 - 9x = \frac{1}{2}(1+1) - \frac{1}{2}(8x+10x) = \frac{1}{2}(1-8x) + \frac{1}{2}(1-10x).$$

Hence, the function $g(x)$ can be written as

$$\begin{aligned}
g(x) &= \frac{\frac{1}{2}(1-8x) + \frac{1}{2}(1-10x)}{(1-10x)(1-8x)} \\
&= \frac{1/2}{1-10x} + \frac{1/2}{1-8x} \;=\; \frac{1}{2}\left(\frac{1}{1-10x} + \frac{1}{1-8x}\right).
\end{aligned}$$

Using the geometric series (5.7), we have

$$g(x) = \frac{1}{2}\left(\sum_{k=0}^{\infty}(10x)^k + \sum_{k=0}^{\infty}(8x)^k\right) = \sum_{k=0}^{\infty} \underbrace{\frac{1}{2}\left(10^k + 8^k\right)}_{T(k)} x^k.$$

Therefore,

$$T(n) = \frac{1}{2}(10^k + 8^k)$$

for any $n \geq 0$.

■

In summary, the iteration method is a powerful technique for resolving recurrence relations. By breaking down these recurrences into a sequence of individual terms and deriving expressions for each term based on preceding ones, we can systematically unveil the solutions to these recursive problems. This approach not only helps us understand the underlying patterns but also provides a practical way to derive closed-form expressions for recurrence relations.

5.4 Recursion-tree

The recursion-tree method is a strategy employed for tackling recurrence relations, and it involves a systematic approach. To utilize this method, we transform the given recurrence into a tree structure, where each node within the tree signifies the cost incurred at different levels of recursion. This tree effectively helps us visualize and break down the recurrence relation, allowing us to analyze and sum up the costs associated with each level of the recursion.

In order to grasp the practical application of the recursion-tree method, it is often best comprehended through an illustrative example. The following example serves as a practical demonstration of how the method operates, offering a clear and tangible instance of how the tree structure is employed to calculate and sum the costs at various recursion levels. Through this illustration, readers can gain a deeper understanding of the methodology and its utility in resolving recurrence relations. Example 5.11 is extracted with some modifications from [Cormen et al., 2001, Section 4.4].

Example 5.11 Use the recursion tree method to determine a good upper bound on each of the following recurrences. (Here c is a constant).

(*a*) $T(n) = 2T(n/2) + n^2$.

(*b*) $T(n) = 3T(n/4) + cn^2$.

(*c*) $T(n) = T(n/3) + T(2n/3) + n$.

Solution (*a*) For convenience, we assume that n is an exact power of 2 so that all subproblem sizes are integers. In Figure 5.1, we show in detail the construction of a recursion tree for the recurrence $T(n) = 2T(n/2) + n^2$. Note that we continue expanding each node in the tree by breaking it into its constituent parts as determined by the recurrence.

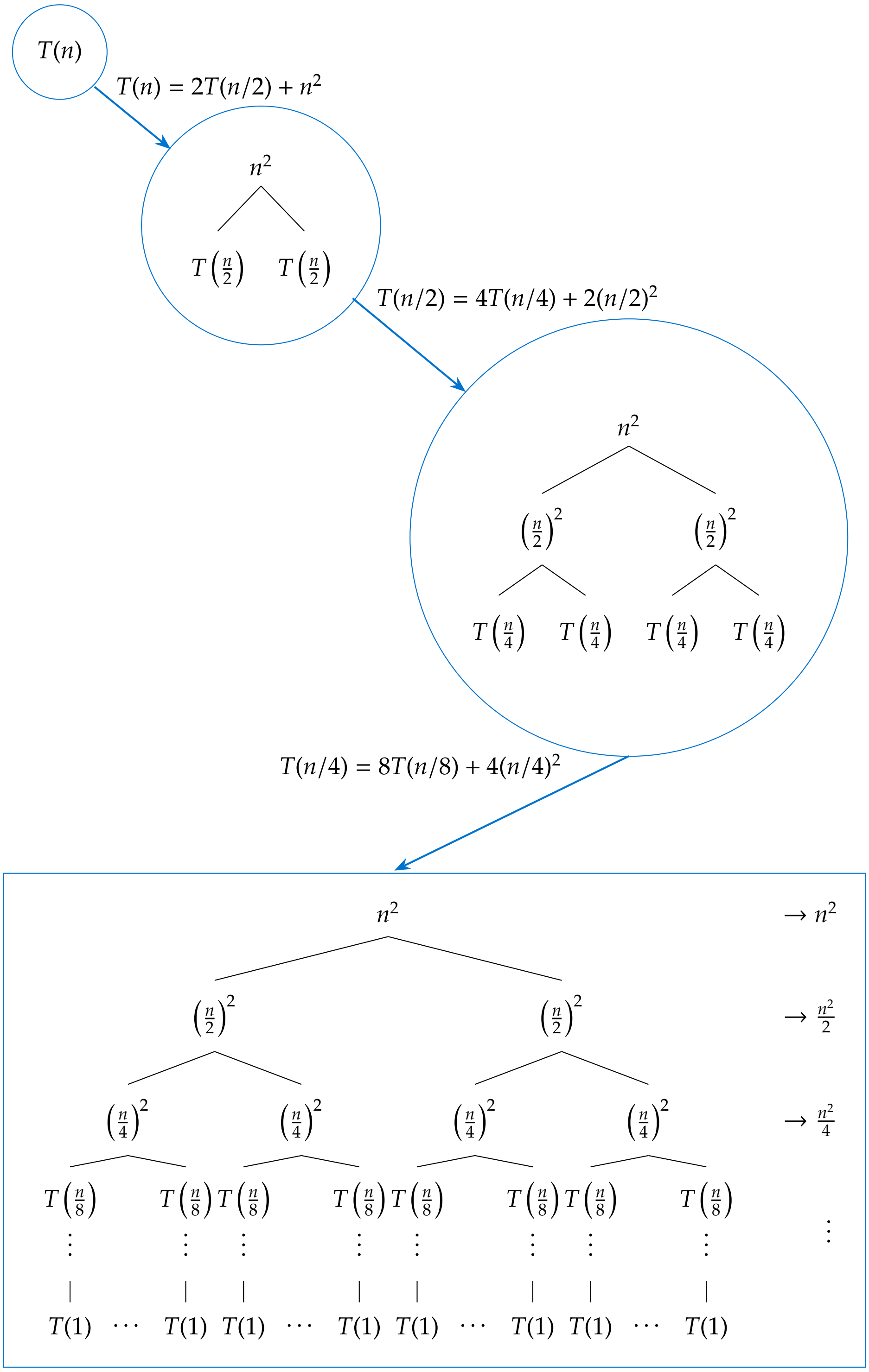

Figure 5.1: Constructing a recursion tree for $T(n) = 2T(n/2) + n^2$.

Let h be the height of the recursion tree. At each level $i = 0, 1, 2, \ldots, h$, we have

- subproblem size $= n/2^i$,
- cost of each node $= (n/2^i)^2$,
- number of nodes $= 2^i$.

The total cost at all levels is $T(n) = \sum_{i=0}^{h}$ total cost at level i, where

$$\begin{aligned}\text{total cost at level } i &= (\text{\# of node at level } i) \cdot (\text{\# cost of each node at level } i)\\ &= 2^i \left(\frac{n}{2^i}\right)^2 = \frac{n^2}{2^i}.\end{aligned}$$

Now, we want to know how to find the height of the tree h. Because subproblem sizes decrease by a factor of 2 each time we go down one level, we eventually must reach a boundary condition. Essentially, we want to know how far from the root do we reach one. Note that the subproblem size for a node at level h is $n/2^h$. Therefore, starting from the root, we reach one when $n/2^h = 1$ (as $T(n/2^h) = T(1)$). Hence, the height of the tree is $h = \log n$.

As a result, the total cost at all levels is

$$T(n) = \sum_{i=0}^{\log n} \frac{n^2}{2^i} = n^2 \sum_{i=0}^{\log n} \left(\frac{1}{2}\right)^i \le n^2 \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i \le n^2 \frac{1}{1-\frac{1}{2}} = 2n^2,$$

where the last inequality follows from the geometric series (3.5). Thus $T(n) \le 2n^2$.

(*b*) For convenience, we assume that n is an exact power of 4 so that all subproblem sizes are integers. In Figure 5.2, we show the recursion tree for the recurrence $T(n) = 3T(n/4) + cn^2$.

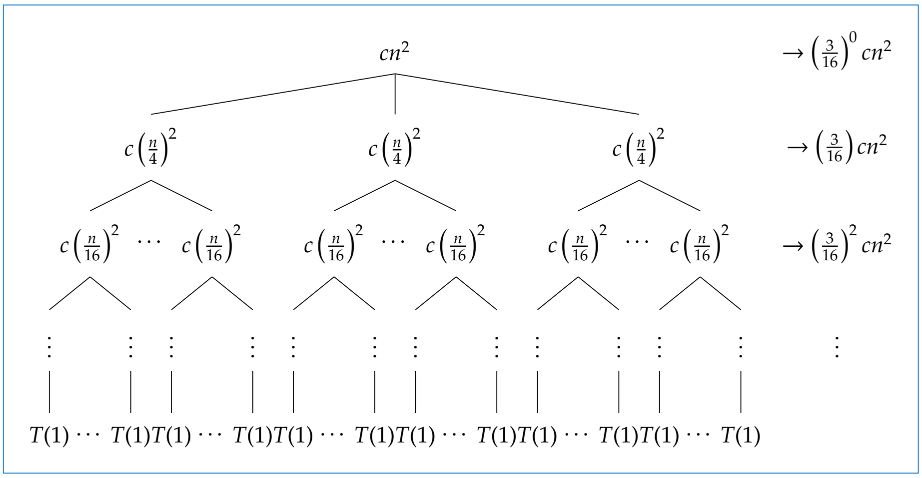

Figure 5.2: The recursion tree for $T(n) = 3T(n/4) + cn^2$.

Let h be the height of the recursion tree. At each level $i = 0, 1, 2, \ldots, h$, we have

- subproblem size = $n/4^i$,
- number of nodes = 3^i.
- cost of each node = $c(n/4^i)^2$,

The total cost at all levels is $T(n) = \sum_{i=0}^{h}$ total cost at level i, where

$$\begin{aligned}\text{total cost at level } i &= (\#\text{ of node at level } i) \cdot (\text{cost of each node at level } i)\\ &= c\left(\frac{n}{4^i}\right)^2 3^i = cn^2\left(\frac{3}{16}\right)^i.\end{aligned}$$

To know how far from the root do we reach one, we note that the subproblem size for a node at level h is $n/4^h$. Therefore, starting from the root, we reach one when $n/4^h = 1$. Hence, the height of the tree is $h = \log_4 n$.

As a result, the total cost at all levels is

$$T(n) = \sum_{i=0}^{\log_4 n} cn^2\left(\frac{3}{16}\right)^i \leq cn^2\sum_{i=0}^{\infty}\left(\frac{3}{16}\right)^i = cn^2\frac{1}{1-\left(\frac{3}{16}\right)} = \frac{16}{13}cn^2,$$

where the last inequality follows from the geometric series. Thus, we have $T(n) \leq (16/3)cn^2$.

(c) Figure 5.3 shows the recursion tree for $T(n) = T(n/3) + T(2n/3) + n$. Note that, unlike in the trees of (a) and (b), in this tree the paths from the root to the leaves have different lengths. Let h be the height of the recursion tree. Note also that the total cost at each level i ($i = 0, 1, 2, \ldots, h$) equals n, and that the longest path from the root to a leaf is

$$n \longrightarrow \left(\frac{2}{3}\right)n \longrightarrow \left(\frac{2}{3}\right)^2 n \longrightarrow \left(\frac{2}{3}\right)^3 n \longrightarrow \cdots \longrightarrow 1.$$

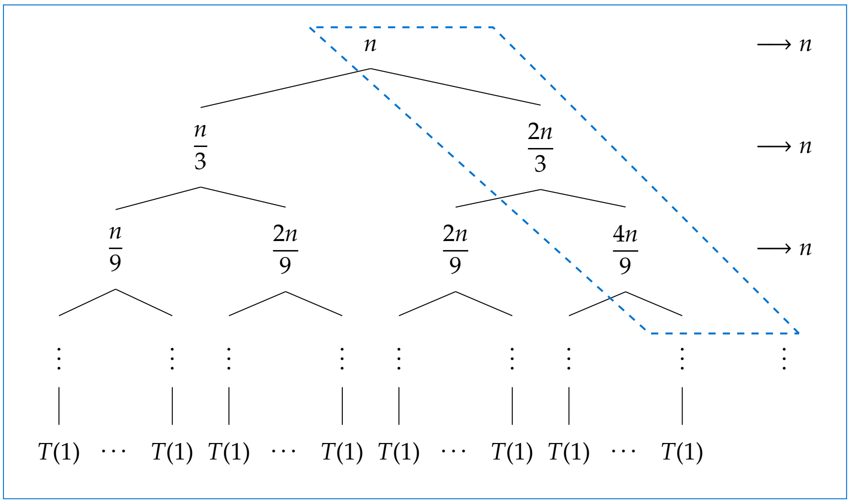

Figure 5.3: The recursion tree for $T(n) = T(n/3) + T(2n/3) + n$.

Starting from the root, we reach one when $(2/3)^h n = 1$. Hence, the height of the tree is $h = \log_{3/2} n$. It follows that the total cost at all levels is

$$T(n) \leq \sum_{i=0}^{\log_{3/2} n} n = \left(1 + \log_{3/2} n\right) n = \left(1 + \frac{\log n}{\log 3/2}\right) n \leq k\, n\, \log n,$$

for some positive constant k. Thus, $T(n) \leq k\, n \log n$.

Note that in Example 5.11 (c), we were looking for the longest path from the root to a leaf because we need an asymptotic upper bound. However, if an asymptotic lower bound is required, we should be looking the shortest path from the root to a leaf as in Exercise 5.9 (b).

Exercises

5.1 Choose the correct answer for each of the following multiple-choice questions/items.

(a) If the guess-and-confirm method is used to solve the recurrence relation $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$, we find that $T(n)$ is (here c is some constant)

(i) cn. (ii) $cn\sqrt{\log n}$. (iii) $cn \log n$. (iv) $cn \log(\log n)$.

(b) If we use the iteration method to solve the following recurrence.

$$T(n) = \begin{cases} T(n-1) + n, & \text{if} \quad n > 1; \\ 1, & \text{if} \quad n = 1, \end{cases}$$

we find that $T(n)$ is

(i) $1+2\cdots+(n-2)+(n-1) = n(n-1)/2$.
(ii) $1+2+\cdots+(n-1)+n = n(n+1)/2$.
(iii) $1+2+\cdots+n+(n+1) = (n+1)(n+2)/2$.
(iv) $1+2+\cdots+n+n^2 = n(3n+1)/2$.

(c) The generating function for the sequence $1, -1, 1, -1, 1, -1, \ldots$ is

(i) $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $|x| < 1$.
(ii) $\sum_{k=0}^{\infty} x^k = \frac{1}{1+x}$ for $|x| < 1$.
(iii) $\sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x}$ for $|x| < 1$.
(iv) $\sum_{k=0}^{\infty} (-1)^{k+1} x^k = \frac{1}{1+x}$ for $|x| < 1$.

(d) The height of the recursion tree drawn to give a good asymptotic upper bound on the recurrence relation $T(n) = T(n/5) + T(4n/5) + n$ is

(i) $\log_4 n$. (ii) $\log_5 n$. (iii) $\log_{5/4} n$. (iv) $\log_{4/5} n$.

(e) The height of the recursion tree drawn to give a good lower bound on the recurrence relation $T(n) = T(n/3) + T(2n/3) + n$ is

(i) $\log_{2/3} n$. (ii) $\log_2 n$. (iii) $\log_{3/2} n$. (iv) $\log_3 n$.

5.2 Use the guess-and-confirm method to solve the following recurrences.

(a) $T(n) = 2T(n/2) + n$, and $T(1) = 1$. (b) $T(n) = 3T(n-1) + 4$, and $T(1) = 1$.

5.3 Use the guess-and-confirm method to solve the following recurrence.

$$T(n) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n = 1, 2; \\ 5T(n-1) - 8T(n-2) + 4T(n-3), & \text{if } n \geq 3. \end{cases}$$

5.4 Use the iteration method to solve the following recurrences.

(a) $T(n) = 5T(n-1)$, where $T(0) = 3$.

(b) $T(n) = 2T(n/2) + n \log n$, where $T(1) = c$ for some constant c.

5.5 Use the generating function method to solve the following recurrence.

$$T(n) = \begin{cases} 1, & \text{if } n = 0; \\ 3, & \text{if } n = 1; \\ -T(n-1) + 6T(n-2), & \text{if } n \geq 2. \end{cases}$$

5.6 Use the generating function method to solve the Fibonacci recurrence:

$$T(n) = \begin{cases} 0, & \text{if } n = 0; \\ 1, & \text{if } n = 1; \\ T(n-1) + T(n-2), & \text{if } n \geq 2. \end{cases}$$

5.7 Use the recursion tree method to solve the recurrence $T(n) = 2T(n-1) + 1$.

5.8 Use the recursion tree method to determine a good upper bound on each of the following recurrences.

(a) $T(n) = T(n/2) + n^2$. (b) $T(n) = T(n-1) + T(n/2) + n$.

5.9 Use the recursion tree method to determine a good lower bound on each of the following recurrences.

(a) $T(n) = 4T((n/2) + 2) + n$.

(b) $T(n) = T(n/3) + T(2n/3) + cn$, where c is a constant.

Notes and sources

The study of recurrences, or recurrence relations, has a long history dating back to ancient civilizations. One of the earliest known instances of recurrence relations is in the work of Fibonacci, an Italian mathematician from the 13th century, who introduced the famous Fibonacci sequence to model the population growth of rabbits; see Sigler [2003]. Another noteworthy historical figure in the development of recurrence relations is Blaise Pascal, who investigated Pascal's triangle, a triangular array of numbers that exhibits numerous recurrence patterns, in the 17th century; see Pascal [1665].

In this chapter, we learned several recurrence-solving techniques, like guess-and-confirm, recursion-iteration, generating functions, and recursion-tree. As we conclude this chapter, it is worth noting that the cited references and others, such as Rosen [2002], Cusack and Santos [2021], Mott et al. [1986], Pottenger [1997], Spencer [1994], Lueker [1980], Bentley et al. [1980], Grimaldi [1999], Liu [1968], Greene and Knuth [2009], also serve as valuable sources of information pertaining to the subject matter covered in this chapter. Exercise 5.9 is due to Cormen et al. [2001].

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CHAPTER 6

COUNTING

Chapter overview: This chapter equips the reader with the essential tools and knowledge needed for counting and effective enumeration. It offers a comprehensive exploration of fundamental concepts in counting. More specifically, we introduce Binomial coefficients and their manipulation and identities. Then we delve into the principles of counting, gaining insight into various counting techniques, including permutations and combinations. The chapter concludes with a set of exercises to encourage readers to apply their knowledge and enrich their understanding.

Keywords: Counting, Binomial coefficients, Principles of counting, Permutations and combinations

Contents

B. Alzalg, *Combinatorial and Algorithmic Mathematics: From Foundation to Optimization*

Readers after this chapter will gain the necessary tools and knowledge required for proficient counting and enumeration. The first section of this chapter introduces the binomial series, which is needed before we start with the counting methods. This also includes the essence of a combinatorial proof and some binomial coefficients and identities.

6.1 Binomial coefficients and identities

A polynomial represents an algebraic expression comprising variables and coefficients, where the permitted operations include addition, subtraction, multiplication, and the nonnegative integer exponentiation of variables. For example, $x^3 + 4xyz^2 + 3y^2z$ is a polynomial of three variables. A polynomial of one variable has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where n is a nonnegative integer and the constant numbers $a_0, a_1, a_2, \ldots, a_n$ are the coefficients. A monomial is a polynomial which has only one term. For example, $2x$, $4x^3$ and $5x^2y^4$ are monomials. A binomial is a polynomial that is the sum of two terms, each of which is a monomial. For example, $2x + y$ is a binomial. The function $3x^2 + 4y^3z$ is also a binomial where the constants 3 and 4 are the binomial coefficients. A term can be a product of constants and variables, such as $5x^2y^4z^3$, but that does not concern us in our context here. In this section, we establish some identities that express relationships among the so-called binomial coefficients.

Most proofs that are given in this section are combinatorial proofs. A combinatorial proof of an identity is a proof that uses either two types of mathematical proof:

- A double counting proof: A proof that employs counting principles to demonstrate that both sides of the identity enumerate the same objects, albeit through distinct methods.

- A bijective proof: A proof that relies on establishing the existence of a one-to-one correspondence (bijection) between the sets of objects counted by the two sides of the identity.

An example of a combinatorial proof is the proof of Theorem 6.1 presented in the following subsection, which is variously known as the binomial theorem.

The binomial theorem and coefficients

The binomial theorem gives a formula for expanding the binomial power $(x + y)^n$ for any positive integer n. The symbol $\binom{n}{k}$ that appears in the following theorem is read as "n choose k", where n and k are integers that satisfy $n \geq k \geq 0$. The symbol $\binom{n}{k}$ represents the number of ways to choose k objects from a set of n objects, and is given by the formula[1]

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}.$$

[1]The factorial of a positive integer n, denoted by $n!$, is defined as $n! = n(n-1)\cdots 2{\cdot}1$. For example, $4! = 4{\cdot}3{\cdot}2{\cdot}1 = 24$. Note that 0!=1 and 1!=1.

It is easy to see that, for any integers $n \geq k \geq 0$, we have

$$\binom{n}{k} = \binom{n}{n-k}. \tag{6.1}$$

Note also that $\binom{k}{1} = k$ and $\binom{k}{k} = \binom{k}{0} = 1$ for any integer $k \geq 0$. This includes $\binom{1}{1} = \binom{0}{0} = 1$.

The symbol $\binom{n}{k}$ occurs as coefficients in the binomial theorem stated below. These coefficients are commonly called the binomial coefficients.

Theorem 6.1 (The binomial theorem) *Let x and y be variables and n be a nonnegative integer, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Proof When the product is expanded, its terms take the form of $x^{n-k}y^k$ for values of k ranging from 0 to n. To determine the count of terms in this form, consider that in order to obtain such a term, you must select $n-k$ instances of x from the total of n terms (leaving the remaining k as y terms). Hence, the coefficient of $x^{n-k}y^k$ is $\binom{n}{n-k}$, which is equivalent to $\binom{n}{k}$. ■

The series presented in the binomial theorem is known as the binomial series. Some computational uses of the binomial theorem are illustrated in the following examples.

Example 6.1 Find the expansion of each of the following expressions.

(*a*) $(x+y)^3$. (*b*) $(x+y)^4$. (*c*) $(x+y)^5$.

Solution (*a*) From the binomial theorem, it follows that

$$\begin{aligned}(x+y)^3 &= \sum_{k=0}^{3} \binom{3}{k} x^{3-k} y^k \\ &= \binom{3}{0} x^3 + \binom{3}{1} x^2 y + \binom{3}{2} x y^2 + \binom{3}{3} y^3 = x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

The same result can be obtained by simple computations as follows.

$$(x+y)^3 = (x+y)(x+y)^2 = (x+y)(x^2+2xy+y^2) = x^3 + 3x^2y + 3xy^2 + y^3.$$

(*b*) From the binomial theorem it follows that

$$\begin{aligned}(x+y)^4 &= \sum_{k=0}^{4} \binom{4}{k} x^{4-k} y^k \\ &= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

(*c*) The proof of this item is left as an exercise for the reader (see Exercise 6.2). ■

Example 6.2 Find the coefficient of x^7y^{12} in each of the following expressions.

(a) $(x+y)^{19}$. (b) $(3x-2y)^{19}$.

Solution (a) From the binomial theorem, it follows that

$$\binom{19}{12} = \frac{19!}{12!\,7!} = 503,88.$$

(b) By the binomial theorem, we have

$$(3x-2y)^{19} = (3x+(-2y))^{19} = \sum_{k=0}^{19}\binom{19}{k}(3x)^{19-k}(-2y)^k.$$

Consequently, the coefficient of x^7y^{12} in the expression is obtained when $k=12$, namely,

$$\binom{19}{12}3^7(-2)^{12} = \frac{19!}{12!\,7!}3^7 2^{12} = 451,373,285,376.$$

Binomial identities

The binomial theorem serves as a powerful tool in mathematics, offering a method to establish and validate numerous binomial identities. By leveraging this theorem, mathematicians can unravel and confirm relationships among binomial expressions, providing a systematic approach to understanding the structure and properties of these identities. The following corollary, derived from the binomial theorem, contributes to the depth and applicability of binomial identities in mathematical exploration and problem-solving.

Corollary 6.1 *Let n be a nonnegative integer. We have*

(a) $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n.$

(b) $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$

(c) $\displaystyle\sum_{k=0}^{n}\binom{n}{2k} = \sum_{k=0}^{n}\binom{n}{2k+1}.$

(d) $\displaystyle\sum_{k=0}^{n}2^k\binom{n}{k} = 3^n.$

Proof (a) Using the binomial theorem with $x=1$ and $y=1$, we have

$$2^n = (1+1)^n = \sum_{k=0}^{n}\binom{n}{k}1^k1^{n-k} = \sum_{k=0}^{n}\binom{n}{k}.$$

(b) Using the binomial theorem with $x=-1$ and $y=1$, we have

$$0 = 0^n = ((-1)+1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k1^{n-k} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k.$$

(*c*) From item (*b*), we have

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}-\binom{n}{5}+\cdots=0,$$

and hence

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots.$$

This proves item (*c*).

(*d*) The proof of this item is left as an exercise for the reader (see Exercise 6.4).

■

The binomial coefficients satisfy some important recurrences and identities. We first introduce the so-called Pascal's identity or Pascal's recurrence formula, which is defined as $T(n,k) = T(n-1,k-1) + T(n-1,k)$, where $T(n,0) = T(n,n) = 1$. It can be shown that the solution of this recurrence is the binomial coefficient $T(n,k) = \binom{n}{k}$. We have the following theorem.

Theorem 6.2 (Pascal's identity) *Let n and k be integers with $n \geq k \geq 0$. Then*

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}.$$

Proof Note that

$$\begin{aligned}
\binom{n-1}{k-1}+\binom{n-1}{k} &= \frac{(n-1)!}{(n-k)!\,(k-1)!}+\frac{(n-1)!}{(n-1-k)!\,k!}\\
&= \frac{(n-1)!}{(n-k)!\,(k-1)!}\,\frac{k}{k}+\frac{(n-1)!}{(n-1-k)!\,k!}\,\frac{n-k}{n-k}\\
&= \frac{(n-1)!\,k}{(n-k)!\,k!}+\frac{(n-1)!\,(n-k)}{(n-k)!\,k!}\\
&= \frac{(n-1)!\,(n-k)}{(k+n-k)!\,k!}=\binom{n}{k}.
\end{aligned}$$

This proves the theorem. ■

Pascal's identity is the basis for a geometric arrangement of the binomial coefficients in a triangle, as shown in Figure 6.1.

The ith row in the triangle consists of the binomial coefficients

$$\binom{i}{0},\binom{i}{1},\binom{i}{2},\ \cdots,\binom{i}{i}.$$

This triangle is known as Pascal's triangle which is due to Blaise Pascal (1623-1662). Pascal's identity illustrates that when you add two consecutive binomial coefficients within this triangular array, the result is the binomial coefficient found in the next row, positioned between these two coefficients.

$$\begin{array}{ccccccccccc}
 & & & & & \binom{n}{0} & & & & & \\
 & & & & \binom{n}{0} & & \binom{n}{1} & & & & \\
 & & & \binom{n}{0} & & \binom{n}{1} & & \binom{n}{2} & & & \\
 & & \binom{n}{0} & & \binom{n}{1} & & \binom{n}{2} & & \binom{n}{3} & & \\
 & \binom{n}{0} & & \binom{n}{1} & & \binom{n}{2} & & \binom{n}{3} & & \binom{n}{4} & \\
\binom{n}{0} & & \binom{n}{1} & & \binom{n}{2} & & \binom{n}{3} & & \binom{n}{4} & & \binom{n}{5} \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

Figure 6.1: Pascal's triangle.

The identity in the following theorem is due to Alexandre-Theophile Vandermonde (1735-1796).

Theorem 6.3 (Vandermonde's identity) *Let m, n and r be nonnegative integers with r not exceeding either m or n. Then*

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Proof Using the binomial theorem, we have

$$\begin{aligned}
\sum_{r=0}^{m+n} \binom{m+n}{r} x^r &= (1+x)^{m+n} \\
&= (1+x)^m (1+x)^n \\
&= \left(\sum_{r=0}^{m} \binom{m}{i} x^i\right)\left(\sum_{j=0}^{n} \binom{n}{j} x^j\right) = \sum_{r=0}^{m+n}\left(\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}\right) x^r,
\end{aligned}$$

where we used (3.6) to obtain the last equality.[2]

By comparing coefficients of x^r, the identity follows for all integers r with $0 \le r \le m+n$. For larger integers r, both sides of the identity are zero due to the definition of binomial coefficients. The proof is complete. ■

The following is a corollary of Vandermonde's identity.

[2]Note that (3.6) can be used because the binomial coefficients $\binom{m}{i}$ and $\binom{n}{j}$ give zero for all $i > m$ and $j > n$, respectively.

Corollary 6.2 *For any nonnegative integer n, we have*

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Proof The result follows immediately from Vandermonde's identity with $m = r = n$ and the identity $\binom{n}{k} = \binom{n}{n-k}$. ■

6.2 Fundamental principles of counting

Counting has applications in a variety of areas, including computer science, probability and statistics. In this section, we present some fundamental counting principles.

The product principle of counting

We start with the product principle which is stated below.

Principle 6.1 *Assume there is a procedure that can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then the procedure can be done in $n_1 \times n_2$ ways.*

The following example shows how the product principle is used.

Example 6.3 In the country of Jordan, there are some amazing restaurants worth to visit in some cities, such as Amman city, Petra city, and Jerash city. Suppose that one can take 3 routes from Jerash to Amman, and can take 5 routes from Amman to Petra. See Figure 6.2.

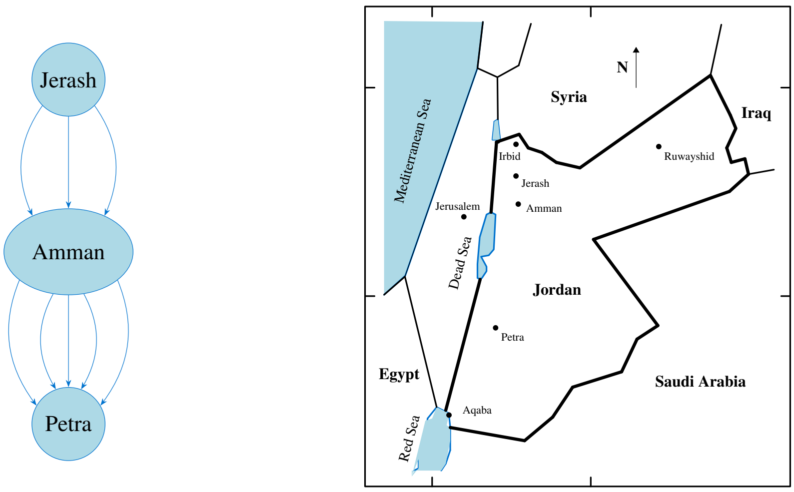


Figure 6.2: The possible routes that one can take to get from Jerash city to Amman city, and those to get from Amman city to Petra city in the country of Jordan.

How many possible routes can one take to get from Jerash to Petra if every Jerash-Petra route consists of two routes that are linked in Amman?

Solution Every Jerash-Petra route consists of a Jerash-Amman route, which can be done using 3 possible routes, and an Amman-Petra route, which can be done using 5 possible routes. Therefore, by the product principle, there are $3 \times 5 = 15$ possible routes can one take to get from Jerash city to Petra city. ■

The following theorem is the extended version of Principle 6.1.

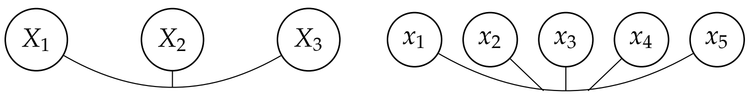
> **Theorem 6.4** *Assume that a procedure can be broken down into a sequence of m tasks, say $T_1, T_2, \ldots, T_m$. If each task $T_i, i = 1, 2, \ldots, m$, can be done in n_i ways, regardless of how the previous task was done, then the procedure can be done in $n_1 \times n_2 \times \cdots \times n_m$ ways.*

Theorem 6.4 can be proved using induction on m (see Exercise 6.7). The following examples show how Theorem 6.4 is applied.

Example 6.4 Each Jordanian national ID card contains an ID number in the back, which consists of three uppercase English letters followed by five digits. What is the number of possible Jordanian national ID cards?

Solution There are 26 choices for each of the three uppercase English letters and ten choices for each of the five digits (see the graph shown below). Hence, by Theorem 6.4, the number of possible Jordanian national ID cards is

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 \times 10 = 1,757,600,000.$$



26 choices for each letter. 10 choices for each digit. ■

Example 6.5 Use the product principle of counting to prove that the cardinality of the powerset of a finite set A is equal to 2^n if the cardinality of A is n.[3]

Solution When constructing a subset of the set A, it involves performing n distinct tasks, where each task entails making a decision about whether to include a particular element in the subset. As each task has two possible outcomes (either including or excluding the element), there are a total of 2^n ways to complete this process. Consequently, the set of all possible subsets of A, that is $\mathcal{P}(A)$, contains precisely 2^n elements. This concludes the intended proof. ■

Example 6.6 What is the value of "sum" after the fragment[4] given in Algorithm 6.1, where n, m, p and q are positive integers, has been executed?

Solution Let T_s be the number of times of traversing the statement (and only the statement) given in line (s) for each $s = 1, 2, \ldots, 6$.

[3]The reader is asked to prove this fact in Exercise 2.4 by mathematical induction.
[4]"For-statement" written in the fragment is formally introduced in Section 7.1.

Algorithm 6.1: The algorithm of Example 6.6

```
1:  sum = 0
2:  for (i = 1; i ≤ n; i + +) do
3:  |   for (j = 0; j < m; j + +) do
4:  |   |   for (k = p; k ≥ 1; k − −) do
5:  |   |   |   for (r = q; r > 0; r − −) do
6:  |   |   |   |   sum = sum + 1
7:  |   |   |   end
8:  |   |   end
9:  |   end
10: end
```

The number of times of traversing the statement given in line (s) is the number of ways to do the task T_s for $s = 1, 2, \ldots, 6$. These numbers are listed in Table 6.1.

Also, let P_v be the procedure of executing the body of the "for" loop in line (r) for each $r = 2, \ldots, 5$. Let us add a column for the number of ways to carry out the procedure $P_v, v = 2, \ldots, 5$ (see the most-right column in Algorithm 6.2). The initial value of "sum" is zero. Each time the "sum" statement in line (6) is executed, 1 is added to "sum". Therefore, the value of "sum" after the fragment given in Algorithm 6.1 has been executed is equal to the number of times to execute the body of the "for" loop in line (5), which is the number of ways to carry out the procedure P_5.

Line	Statement	Task	# ways to do task T_s
1	`sum=0`	T_1	`1`
2	`for (i=1; i<=n; i++)`	T_2	`n`
3	`for (j=0; j<m; j++)`	T_3	`m`
4	`for (k=p; k>=1; k-)`	T_4	`p`
5	`for (r=q; r>0; r-)`	T_5	`q`
6	`sum=sum+1`	T_6	`1`

Table 6.1: Numbers of ways to do the tasks of going over simple and for statements.

Algorithm 6.2: Algorithm 6.1 revisited

```
1:  sum = 0
2:  for (i = 1; i ≤ n; i + +) do                  // Procedure P2 is done in n ways
3:  |   for (j = 0; j < m; j + +) do              // Procedure P3 is done in n × m ways
4:  |   |   for (k = p; k ≥ 1; k − −) do          // P4 is done in n × m × p ways
5:  |   |   |   for (r = q; r > 0; r − −) do      // P5 is done in n × m × p × q ways
6:  |   |   |   |   sum = sum + 1
7:  |   |   |   end
8:  |   |   end
9:  |   end
10: end
```

The following points are noted, which lead us to conclude the number of ways to carry out the procedure P_5.

- How many times do we execute the body of the "for" statement in line (2)? The answer is from 1 to n, which is n times. Note that when $i = n + 1$ is checked, the inequality is false. Hence, the number of ways to carry out the procedure P_2 is n.
- How many times do we execute the body of the "for" statement in line (3)? If the "for" statement in line (3) was not nested, it would execute m times just like the "for" statement in line (2). Since the "for" statement in line (3) is nested, its body is executed m times for *each* time we execute the body of the "for" statement in line (2) (which is n times). In other words, by the product principle of counting, the number of ways to carry out the procedure P_3 is equal to the number of ways to do the task T_2 multiplied by the number of ways to do the task T_3, which is $n \times m$.
- Similarly, since the "for" statement in line (4) is nested, its body is executed p times for *each* time we execute the body of the "for" statement in line (3) (which is $n \times m$ times). In other words, by the product principle of counting, the number of ways to carry out the procedure P_4 equals the number of ways to do the task T_2 times the number of ways to do the task T_3 times the number of ways to do the task T_4 which is $n \times m \times p$.
- Again and similarly, since the "for" statement in line (5) is nested, its body is executed q times for *each* time we execute the body of the "for" statement in line (4) (which is $n \times m \times p$ times). In other words, by the product principle of counting, the number of ways to carry out the procedure P_5 equals the number of ways to do the task T_2 times the number of ways to do the task T_3 times the number of ways to do the task T_4 times the number of ways to do the task T_5, which is $n \times m \times p \times q$.

Thus, the value of "sum" after the fragment given in Algorithm 6.1 has been executed is equal to $n \times m \times p \times q$. ■

The product principle is also given in terms of sets. We have the following remark.

Remark 6.1 *The number of elements in the Cartesian product of finite sets is the product of the number of elements in each set.*

Proof Let $A_1, A_2, \ldots, A_m$ be finite sets. Note that the task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2, ..., and an element in A_m. By the product principle, it follows that

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|.$$

The proof is complete. ■

The sum principle of counting

In this part, we present the sum principle of counting.

Principle 6.2 *If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.*

The following example shows how the sum principle is used.

Example 6.7 Sara has decided to study a Bachelor's degree in Mathematics at one of the Jordanian universities, either in Irbid city or in Amman city. If Sara decides to go to Irbid, she will study at either Yarmouk University, Jordan University of Science and Technology, Irbid National University, or Jadara University. If Sara decides to go to Amman, she will study at either University of Jordan, Philadelphia University, Petra University, Al-Zaytoonah University, or Al-Ahliyya Amman University. What is the possible number of Jordanian universities Sara could choose?

Solution There are 4 possible universities (4 ways) Sara could go to Irbid city, and 5 possible universities (5 ways) Sara could go to in Amman city. Thus, by the sum principle of counting, there are 4+5=9 possible universities Sara could choose to study in Irbid or Amman. ■

The following theorem is the extended version of Principle 6.2.

Theorem 6.5 *Assume that a procedure can be done in one of n_1 ways, in one of n_2 ways, ..., or in one of n_m ways, where none of the set of n_i ways of doing the procedure is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m$, then the total number of ways to do the procedure is $n_1 + n_2 + \cdots + n_m$ ways.*

Theorem 6.5 can be proved using induction on m (see Exercise 6.8). The following examples show how Theorem 6.5 is applied.

Example 6.8 Zaid won a scholarship to study an undergraduate degree at one of the following universities in Ohio: Case Western Reserve University (CWRU), Kent State University (KSU), Ohio State University (OSU), and University of Cincinnati (UC). At CWRU, the scholarship requires that Zaid chooses either veterinary medicine, biomedical engineering, or biology. At KSU, it requires that Zaid studies clinical nutrition. At OSU, it requires that Zaid chooses either medicine, dentistry, pharmacy, or public health. At UC, it requires that Zaid chooses either nursing or chemistry. How many possible undergraduate degree programs that Zaid can choose from?

Solution Zaid can choose an undergraduate degree program by selecting a program at CWRU, KSU, OSU, or UC. There are 3 ways to choose a program at CWRU, one way to choose a program at KSU, 4 ways to choose a program at OSU, and 2 ways to choose a program at UC. Therefore, by the sum principle of counting, there are $3 + 1 + 4 + 2 = 10$ ways to choose a program. ■

Example 6.9 What is the value of "sum" after the fragment given in Algorithm 6.3, where n, m, p and q are positive integers, has been executed?

Solution Let T_s be the task of executing the statement in line (s) for each $s = 1, 2, 3, 5, 6, 8, 9, 11, 12$. Let us add comments showing the number of ways to carry out the task T_s, $s = 1, 2, 3, 5, 68, 9, 11, 12$ (see the comments in gray in Algorithm 6.1). The initial value of "sum" is zero. For each time each "sum" statement in lines (3), (6), (9) and (12) is executed, 1 is added to "sum". Therefore, by the sum principle of counting, the value of "sum" after the fragment given in Algorithm 6.1 has been executed is equal to the number of ways to carry out the task T_3, plus the number of ways to carry out the task T_6, plus the number of ways to carry out the task T_9, plus the number of ways to carry out the task T_{12}.

Algorithm 6.3: The algorithm of Example 6.9

```
1:  sum = 0                         // Task T_1 is carried out in 1 way
2:  for (i = 1; i ≤ n; i + +) do    // Task T_2 is carried out in n + 1 ways
3:  |   sum = sum + 1               // Task T_3 is carried out in n ways
4:  end
5:  for (j = 0; j < m; j + +) do    // Task T_5 is carried out in m + 1 ways
6:  |   sum=sum+1                   // Task T_6 is carried out in m ways
7:  end
8:  for (k = p; k ≥ 1; k − −) do    // Task T_8 is carried out in p + 1 ways
9:  |   sum = sum + 1               // Task T_9 is carried out in p ways
10: end
11: for (r = q; r > 0; r − −) do    // Task T_11 is carried out in q + 1 ways
12: |   sum = sum + 1               // Task T_12 is carried out in q ways
13: end
```

Note that the number of ways to carry out the task T_3 equals the number of times to execute the body of the "for" statement in line (2). How many times the "for" statement in line (2) executes? 1 to n, which is n times, plus 1 for the last time that i is checked and the inequality is false, that is, $n + 1$ times. Hence, the number of times the statement in line (3) (i.e., body of the "for" statement in line (2)) is n. Thus, the number of ways to carry out the task T_3 is n. Similarly, the tasks T_6, T_9 and T_{12} are carried out in m, p and q ways, respectively.

Therefore, by the sum principle of counting, the value of "sum" after the fragment given in Algorithm 6.1 has been executed is equal to $n + m + p + q$. ■

The sum principle is also given in terms of sets. We have the following remark.

Remark 6.2 *The number of elements in the union of pairwise disjoint finite sets is the sum of the number of elements in each set.*

Proof Consider the sets $A_1, A_2, \ldots, A_m$, which are pairwise disjoint, meaning that their intersections satisfy $A_i \cap A_j = \emptyset$ for all i, j. It is important to note that within each set A_i, there are precisely $|A_i|$ ways to select an element, where i ranges from 1 to m. Due to the pairwise disjoint nature of these sets, selecting an element from one set, say A_i, does not simultaneously entail selecting an element from any other set, such as A_j. Consequently, in accordance with the sum principle, since we cannot select an element from two different sets simultaneously, the total number of ways to choose an element from the union of these sets, denoted as $|A_1 \cup A_2 \cup \cdots \cup A_m|$, is given by the sum of the individual set sizes:

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|.$$

This equality holds true when $A_i \cap A_j = \emptyset$ for all i, j. The proof is complete. ■

The subtraction principle of counting

In this part, we present the subtraction principle which is also known as the principle of inclusion-exclusion.

Principle 6.3 *If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.*

The subtraction principle is also given in terms of sets. The following remark follows immediately from Principle 6.3.

Remark 6.3 (Inclusion-exclusion for two sets) *The number of elements in the union of finite sets A_1 and A_2 is the sum of the number of elements in each set, minus the number of ways to select an element that is in both A_1 and A_2. That is,*

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|. \tag{6.2}$$

We have the following example.

Example 6.10 Let S be a space or universe of two sets A and B, where $|S| = 60, |A| = 34, |B| = 22$ and $|A \cap B| = 8$. Find $|(A \cup B)'|$.

Solution By using (6.2), we have

$$|(A \cup B)'| = |S| - |A \cup B| = |S| - (|A| + |B| - |A \cap B|) = 60 - (34 + 22 - 8) = 12.$$

This responds to the example's query. ■

Note that Equation (6.2) can follow directly from Venn diagrams. Also note that the following basic formula is also followed directly from Venn diagrams.

$$|A_1 - A_2| = |A_1| - |A_1 \cap A_2|.$$

The subtraction principle can be extended to find the number of ways to do one of n different tasks or, equivalently, to find the number of elements in the union of n finite sets (see Rosen [2002] for more detailed presentations).

We have introduced the product, sum, and subtraction principles of counting. There is also a division principle of counting which is out the scope of this book. See Rosen [2002] for a good presentation of the division principle of counting.

6.3 The pigeonhole principle

Consider a situation in which there are 10 pigeons and only 9 pigeonholes available. In such a scenario, it is a certainty that at least one of the 9 pigeonholes will contain a minimum of two pigeons. To grasp the reason behind this assertion, observe that if every pigeonhole could accommodate, at most, a single pigeon, then the total number of pigeons that could be accommodated would be limited to a maximum of 9 (with one pigeon per hole). See Figure 6.3.

The pigeonhole principle can be stated as follows.

Principle 6.4 *If there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.*

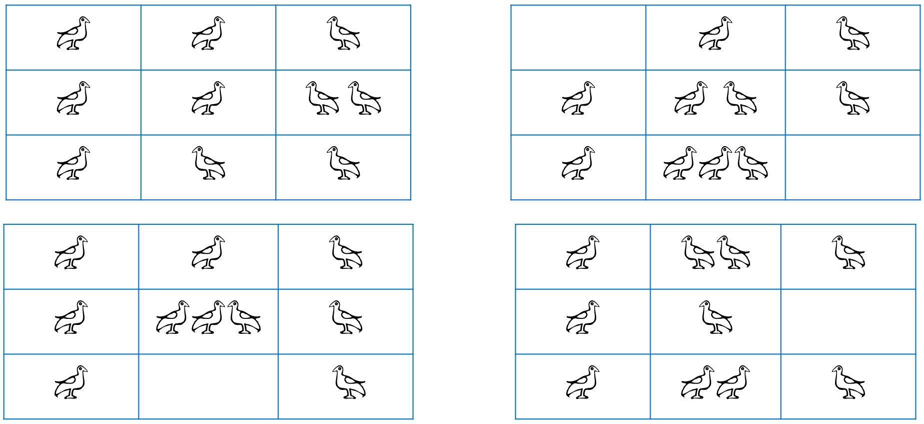


Figure 6.3: Illustrating the pigeonhole principle: There are 10 pigeons but only 9 pigeonholes, at least one of these 9 pigeonholes must have at least two pigeons in it.

More generally, we have the following theorem.

Theorem 6.6 *Let k be a positive integer. If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.*

Proof Suppose, on the contrary, that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This contradicts the fact that there are at least $k + 1$ objects. ■

The following examples show how Theorem 6.6 is applied.

Example 6.11 If 20 different algorithms exist to solve 19 different problems, then there is at least one problem that can be solved by two different algorithms. ■

Example 6.12 Among any group of 367 people infected by COVID-19 in 2020, there must be at least two cases who were diagnosed on the same day, because there are only 366 possible days in the year. In addition, by the same reasoning, there must be at least two of them with the same birthday. ■

The following theorem is the extended version of Theorem 6.6.

Theorem 6.7 *Let n and k be positive integers. If $k(n-1)+1$ or more objects are placed into k boxes, then there is at least one box containing n or more objects.*

Proof Suppose, in the contrary, that none of the k boxes contains more than $n - 1$ objects. Then the total number of objects would be at most $k(n-1)$. This contradicts the fact that there are a total of $k(n - 1) + 1$ objects. ■

Another version of Theorem 6.7 is the following theorem, which can also be proved using contradiction (see Exercise 6.9).

Theorem 6.8 *Let m and k be positive integers. If m objects are placed into k boxes, then there is at least one box containing at least $\lceil m/k \rceil$ objects.*[a]

[a]For a real number x, the ceiling of x is denoted as $\lceil x \rceil$ and is defined as the smallest integer that is not smaller than x. For example, $\lceil 5 \rceil = 5$ and $\lceil \sqrt{5} \rceil = 3$.

The following examples show how Theorems 6.8 and 6.7 are applied.

Example 6.13 As per the definition provided by the Pew Research Center, a millennial is characterized as an individual born within the timeframe spanning from 1981 to 1996. Consequently, in the year 2021, this would encompass millennials ranging in age from 24 to 40. Among a group of 145 millennials, it is guaranteed that there are at least 10 individuals who share the same birth year. This can be seen by applying Theorem 6.7, taking into account the equation $145 = 16(10 - 1) + 1$. ■

Example 6.14 A Nobel Prize-winning scientist was invited to give plenary talks at 25 international conferences which were held in 2019. From Theorem 6.8, among these 25 conferences, there are at least $\lceil 25/12 \rceil = 3$ which were held in the same month. ■

The pigeonhole principle is also given in terms of sets. We have the following remark.

Remark 6.4 *Let k be a positive integer. If A is a finite set with $k + 1$ or more elements and $A_1, A_2, \ldots, A_k$ are subsets of A that form a partition of A, then there exists at least one subset A_i, where $i \in \{1, 2, \ldots, k\}$, such that $|A_i| \geq 2$.*

Rather than proving Remark 6.4, we state and prove the following theorem which generalizes Remark 6.4.

Theorem 6.9 *Let $r_1, r_2, \ldots, r_k$ be positive integers. If A is a finite set with $(\sum_{i=1}^{k} r_i) - k + 1$ or more elements and $A_1, A_2, \ldots, A_k$ are subsets of A that form a partition of A, then we have $|A_i| \geq r_i$ for some $i \in \{1, 2, \ldots, k\}$.*

Proof We prove the theorem by contradiction. Suppose that for each $i \in \{1, 2, \ldots, k\}$ we have that $|A_i| \leq r_i - 1$. By the sum principle, we have

$$\begin{aligned} |A| &= |A_1| + |A_2| + \cdots + |A_k| \\ &\leq (r_1 - 1) + (r_2 - 1) + \ldots + (r_k - 1) \\ &= (\textstyle\sum_{i=1}^{k} r_i) - k, \end{aligned}$$

which contradicts the fact that A has more than $(\sum_{i=1}^{k} r_i) - k$ elements. Thus, $|A_i| \geq 2$ for some $i \in \{1, 2, \ldots, k\}$. The proof is complete. ■

Note that Remark 6.4 is a special case of Theorem 6.9 which occurs when $r_1 = r_2 = \cdots = r_k = 2$.

Ramsey Theory refers to the study of partitions of large structures, and generalizes the pigeonhole principle. The Ramsey number $R(m, n)$ gives the solution to the party problem, which asks for the minimum number of guests $R(m, n)$ that must be invited so that at least m will know each other or at least n will not know each other. For example, one can show that $R(3, 3) = 6$. By symmetry, it is true that $R(m, n) = R(n, m)$. It is possible to prove some useful properties about Ramsey numbers, but for the most part, it is difficult to find their exact values. See, for example, Radziszowski [2011], Lidický and Pfender [2021], Jaradat M. [2007, 2008] and the references contained therein.

6.4 Permutations

In order to find the number of possible arrangements of a set of objects, we use a concept called permutations. There are methods for calculating permutations, and it is important to understand the difference between a set with and without repetition.

Permutations without repetition

Many counting problems can be resolved by determining the count of arrangements possible for a specified number of distinct elements drawn from a set of a specific size, with an emphasis that the order of these elements matters. For example, in how many ways can we choose three singers from a group of five singers to perform three different songs in a concert, where each singer will perform exactly one song individually? In this section, we develop the present rules to solve counting problems such as this.

To answer the question posed in the previous paragraph, note that the order in which we choose the singers matters. There are five ways to choose the first singer for the first song that will be performed at the start of the concert. Once this singer has been chosen, there are four ways to choose the second singer for the second song that will be performed in the middle of the concert. Once the first and second singers have been chosen, there are three ways to choose the third singer who will perform at the end of the concert. By the product rule, there are $5 \cdot 4 \cdot 3 = 60$ ways to choose three singers from a group of five singers to perform three different songs in a concert, where each singer will perform exactly one song individually.

A permutation of a set of distinct objects is an ordered arrangement of some or all of these objects. Formally, we have the following definition.

Definition 6.1 *Let n and r be integers with $0 \leq r \leq n$. An ordered arrangement of r elements of a set with n distinct elements is called an r-permutation and is denoted by $P(n, r)$.*

The product principle of counting can be used to find a formula for $P(n, r)$.

Theorem 6.10 *Let n and r be integers with $0 \leq r \leq n$. Then*

$$P(n, r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1).$$

Proof We will employ the product rule to establish the correctness of this formula. Initially, there are n ways to select the first element of the permutation, given that there are precisely n elements in the set. Subsequently, for the second element of the permutation, there are $n-1$ choices, as there remain $n-1$ elements after utilizing the one picked for the first position. Following this pattern, the third element has $n-2$ options, and so forth, until there are precisely $n-(r-1) = n-r+1$ ways to choose the rth element. Consequently, in accordance with the product rule, there are

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

r-permutations of the set. This proves the theorem. ■

Let n be a nonnegative integer. Note that $P(n,n) = n!$. Note also that $P(n,0) = 1$ because there is exactly one way to order zero elements. That is, there is exactly one list with no elements in it, namely the empty set.

We give some examples as direct applications of Theorem 6.10.

Example 6.15 In how many ways can we arrange all singers of a group of five singers to perform different songs in a concert, where each singer will perform exactly one song individually?

Solution By Theorem 6.10, there are

$$P(5,5) = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

ways to select five singers from a group of five singers to perform different songs in a concert, where each singer will perform exactly one song individually. ■

Example 6.16 In how many ways can we form four distinct letter passwords from the letters:

(*a*) A, B, C, D?

(*b*) A, B, C, D, E, F?

Solution (*a*) By Theorem 6.10, there are $P(4,4) = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to form four distinct letter passwords from the letters A, B, C, D.

(*b*) By Theorem 6.10, there are $P(6,4) = 6 \cdot 5 \cdot 4 \cdot 3 = 360$ ways to form four distinct letter passwords from the letters A, B, C, D, E, F.

■

Permutations with repetition

In many counting problems, elements may be used repeatedly. For instance, a letter may be used more than once on a password. Permutations when repetition of elements is allowed can be easily counted by using the product principle of counting. We have the following theorem.

Theorem 6.11 *The number of r-permutation of a set of n objects with repetition allowed is n^r.*

Proof When repetition is permitted in forming an r-permutation, there exist n options to pick an element from the set for each of the r positions. This is due to the fact that, for each selection, all n objects remain available for consideration. Consequently, by applying the product rule, it follows that there are a total of n^r possible r-permutations when repetition is allowed. This proves the theorem. ■

The following examples are direct applications of Theorems 6.10 and 6.11.

Example 6.17

(*a*) In how many ways can ten boys have ten different birthdays?

(*b*) In how many ways can ten boys have ten birthdays?

Ignore the existence of the leap year, so a year has 365 days.

Solution (*a*) By Theorem 6.10, there are

$$P(365, 10) = 365 \cdot 364 \cdot 363 \cdot 362 \cdot 361 \cdot 360 \cdot 359 \cdot 358 \cdot 357 \cdot 356$$

ways ten boys have ten different birthdays.

(*b*) By Theorem 6.11, there are 365^{10} ways ten boys have ten birthdays.

Example 6.18 Suppose that there are three daily round-trip bus routes between Columbus and Cleveland, five daily round-trip bus routes between Cleveland and Buffalo, and four daily round-trip bus routes between Buffalo and Toronto. See Figure 6.4. A passenger would like to take a round-trip that departs from (and returns to) Columbus, and runs as follows: Columbus - Cleveland - Buffalo - Toronto - Buffalo - Cleveland - Columbus. In how many ways can the passenger take such a round trip in each of the following two cases:

(*a*) The passenger cannot use the same bus more than once.

(*b*) The passenger may use the same bus more than once.

Solution (*a*) If the passenger cannot use the same bus more than once, then there are

$$3 \cdot 5 \cdot 4 \cdot 3 \cdot 4 \cdot 2 = 1440$$

ways to take the planned round trip. This is a direct consequence of Theorem 6.10 since $1440 = 3 \cdot (3 - 1) \cdot 5 \cdot (5 - 1) \cdot 4 \cdot (4 - 1)$.

(*b*) If the passenger may use the same bus more than once, then there are

$$3 \cdot 5 \cdot 4 \cdot 4 \cdot 5 \cdot 3 = 3600$$

ways to take the planned round trip. This is a direct consequence of Theorem 6.11 since $3600 = 3^2 \cdot 5^2 \cdot 4^2$.

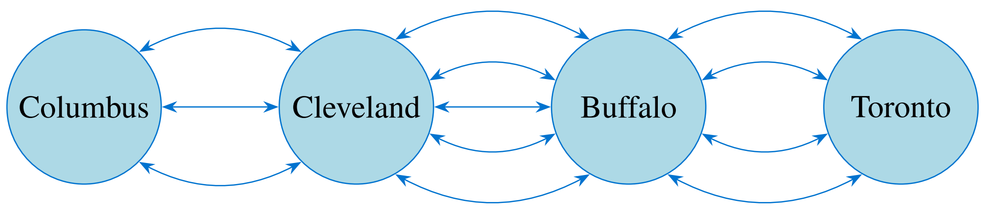


Figure 6.4: Daily round-trips between Columbus and Toronto, with connections in Cleveland and Buffalo.

6.5 Combinations

In order to find the number of ways to select a particular number of elements from a set of a particular size, we use a concept called combinations. There are methods for calculating combinations, and it is important to understand the difference between a set with and without repetition.

Combinations without repetition

Many counting problems can be resolved by determining the count of arrangements possible for a specified number of distinct elements drawn from a set of a specific size, with an emphasis that the order of the elements selected does not matter. For example, how many different bands of three singers can be formed from a group of four singers? In this section, we develop present rules to solve counting problems such as this.

To answer the question posed in the previous paragraph, note that the order in which we choose the singers does not matter. We need to find the number of subsets with three elements from the set containing the four singers. We see that there are four such subsets, one for each of the four singers, because choosing three singers is the same as choosing one of the four singers to leave out the group. This means that there are four ways to choose three singers for the band, where the order in which these singers are chosen does not matter.

A combination of a set of distinct objects is an unordered arrangement of some or all of these objects. Formally, we have the following definition.

Definition 6.2 *Let n and r be integers with $0 \leq r \leq n$. An unordered selection of r elements from a set with n distinct elements is called an r-combination and is denoted by $C(n, r)$.*

The product principle of counting can be used to find a formula for $C(n, r)$.

Theorem 6.12 *Let n and r be integers with $0 \leq r \leq n$. Then*

$$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}.$$

Proof The r-permutations, $P(n, r)$, for the set can be derived by first forming the r-combinations, $C(n, r)$, and then arranging the elements within each r-combination. The arrangement of elements within an r-combination can be done in exactly $P(r, r)$ ways. Consequently, by applying the product rule, it follows that $P(n, r) = C(n, r) \cdot P(r, r)$. This relationship implies the desired result that

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!\,r!}.$$

■

Returning to the question posed at the beginning of this section, we find that we can form $C(4, 3) = \binom{4}{3} = 4$ different bands of three singers from a group of four singers. As we mentioned earlier, choosing three singers is the same as choosing one of the four singers to leave out the group. That is, $C(4, 3) = C(4, 1) = \binom{4}{1} = 4$. More generally, we have the following corollary.

Corollary 6.3 *Let n and r be integers with $0 \le r \le n$. Then $C(n,r) = C(n, n-r)$.*

Proof The result follows immediately from Theorem 6.12 and the identity given in (6.1). ■

The following examples are direct applications of Theorem 6.12.

Example 6.19 How many pairs can be chosen from a group of six people?

Solution By Theorem 6.12, we can choose $C(6,2) = \binom{6}{2} = 15$ pairs from a group of six people. ■

Example 6.20 Suppose that a class consists of six boys and four girls.

(*a*) How many ways are there to choose a group of five students?

(*b*) How many ways are there to choose a group of five students if the group consists of three boys and two girls?

Solution (*a*) By Theorem 6.12, the number of ways to select the group is

$$C(10,5) = \binom{10}{5} = 252.$$

(*b*) By the product principle of counting, the answer is the product of the number of 3-combinations of boys and the 2-combinations of girls. By Theorem 6.12, the number of ways to select the group is

$$C(6,3) \cdot C(4,2) = \binom{6}{3}\binom{4}{2} = (20)(6) = 120.$$

This answers the questions raised in the example. ■

Combinations with repetition

Let us say there are five flavors of ice cream: banana (B), chocolate (C), lemon (L), strawberry (S), and vanilla (V). We can have three scoops. Example selections include:

- {C, C, C}, which means that three scoops of chocolate were selected.
- {B, L, V}, which means that one each of banana, lemon and vanilla was selected.
- {B, V, V}, which means that one of banana, two of vanilla were selected.

We are interested in how many variations there will be. This is an example of a combination with repetition of elements allowed. Note that there are five things to choose from, and we choose three of them. Also note that the order in which the scoops are selected does not matter, and we can repeat! Therefore, this example involves counting 3-combinations with repetition allowed from a 5-element set. Below we show a technique for solving this counting problem, which leads us to a general method for counting the r-combinations with repetition allowed from an n-element set.

Let us think about the ice cream problem posed above as ice cream being in boxes (see Figure 6.5) and a robot being ordered remotely to get the desired selections. For instance:

- To select {C, C, C}, we send the orders "move past the first box, then take three scoops, then move along three more boxes to the end". In other words, we send the orders "move once, then take three scoops, then move thrice to the right".
- To select {B, L, V}, we send the orders "take a scoop, then move twice to the right, then take a scoop, then move twice to the right, then take a scoop".
- To select {B, V, V}, we send the orders "take a scoop, then move quarce to the right, then take two scoops".

B	C	L	S	V

Figure 6.5: Five different kinds of ice cream in five boxes (containers).

Let us write "$\rightarrow$" to mean that "move once to the right", and "$\bigcirc$" to mean that "take one scoop". The above three selections can be written as follows:

- The selection {C, C, C} is represented as "$\rightarrow \bigcirc\bigcirc\bigcirc \rightarrow \rightarrow \rightarrow$".
- The selection {B, L, V} is represented as "$\bigcirc \rightarrow \rightarrow \bigcirc \rightarrow \rightarrow \bigcirc$".
- The selection {B, V, V} is represented as "$\bigcirc \rightarrow \rightarrow \rightarrow \rightarrow \bigcirc \bigcirc$".

From the above discussion, the above counting question can be simplified as follows: How many different ways can we arrange arrows and circles?

Note that there are always 3 circles we always have 3 scoops of ice cream, and there are always 4 arrows because we need to move 4 times to go from the first container to the fifth container. Therefore, there are 7 positions, and we want to choose 3 of them to have circles. That is, we have 3 + (5−1) positions and want to choose 3 of them. So, the number of ways of having 3 scoops from 5 flavors of ice cream is

$$\binom{3+5-1}{3} = \binom{7}{3} = \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3} = 35.$$

We can also look at the arrows instead of the circles, and say that we have 3 + (5−1) positions and want to choose (5−1) of them to have arrows. The following theorem generalizes this discussion.

Theorem 6.13 *There are*

$$C(n+r-1, n-1) = C(n+r-1, r) = \binom{r+n-1}{r}$$

r-combinations from a set with n-elements when repetition of elements is allowed.

Proof To represent each r-combination from a set with n elements when repetition is allowed, a combination can be depicted using $n-1$ right arrows and r circles. These $n-1$ right arrows are employed to separate n distinct cells, where each cell corresponds to an element in the set, and the ith cell contains a circle for each occurrence of the ith element within the combination. For instance, consider a 5-combination from a set of four elements, which is represented as three arrows and five circles:

$$\bigcirc\bigcirc \rightarrow \rightarrow \bigcirc \rightarrow \bigcirc\bigcirc$$

This representation signifies a combination comprising exactly two occurrences of the first element, none of the second element, one of the third element, and two of the fourth element from the set.

As demonstrated, each unique list consisting of $n-1$ arrows and r circles corresponds to an r-combination derived from the set with n elements, permitting repetition. The count of such lists can be computed as $C(n-1+r, r)$, as each list corresponds to selecting r positions out of the $n-1+r$ positions that encompass both circles and arrows. This count also equals $C(n-1+r, n-1)$ because each list corresponds to choosing $n-1$ positions to place the $n-1$ arrows. ■

The following examples show how Theorem 6.13 is applied.

Example 6.21 Suppose there are four varieties of donuts: Chocolate (C), Glazed (G), Pumpkin (P), and Raspberry (R).

(*a*) Find the number of ways one can select 30 donuts.

(*b*) Find the number of ways one can select 30 donuts such that the selection includes at least 2 C donuts, 2 G donuts, 3 P donuts, and 4 R donuts.

Solution Note that the order in which the donuts can be selected does not matter, and the donuts can be repeated. The total number of varieties of donuts is $n = 4$. The number of donuts to be selected is $r = 30$.

(*a*) This count would be the number of 30-combinations with repetition allowed from a set with six elements. From Theorem 6.13, the donuts can be selected in the following different ways:

$$C(4+30-1, 30) = C(33, 30) = \frac{33 \cdot 32 \cdot 31}{1 \cdot 2 \cdot 3} = 5456.$$

(*b*) To satisfy the requirements, we preselect the minimum number of each type: 2 C, 2 G, 3 P, 4 R. We then have at most $30-2-2-3-4 = 19$ donuts left to choose from, now without any restrictions. This count would be

$$C(4+19-1, 19) = C(22, 19) = \frac{22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3} = 1540.$$

This answers the questions raised in the example. ■

Example 6.22 How many solutions does the equation $x_1 + x_2 + x_3 + x_4 = 13$ have, where x_1, x_2, x_3, and x_4 are nonnegative integers?

Solution To determine the count of solutions, we can observe that a solution corresponds to a manner of selecting 13 items from a set containing four distinct elements, with x_1 items of type one, x_2 items of type two, x_3 items of type three, and x_4 items of type four. In this context, the number of solutions equates to the count of 13-combinations with repetition allowed from a set comprising four elements. Applying Theorem 6.13, we conclude that there are

$$C(4+13-1,13) = C(16,13) = C(16,3) = \frac{16\cdot 15\cdot 14}{1\cdot 2\cdot 3} = 560$$

possible solutions. ■

Example 6.23 What is the value of "sum" after the fragment given in Algorithm 6.4, where n, m, p and q are positive integers, has been executed?

Solution Note that the initial value of "sum" is 0 and that 1 is added to "sum" each time the nested loop is traversed with a sequence of four letters i, j, k, and r such that

$$1 \le m \le k \le j \le i \le n.$$

The number of such sequences of integers is the number of ways to choose 4 integers from $\{1, 2, \ldots, n\}$, with repetition allowed.

Hence, from Theorem 6.13, it follows that

$$\text{sum} = C(n+4-1,4) = \binom{n+3}{4} = \frac{1}{24}(n^4 + 6n^3 + 11n^2 + 6n)$$

after this code has been executed. ■

The formulas for the numbers of ordered and unordered selections of r elements, chosen with and within repetition allowed from a set with n elements, are shown in Table 6.2.

Algorithm 6.4: The algorithm of Example 6.23

```
sum = 0
for (i = 1; i ≤ n; i + +) do
    for (j = 1; j ≤ i; j + +) do
        for (k = 1; k ≤ j; k + +) do
            for (m = 1; m ≤ k; m + +) do
                sum = sum + 1
            end
        end
    end
end
```

	Without repetition	With repetition
r-permutations	$\frac{n!}{(n-r)!\,r!}$	n^r
r-combinations	$\frac{n!}{(n-r)!\,r!}$	$\frac{(n+r-1)!}{(n-1)!\,r!}$

Table 6.2: Permutations and combinations with and without repetition.

Using combinations to find permutations with indistinguishable objects Some elements may be indistinguishable in counting problems. When this is the case, care must be taken to avoid counting things more than once. In particular, we can use combinations to find the number of different permutations with indistinguishable objects. We have the following theorem.

Theorem 6.14 *The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., n_k indistinguishable objects of type k, is*

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

Proof To calculate the count of permutations, we can start by observing that the n_1 objects of type 1 can be arranged among the n positions in $C(n, n_1)$ ways, which leaves $n - n_1$ positions unoccupied. Following this, the objects of type 2 can be positioned in $C(n - n_1, n_2)$ ways, resulting in $n - n_1 - n_2$ available positions. This process continues for objects of type 3, and so on, up to type $k - 1$, until finally, n_k objects of type k can be accommodated in $C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$ ways. Therefore, by applying the product principle of counting, the total number of distinct permutations can be calculated as follows:

$$\begin{aligned} & C(n, n_1)C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k) \\ &= \frac{n!}{n_1!\,(n - n_1)!} \frac{(n - n_1)!}{n_2!\,(n - n_1 - n_2)!} \cdots \frac{(n - n_1 - \cdots n_{k-1})!}{n_k!\, 0!} = \frac{n!}{n_1!\, n_2! \cdots n_k!}. \end{aligned}$$

The proof is complete. ∎

The following example shows how Theorem 6.14 is applied.

Example 6.24 Determine the number of ways to arrange 6 letters of the word BANANA.

Solution We count permutations of 3 A's, 2 N's, and 1 B, a total of 6 symbols. By Theorem 6.14, the number of these is $\frac{6!}{3!\,2!\,1!} = 60$. ∎

Distributing objects into distinguishable boxes

Many counting problems can be solved by enumerating the ways objects, distinguishable (i.e., different) or indistinguishable (i.e., identical), can be placed into boxes that are distinguishable (often called labeled) or indistinguishable (often called unlabeled).

There are closed formulas for counting the ways to distribute objects, distinguishable or indistinguishable, into distinguishable boxes, but there are no closed formulas for counting the ways to distribute objects, distinguishable or indistinguishable, into indistinguishable boxes.

Distributing distinguishable objects into distinguishable boxes Counting problems that involve distributing distinguishable objects into distinguishable boxes can be solved using the following theorem.

Theorem 6.15 *The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals*

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

The proof of Theorem 6.15 uses the product principle, similar to the proof of Theorem 6.14, and it is therefore omitted. It can also be proved by setting up a one-to-one correspondence between the permutations counted in Theorem 6.14 and the ways to distribute distinguishable objects counted by Theorem 6.15. The following example is taken from [Rosen, 2002, Section 6.5].

Example 6.25 How many ways are there to distribute hands of 5 cards to each of four players from a card game of 52 cards[5]?

Solution This counts the number of ways to distribute 52 distinguishable objects into 5 distinguishable boxes so that 5 objects are placed into the first box, 5 objects are placed into the second box, 5 objects are placed into the third box, 5 objects are placed into the fourth box, and 32 objects (which represent $52 - 4 \times 5$ remaining cards) are placed into the fifth box. By Theorem 6.15, these are $52!/(5!\,5!\,5!\,5!\,32!)$ ways. ■

Distributing indistinguishable objects into distinguishable boxes Counting problems that involve distributing indistinguishable objects into distinguishable boxes can be solved using the following theorem.

Theorem 6.16 *There are*

$$C(n + r - 1, n - 1) = C(n + r - 1, r) = \binom{n + r - 1}{r}$$

ways to place r indistinguishable objects into n distinguishable boxes.

Theorem 6.16 is similar to the proof Theorem 6.13 and it is therefore omitted. It can also be proved by setting up a one-to-one correspondence between the combinations counted in Theorem 6.13 and the ways to place distinguishable objects counted by Theorem 6.16.

Example 6.26 How many ways to place 12 indistinguishable homemade doughnuts into 9 distinguishable plates?

Solution This counts the number of ways to place 12 indistinguishable objects into 9 distinguishable boxes. By Theorem 6.16, these are

$$C(9 + 12 - 1, 9 - 1) = C(20, 8) = \binom{20}{8} = 125970$$

ways. ■

[5]The standard deck consists of 52 cards.

Exercises

6.1 Choose the correct answer for each of the following multiple-choice questions/items.

(*a*) The coefficient of $x^{13}y^{12}$ in the expression $(2x - 3y)^{25}$ is

(*i*) $-\frac{25!}{12!\,13!}2^{12}3^{13}$. (*ii*) $-\frac{25!}{13!\,12!}2^{13}3^{12}$. (*iii*) $\frac{25!}{12!\,13!}2^{12}3^{13}$. (*iv*) $\frac{25!}{13!\,12!}2^{13}3^{12}$.

(*b*) There are cats in three rooms. The first room has 4 cats, the second room has 5 cats, and the third room has 3 cats. In how many ways to choose a cat from these three rooms.

(*i*) 12. (*ii*) 24. (*iii*) 48. (*iv*) 60.

(*c*) Let A and B be two disjoint sets. If $|A| = 3|B| = 6$, then the cardinality of the powerset of $A \times B$ is

(*i*) 8. (*ii*) 12. (*iii*) 256. (*iv*) 4096.

(*d*) In an exam, there are 7 true/false questions and 8 multiple-choice questions for which the answers can be (*i*), (*ii*), (*iii*), (*iv*). The number of different ways of answering the exam are:

(*i*) $2^7 \times 4^7$. (*ii*) $2^7 \times 4^8$. (*iii*) $2^8 \times 4^7$. (*iv*) $2^8 \times 4^8$.

(*e*) How many even 4 digit whole numbers are there?

(*i*) 256. (*ii*) 625. (*iii*) 4500. (*iv*) 5000.

(*f*) A professor gives a multiple-choice quiz that has eight questions, each with four possible answers (*i*), (*ii*), (*iii*), (*iv*). What is the minimum number of students that must be in the professor's class in order to guarantee that at least four answer sheets must be identical? (Assume that no answers are left blank.)

(*i*) 4^8. (*ii*) $4^8 + 1$. (*iii*) $2 \times 4^8 + 1$. (*iv*) $3 \times 4^8 + 1$.

(*g*) How many ways can we assign four problems to four students to solve them so that each student solves one problem?

(*i*) 12. (*ii*) 16. (*iii*) 24. (*iv*) 48.

(*h*) How many numbers of two digits can be formed with digits 1, 3, 5, 7, and 9?

(*i*) 60. (*ii*) 120. (*iii*) 180. (*iv*) 240.

(*i*) What is the number of diagonals can be drawn in an octagon?

(*i*) 20. (*ii*) 28. (*iii*) 34. (*iv*) 42.

(*j*) Let A, B and C be three disjoint sets. If $|A| = |B| + 1 = |C| + 2 = 4$, then the cordiality of $A \cup B \cup C$ is

(*i*) 9. (*ii*) 24. (*iii*) 512. (*iv*) 4096.

(*k*) The number of subsets of a 99-element set is:

(*i*) 2^{99}. (*ii*) $\binom{99}{2}$. (*iii*) 9^{99}. (*iv*) $\binom{99}{9}$.

(*l*) The number of 9-element subsets of a 99-element set is:

(*i*) 2^{99}. (*ii*) $\binom{99}{2}$. (*iii*) 9^{99}. (*iv*) $\binom{99}{9}$.

(*m*) The number of 7-letter upper-case words is:

(*i*) 26^7. (*ii*) 25^7. (*iii*) $26^7 - 25^7$. (*iv*) $\binom{26}{7}$.

(*n*) The number of 7-letter upper-case words that do not contain the letter B is:

(*i*) 26^7. (*ii*) 25^7. (*iii*) $26^7 - 25^7$. (*iv*) $\binom{26}{7}$.

(*o*) The number of 7-letter upper-case words that contain the letter B is:

(*i*) 26^7. (*ii*) 25^7. (*iii*) $26^7 - 25^7$. (*iv*) $\binom{26}{7}$.

(*p*) The number of 7-letter upper-case words whose letters are distinct and occur in alphabetically increasing order is[6]:

(*i*) 26^7. (*ii*) 25^7. (*iii*) $26^7 - 25^7$. (*iv*) $\binom{26}{7}$.

6.2 Use the binomial theorem to find the expansion of $(x + y)^5$.

6.3 Give a combinatorial proof for item (*a*) of Corollary 6.1.

6.4 Prove item (*d*) of Corollary 6.1.

[6]E.g., ABEL is counted, but not ABLE (since L and E are not in alphabetical order), nor APPEL (since P is repeated).

6.5 Give a combinatorial proof for Pascal's identity (Theorem 6.2).

6.6 Give a combinatorial proof for Vandermonde's identity (Theorem 6.3).

6.7 Use mathematical induction to prove Theorem 6.4.

6.8 Use mathematical induction to prove Theorem 6.5.

6.9 Use contradiction to prove Theorem 6.8

6.10 Let A, B and C be three sets such that $|B \cap C| = 19$ and $|A \cap B \cap C| = 11$. Find $|A' \cap B \cap C|$.

6.11 A survey of 240 people showed that 91 like tea, 70 like coffee, 31 like tea and coffee, 91 like neither coffee nor tea, and in addition do not like milk, and 7 like coffee, tea, and milk. How many like milk only? Justify your answer. (Hint: Use a Venn diagram).

6.12 What is the value of "sum" after the fragment given in Algorithm 6.5, where `n`, `m`, `p` and `q` are positive integers, has been executed?

6.13 Prove that a function f from a set with $k+1$ or more elements to a set with k elements is not an injection.

6.14 Zaid has six different colored shirts. In how many ways can he hang the four shirts in the cupboard?

6.15 Determine the number of ways to arrange:

(a) The 7 letters of the word ONEWORD. (Note that this word has 2 O's, 1 N, 1 E, 1 W, 1 R, and 1 D).

(b) The word ONEWORD if the 2 O's must be adjacent (as in NEWDOOR).

6.16 Determine the number of ways to arrange:

(a) The 9 letters of the word PINEAPPLE. (Note that this word has 3 P's, 2 E's, 1 I, 1 N, 1 A, and 1 L).

(b) The 9 letters of the word PINEAPPLE if the 3 P's must be adjacent (as in APPPLENIE).

Algorithm 6.5: The algorithm of Exercise 6.12

```
sum = 1
for (i = 1; i ≤ n; i + +) do
    for (j = 0; j < m; j + +) do
        sum = sum + 1
        for (k = p; k ≥ 1; k − −) do
            sum = sum + 1
        end
        for (r = q; r > 0; r − −) do
            sum = sum + 1
        end
    end
end
```

Notes and sources

The history of counting is as ancient as human civilization itself. Counting can be traced back to prehistoric times when early humans used simple tally marks, notches, or pebbles to keep track of quantities; see Ifrah [2000]. As societies developed, more advanced systems of counting and numbering emerged. One of the earliest known numerical systems is the Babylonian cuneiform script, dating back to around 2000 BCE, which used a base-60 (sexagesimal) numbering system. The ancient Egyptians also developed their numeral system, hieroglyphs for numbers, and the concept of place value. The Indian subcontinent played a significant role in the history of counting, with the introduction of the decimal system and the concept of zero, which led to the development of modern numeral systems (see, for instance, Smith and Karpinski [2013]). In particular, the Arabic numeral system, with its ingenious use of ten symbols and the concept of zero, has not only revolutionized mathematics but also become the foundation for modern global numerical representation, providing an efficient and universally adopted system for counting.

This chapter provided the reader with the essential tools and knowledge required for counting and effective enumeration. It offered a comprehensive exploration of fundamental concepts in counting. More precisely, we introduced Binomial coefficients and their manipulation and identities. Then, we delved into the principles of counting, gaining insight into various counting techniques, including permutations and combinations.

As we conclude this chapter, it is worth noting that the cited references and others, such as Cusack and Santos [2021], Mott et al. [1986], Liu [1968], Grimaldi [1999], Yellen et al. [1999], Fleming and Pitassi [2021], Beame and Riis [1996], Knuth [1997], also serve as valuable sources of information pertaining to the subject matter covered in this chapter. The code that created Figure 6.1 is due to StackExchange [2011]. The code that created the right-hand side picture in Figure 6.2 was taken from the source file of Alzalg and Alioui [2022]. The source code of the Jordan map in Figure 6.2 is publicly available at http://sites.ju.edu.jo/sites/alzalg/pages/jordanmap.aspx. The code that created Figure 6.3 is due to StackExchange [2016].

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Part III

ALGORITHMS

CHAPTER 7

ANALYSIS OF ALGORITHMS

Chapter overview: In this chapter, we explore the essential concepts of the analysis of algorithms. By delving into asymptotic notation, readers will gain a deep understanding of how to quantify and assess the running time of algorithms, enabling them to evaluate and compare the efficiency of algorithms, ultimately facilitating the selection of efficient algorithms that find more optimized solutions to problems in mathematics and computer science. This chapter concludes with a set of exercises to encourage readers to apply their knowledge and enrich their understanding.

Keywords: Asymptotic notation, Analysis of algorithm, Running time

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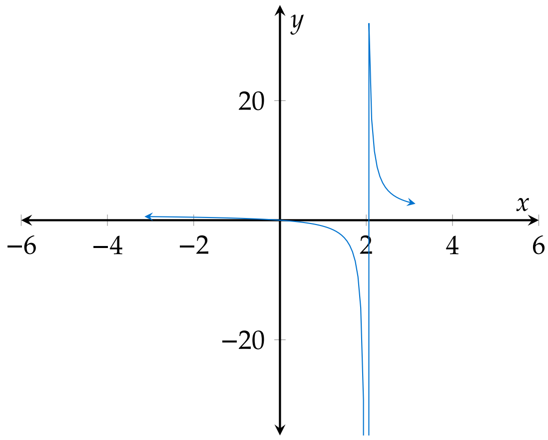


Figure 7.1: The graph of $f(x) = \frac{x}{x-2}$.

An algorithm is a finite set of precise instructions that are performed to solve a problem. So, every algorithm is constructed using a finite sequence of statements. The term "algorithm" has its origins in the work of Muhammad Ibn Musa Al-Khwarizmi (ca. 780–840). His book, which focused on the Arabic numeral system and various computational methods such as addition, subtraction, multiplication, division, handling fractions, and calculating roots, played a groundbreaking role in the development of Western mathematics. This work by Al-Khwarizmi had a profound impact on the field (refer to Khuwārizmī et al. [1963]).

In the first half of this chapter, we present the fundamentals of runtime analysis, and study the asymptotic analysis of algorithms. In the second half of this chapter, we analyze sequential programs, which are programs without function calls. We also analyze recursive and nonrecursive programs, which are programs with function calls. This chapter ends by briefly introducing the complexity classes NP and NP-complete.

In mathematics, asymptotic analysis is a method of describing limiting behavior. For instance, the study of the properties of a function $f(x)$ as x becomes very large. As an example, the graph of the function $f(x) = \frac{x}{x-2}$ approaches 1 as x goes to infinity. See Figure 7.1.

In fact,

$$\lim_{x\to\infty} \frac{x}{x-2} = \lim_{x\to\infty} \frac{x}{x(1-\frac{2}{x})} = \lim_{x\to\infty} \frac{1}{1-\frac{2}{x}} = 1.$$

Therefore, mathematically speaking, we call the line $y = 1$ a horizontal asymptotic for $f(x)$.

In computer science, asymptotic analysis involves assessing the efficiency of an algorithm concerning the input size, denoted as n, particularly when n becomes very large. What is an algorithm? How can we construct an algorithm for performing certain tasks? And how can we compare between various algorithms for solving a certain problem? This is the substance of the first section.

7.1 Constructing and comparing algorithms

In the first part of this section, we present some decision making statements that are used to build blocks of algorithms, and introduce the running time of an algorithm.

Basic tools for constructing algorithms

Decision making statements can be placed into the following categories:

Simple statements: The following are simple statements in the programming language C.

- Expressions: This includes
  - `printf;` (write statement – print formatted data).
  - `scanf;` (read statement – read formatted data).
  - assignment statements (such as "set a=1;").
- Jump statements, such as `goto; break; continue; return;` etc.
- The null statement: This contains only a semicolon ";". It is known that nothing happens when a null statement is executed.

We note that, in the programming languages such as C, simple statements end in a semicolon.

If-statement: The "if-statement" and "if-else-statement" are also-called selection statements. In Algorithm 7.1, we present a simple code for "if-statement". Here statement(s) is (are) executed only if condition is true. See Figure 7.2 (a).

In Algorithm 7.2, we present a simple code for "if-else-statement", where S_1 denotes a set of statement(s), and S_2 denotes another set of statement(s). Here, if condition is true, then S_1 is executed. Otherwise, S_2 is executed. See Figure 7.2 (b).

Example 7.1 In Algorithm 7.3, we present a code for finding the largest of two integers.

Algorithm 7.1: Writing an if-statement

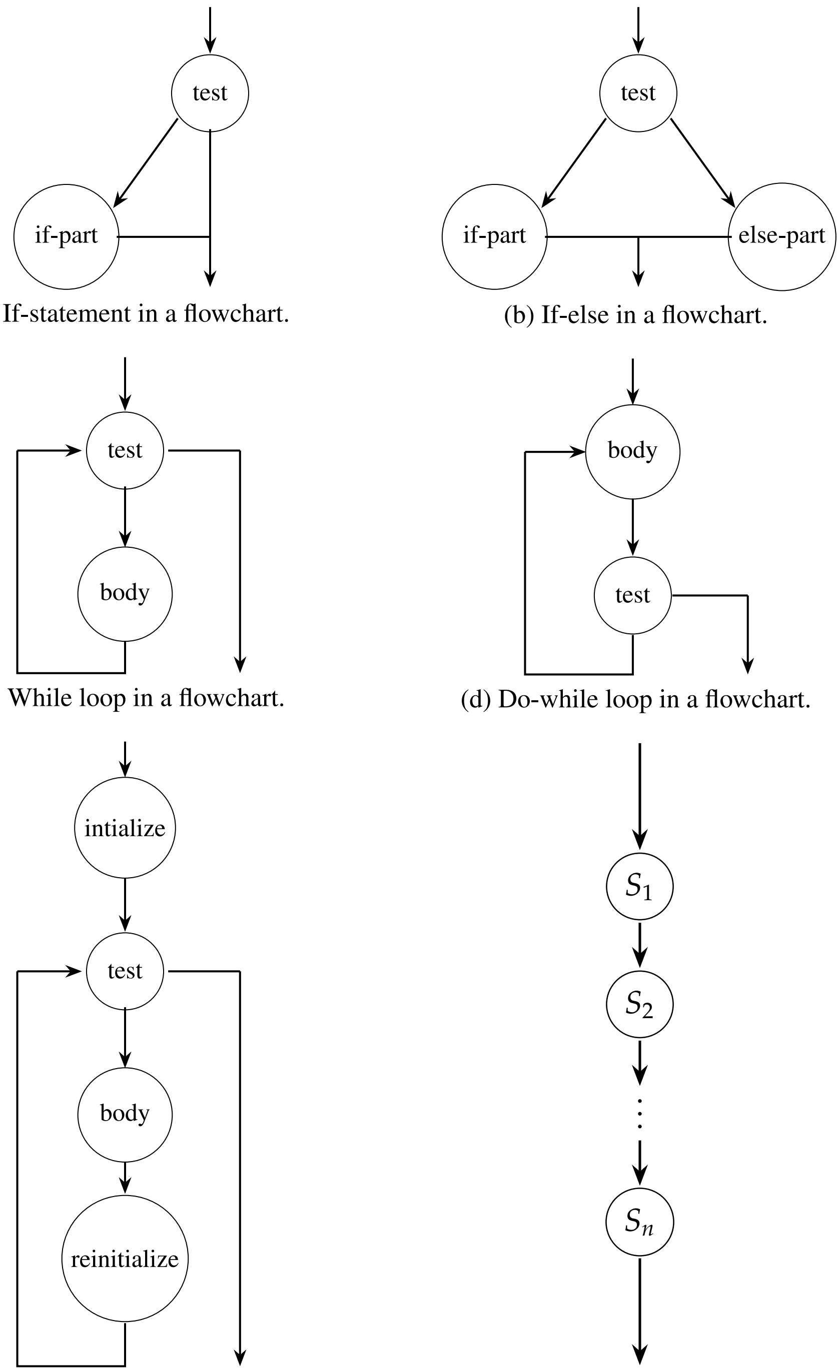
if some condition is true **then**
  do some statement(s) S
end

Algorithm 7.2: Writing an if-else-statement

if some condition is true **then**
  do some statement(s) S_1
end
else
  do some different statements S_2
end

Algorithm 7.3: Finding the largest of two integers

Input: Integers a, b
Output: Bigger value of a, b
if $(a > b)$ **then**
  `printf`("bigger value = "a)
end
else
  `printf`("bigger value = "b)
end



(a) If-statement in a flowchart.

(b) If-else in a flowchart.

(c) While loop in a flowchart.

(d) Do-while loop in a flowchart.

(e) For-loop in a flowchart.

(f) A block in a flowchart.

Figure 7.2: Flowcharts showing basic statements.

For-statement: Algorithm 7.4 shows a simple code that represents "for-statement". See also Figure 7.2 (e).

Algorithm 7.4: Writing a for-statement

```
for (initialize; test; in/decrement) do
    do some statement(s)
end
```

Example 7.2 Algorithm 7.5 presents a code for finding the maximum of integers.

Algorithm 7.5: Finding the maximum of integers

Input: Integers $a_1, a_2, \ldots, a_n$
Output: Largest value of $a_1, a_2, \ldots, a_n$

```
max = a_1
for (i = 0; i < n; i++) do
    if (max < a_i) then
        max = a_i
    end
end
printf("largest value = "max)
```

While-statement: In Algorithm 7.6, we present a simple code for "while statement". Here the body (statement(s)) is executed as long as condition is true. See Figure 7.2 (c).

Algorithm 7.6: Writing a while-statement

```
while some condition is true do
    do some statement(s)
end
```

Example 7.3 Algorithm 7.7 contains a while loop that prints the numbers $1, 2, \ldots, 5$.

Algorithm 7.7: Using a while-statement to print numbers 1 to 5

Input: Integer i
Output: Print numbers from 1 to 5

```
int i = 1
while (i ≤ 5) do
    printf(i)                // Output is 1 2 3 4 5
    ++i
end
return 0
```

Do-while-statement: In Algorithm 7.8, we present a simple code for "do-while statement". Note that the do-while is similar to the while-statement except that the body (statement(s)) is executed at least once.

Algorithm 7.8: Writing a do-while-statement

```
do
    do some statement(s)
while some condition is true
```

Example 7.4 Algorithm 7.9 contains a do-while loop that writes the numbers $1, 2, \ldots, 5$. ■

Algorithm 7.9: Using a do-while-statement to print numbers from 1 to 5

Input: Integer i
Output: Print numbers from 1 to 5

```
int i = 1
do
    printf(i)                    // Output is 1 2 3 4 5
    ++i
while (i ≤ 5)
return 0
```

Block: If $S_1, S_2, \ldots, S_n$ are statements, then the code in Algorithm 7.10 is called a block. See Figure 7.2 (f).

Algorithm 7.10: Writing a block

do a statement S_1
do a statement S_2
$\vdots$
do a statement S_n

The following example shows a block with three statements.

Example 7.5 In Algorithm 7.11, we present a code that writes the letters a, b and c in three lines. Here, the command "$\backslash n$" means jumping to a newline. ■

Algorithm 7.11: Printing the letters a, b and c

```
printf("a\n")            // Output is a
printf("b\n")            // Output is b
printf("c")              // Output is c
```

The following algorithm example is due to Kuchling [2012].

Example 7.6 A straightforward algorithm named `find-max` has been devised for the following problem:

PROBLEM: Given an array of positive numbers, return the largest number of the array.
INPUT: An array `A` of positive numbers. This list must contain at least one number. (Asking for the largest number in a list of no numbers is not a meaningful question.)
OUTPUT: A number that will be the largest number of the list.

(*a*) Construct an algorithmic code for the given approach as follows:

1. Set `max` to 0.
2. For each `x` in the list `L`, compare it to `max`. If `x` is larger, set `max` to `x`.
3. `max` is now set to the largest number in the list.

(*b*) Examine whether the algorithm formulated in item (*a*) possesses well-defined inputs and outputs, ensures termination, and yields accurate results.

Solution (*a*) A code that solves the given problem is stated in Algorithm 7.12.

Algorithm 7.12: Finding the largest number of an array: `find-max`(A, n)

Input: An array $A[0 : n-1]$ of positive integers and length n
Output: The largest number of the array

1: max = 0
2: **for** $(i = 0; i < n; i++)$ **do**
3:     **if** $(A[i] >$ max$)$ **then**
4:         max = $A[i]$
5:     **end**
6: **end**
7: **return** max

(*b*) The algorithm in item (*a*) has well-defined inputs and outputs, guarantees termination, and produces accurate results. Note that the input for Algorithm 7.12 is a finite array of positive numbers whose length is `n`, and the output is the maximum in the array. Note also that Algorithm 7.12 terminates after visiting all `n` elements, and that the "for" loop from line (2) to line (6) searches through the array and assigns any value larger than `max` to `max`.

■

It is not in the scope of this book to provide the reader a comprehensive list of the control statements because this is an asymptotic analytically-oriented book chapter. Such a list can be found in any book about C programming. The switch statement, for instance, is known as one of the selection statements available in computer programming languages.

Choosing and comparing algorithms

Some problems can be solved by more than one algorithm. How, then, should we choose an algorithm to solve a given problem? If we are tasked with developing a program for a one-time use with small data sets, it is advisable to opt for the simplest algorithm available. However, in scenarios where the program is intended for long-term use and will be maintained by multiple individuals, additional factors come into play:

- Simplicity: An algorithm that is straightforward and easy to understand is more likely to be implemented correctly compared to a complex one.

- Clarity: Well-written, well-documented algorithms are more manageable for others to maintain over time.

- Efficiency: Efficiency becomes crucial as the problem size increases. While efficiency often relates to the runtime of a program, it can also encompass other resource considerations, such as storage space. For larger problems, the runtime is a primary factor in choosing an algorithm. We shall, in fact, take the efficiency of an algorithm to be its running time.

Definition 7.1 *The running time of an algorithm is the duration it requires to complete a task, quantified as a function of the input size.*

Algorithm 7.13: Fragment A of Example 7.7

array[0] = 0 // Cost is c_1
array[1] = 0 // Cost is c_1
array[2] = 0 // Cost is c_1
$\vdots$
array[$n-1$] = 0 // Cost is c_1
// Total cost is $c_1 + c_1 + \cdots + c_1 = nc_1$

Algorithm 7.14: Fragment B of Example 7.7

for ($i = 0; i < n; i++$) **do** // Cost is c_0
  array[i] = 0 // Cost is c_1
end
// Total cost is $(n+1)c_0 + nc_1 = (c_0 + c_1)n + c_0$

Examples of input size (number of elements in the input) are:

- Size of an array.
- Degree of a polynomial.
- Vertices and edges of a graph.
- Number of elements in a matrix.

Given n, the size of the input, we can express the running time as a function of the input, say $f(n)$.

Example 7.7 For each of the two fragments in Algorithms 7.13 and 7.14, we associate a "cost" with each statement as follows: c_0 is the cost of checking the for-statement statement's condition at every iteration in Fragment B shown in Algorithm 7.14, and c_1 is the cost of executing the assignment statement array[i] = 0 for each $i = 0, 1, \ldots, n-1$.

We find the "total cost" by finding the total number of times each statement is executed. The total cost of performing Fragment A shown in Algorithm 7.13 is $f(n) = c_1 n$.

The body of the "for" statement is executed `n` times (from 0 to `n-1`). The "for" statement is executed `n` times (from 0 to `n-1`), plus 1 for the last time that `i` is checked and the inequality is false, that is, `n+1` times. Thus, the total cost of performing Fragment B shown in Algorithm 7.14 is $g(n) = (n+1)c_0 + nc_1 = (c_0 + c_1)n + c_0$.

Now, when $c_0 > 0$, can we say that Algorithm 7.13 is more efficient than Algorithm 7.14? Note that both fragments have linear cost-time. We may find an answer to this question at the end of Example 7.8. ■

In general, to compare two algorithms, we compare the running time function for each algorithm. We will see a rough measure that characterizes how fast function grows, that is, the rate of growth. We compare functions for large values of n, that is, we compare functions asymptotically (in the limit).

Example 7.8 Suppose that for some problem we have the choice of using a linear-time program whose running time is $f(n) = 50n$ and a quadratic-time program whose running time is $g(n) = n^2$. The graphs of the running times are shown in Figure 7.3.

The natural question that arises now is which algorithm is faster? Indeed, we are always interested in which is fastest asymptotically as n gets very large (we can think of n as going to infinity). So, clearly, the linear-time program is faster than the quadratic-time program.

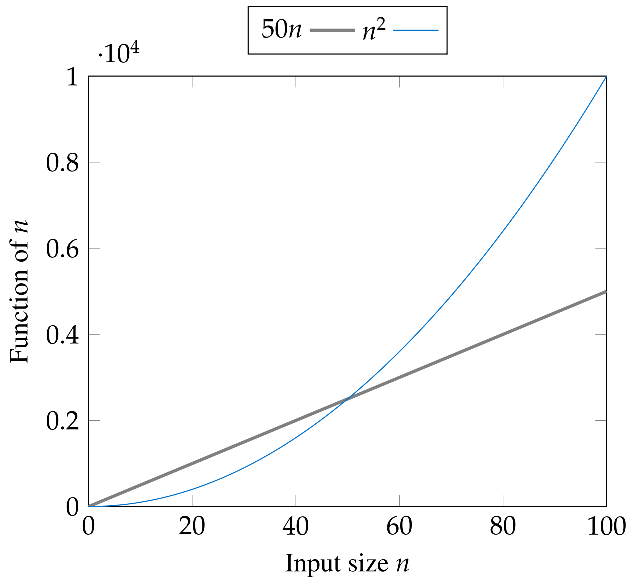


Figure 7.3: $50n$ versus n^2.

Another question that also arises is what impact do the constants have on the running time? Asymptotically speaking, it is clear that the constants have little or no impact on the running time. This might answer the question that we had at the end of Example 7.7. ■

7.2 Running time of algorithms

In Section 7.1, we defined the running time of an algorithm to the amount of time it takes, measured as a function of the size of its input, such as the number of elements in a matrix, number of vertices and edges of a graph, etc. We also took the efficiency of an algorithm to be its running time. Given n, the size of the input, we can express the running time as a function of the input, $f(n)$.

In this section, we study line-by-line runtime analysis of algorithmic fragments, present types of runtime analysis of algorithms, determine upper/lower bounds for running time, and provide more examples on running time.

Line-by-line runtime analysis

The runtime analysis that we did in Example 7.7 is called a line-by-line analysis. In this kind of runtime analysis, we multiply the cost of execution of each line by the number of times we execute that line and sum the results of each line. In this part, we present more examples on performing line-by-line runtime analysis.

Example 7.9 Find the "total cost" needed to perform the fragment given in Algorithm 7.15 by finding the total number of times each statement is executed.

Algorithm 7.15: The algorithm of Example 7.9

```
1: sum = 0                                    // Cost = c1, # times = 1
2: for (i = 0; i < n; i++) do                 // Cost = c2, # times = n + 1
3:     for (j = 0; j < n; j++) do             // Cost = c2, # times = n(n + 1)
4:         sum += array[i][j]                 // Cost = c3, # times = n^2
5:     end
6: end
```

Solution Let us add comments showing the number of times each statement is executed (see the comments in gray in Algorithm 7.15). The following points are noted in order to find the total running time:

- It is clear that the assignment statement in line (1) is executed once.
- How many times the "for" statement in line (2) executes? 0 to $n-1$, which is n times, plus 1 for the last time that i is checked and the inequality is false, that is, $n+1$ times.
- How many times the "for" statement in line (3) executes? If the "for" statement in line (3) was not nested it would execute $n+1$ times just like the "for" statement in line (2). Since the "for" statement in line (3) is nested it executes $n+1$ times for *each* time we execute the body of the "for" statement in line (2) which is n times.
- What about the sum statement in line (4)? If the second loop was not nested, the sum statement in line (4) would execute n times since that is the number of times the body of the second loop would execute. Since the second loop is nested, the sum statement in line (4) executes n times for each time we execute the body of the first loop which is n times. Therefore, $n \times n = n^2$.

Now, the total cost or running time of the algorithm is the sum of the cost multiplied by the number of times for each line.

Running time is $c_1(1) + c_2(n+1) + c_2 n(n+1) + c_3 n^2$. Simplifying to polynomial form, we conclude that the running time is

$$(c_2 + c_3)n^2 + 2c_2 n + (c_1 + c_2),$$

or $\bar{c}n^2 + \hat{c}n + \tilde{c}$, where $\bar{c} = c_2 + c_3, \hat{c} = 2c_2, \tilde{c} = c_1 + c_2$.

We use the predominate term or in other words, the degree of the polynomial, to express the running time. So, we conclude the running time is $\bar{c}n$, where $\bar{c} = c_2 + c_3$. ■

In Example 7.9, we found that the condition in the for-statement "for $(i = 0; i < n; i++)$" was checked $n+1$ times. Similarly, the condition in each of the for-statements "for $(i = n; i > 0; i--)$", "for $(i = 1; i <= n; i++)$", and "for $(i = n; i >= 1; i--)$" is checked $n+1$ times. We have the following example.

Example 7.10 Find the running time of the fragment given in Algorithm 7.16 by finding the total number of times each statement is executed.

Solution Note that this algorithm comprises two independent loops, each performing slightly distinct tasks. We have added comments in gray showing the number of times each statement is executed.

Algorithm 7.16: The algorithm of Example 7.10

```
int q = 0                               // Cost = c1, # times = 1
for (i = 1; i ≤ n; i + +) do            // Cost = c2, # times = n + 1
    q = q + i^2                         // Cost = c3, # times = n
end
for (j = n; j ≥ 1; j − −) do            // Cost = c2, # times = n + 1
    q = q + j                           // Cost = c4, # times = n
end
```

Algorithm 7.17: The algorithm of Example 7.11

```
int q = 0                               // Cost = c1, # times = 1
for (i = 1; i ≤ log n; i + +) do        // Cost = c2, # times = (log n) + 1
    q = q + i^2                         // Cost = c3, # times = log n
end
for (j = 1; j < n; j∗ = 2) do           // Cost = c4, # times = (log n) + 1
    q = q + j^2                         // Cost = c3, # times = log n
end
for (k = n; k > 1; k/ = 2) do           // Cost = c4, # times = (log n) + 1
    q = q + k^2                         // Cost = c3, # times = log n
end
```

The running time of the fragment is the sum of the cost multiplied by the number of times for each line. Therefore,

$$\begin{aligned}\text{Running time} &= c_1 + c_2(n+1) + c_3 n + c_2(n+1) + c_4 n \\ &= (2c_2 + c_3 + c_4)n + (c_1 + 2c_2) = \bar{c}n + \hat{c},\end{aligned}$$

where $\bar{c} = 2c_2 + c_3 + c_4$ and $\hat{c} = c_1 + 2c_2$. Using the predominate term, the running time is $\bar{c}n$. ■

In Example 7.10, we found that the condition in the for-statement "for $(i = 1; i <= n; i{+}{+})$" is checked $n + 1$ times. One can also find that the condition in each of the for-statements "for $(i = 1; i <= logn; i + +)$", "for $(i = 1; i < n; i* = 2)$", and "for $(i = n; i > 1; i/ = 2)$" is checked $(\log n) + 1$ times.

Example 7.11 Find the running time of the fragment given in Algorithm 7.17 by finding the total number of times each statement is executed.

Solution Note that this algorithm has three independent loops, each performing slightly distinct tasks. We have added comments in gray showing the number of time each statement is executed.

The running time of the fragment is the sum of the cost multiplied by the number of times for each line. Therefore,

$$\begin{aligned}\text{Running time} &= c_1 + c_2(1 + \log n) + c_3 \log n + 2c_4(1 + \log n) + 2c_3 \log n \\ &= (c_1 + c_2 + 2c_4) + (c_2 + 3c_3 + 2c_4)\log n = \bar{c}\log n + \hat{c},\end{aligned}$$

where $\bar{c} = c_2 + 3c_3 + 2c_4$ and $\hat{c} = c_1 + c_2 + 2c_4$. Using the predominate term, the running time is $\bar{c}\log n$. ■

Types of runtime analysis

A question that suggests itself is how do we evaluate the running time given there is conditional execution (i.e., selection statement)? Here, we will look at the worst-case running time. Why not the best-case? What would be the worst-case scenario be? There are three types of runtime analysis for a given algorithm:

- Worst-case analysis: This type provides an upper bound on running time guaranteeing that the algorithm would not run longer.
- Average-case analysis: This type provides a prediction about running time, assuming input is random.
- Best-case analysis: This type provides a lower bound on running time on input for which algorithm runs the fastest.

Note that the average running time can sometimes provide a more practical estimate of performance in real-world scenarios. However, calculating it is often significantly more challenging than determining the worst-case running time.

Example 7.12 The code in Algorithm 7.18 sets small to the index of the smallest element found in the portion of the array A from $A[i]$ through $A[n-1]$. Find the running time equipped with worst-case performance.

Solution We are looking at the worst-case running time. We have added comments in gray showing the number of times each statement is executed.

We consider the body of the "for" statement, the "if" statement (lines (3) and (5)). The test of line (3) is always executed, but the assignment at line (4) is executed only if the test succeeds. Thus, the body (lines (3) and (4)) takes either $c_3 + c_1$ or c_3 time costs, depending on the data in array A. If we want to take the worst-case, then we can assume that the body takes $c_3 + c_1$ time cost.

The (worst-case) total cost or running time of the code is the sum of the cost multiplied by the number of times for each line. Therefore

$$\begin{aligned} \text{Running time} &= c_1 + c_2(n-i) + c_3(n-i-1) + c_1(n-i-1) \\ &= (c_1 + c_2 + c_3)(n-i) - c_3. \end{aligned}$$

It is natural to regard the size "$\bar{n}$" of the data on which the code operates as $\bar{n} = n - i$, since that is the length of the array $A[i : n-1]$ on which it operates. Then the running time, which is $(c_1 + c_2 + c_3)(n-i) - c_3$, equals $\bar{c}\bar{n} - c_3$ where $\bar{c} = c_1 + c_2 + c_3$. To sum up, using the predominate term, the running time is $\bar{c}\bar{n}$ where $\bar{n} = n - i$ and $\bar{c} = c_1 + c_2 + c_3$. ■

Algorithm 7.18: The algorithm of Example 7.12

```
1: sum = i                                  // Cost = c1, # times = 1
2: for (j = i + 1; j < n; j++) do           // Cost = c2, # times = n − i
3:     if (A[i] < A[small]) then            // Cost = c3, # times = n − i − 1
4:         sum = j                          // Cost = c1, # times = n − i − 1 [worst-case]
5:     end
6: end
```

Algorithm 7.19: The algorithm of Example 7.13

```
for (i = 1; i < m; i++) do                    // Cost = c1, # times = m
    if (i < n/2) then                         // Cost = c2, # times = m − 1
        while (n > 1) do                      // Cost = c3, # times = (m − 1)n [worst case]
            printf(n)                         // Cost = c4, # times = (m − 1)(n − 1) [worst case]
            n−−                               // Cost = c5, # times = (m − 1)(n − 1) [worst case]
        end
    end
    while (n > 1) do                          // Cost = c6, # times = (m − 1)((log10 n) + 1)
        printf(n)                             // Cost = c7, # times = (m − 1) log10 n
        n/ = 10                               // Cost = c8, # times = (m − 1) log10 n
    end
end
```

Example 7.13 Compute the worst-case and best-case runtime complexities of the program snippet shown in Algorithm 7.19.

Solution First, we find the worst-case running time. The most-right column adds the number of times each statement is executed.

The worst-case running time of the fragment is the sum of the cost multiplied by the number of times for each line. Therefore,

$$\begin{aligned}\text{Run time} &= c_1 m + c_2(m-1) + c_3(m-1)n + (c_4+c_5)(m-1)(n-1)\\ &\quad + c_6(m-1)(1+\log_{10} n) + c_7(m-1)\log_{10} n + c_8(m-1)\log_{10} n\\ &= c_1 m + (c_2+c_6)(m-1) + c_3(m-1)n + (c_4+c_5)(m-1)(n-1)\\ &\quad + (c_6+c_7+c_8)(m-1)\log_{10} n.\end{aligned}$$

Using the predominate term, the running time is $\bar{c}nm$ where $\bar{c} = c_3 + c_4 + c_5$.

The best-case running time of the fragment is the sum of the cost multiplied by the number of times for each line. Therefore,

$$\begin{aligned}\text{Running time} &= c_1 m + c_2(m-1) + c_6(m-1)(1+\log_{10} n) + c_7(m-1)\log_{10} n\\ &\quad + c_8(m-1)\log_{10} n\\ &= c_1 m + (c_2+c_6)(m-1) + (c_6+c_7+c_8)(m-1)\log_{10} n.\end{aligned}$$

Using the predominate term, the running time is $\hat{c}m \log n$ where $\hat{c} = (c_6 + c_7 + c_8)/\log 10$. ■

Another example for computing different case complexities is shown in Insertion Sort which will be introduced and analyzed in Chapter 8.

Summation representations for looping

In all the illustrative instances we have examined thus far concerning the assessment of running times, our approach has consistently involved a detailed examination of the code, performing a line-by-line analysis to determine the time complexities involved. In this part, we depart from the conventional practice of inspecting each line's execution time individually. Instead, we pivot towards a more abstract perspective, providing additional illustrative examples that showcase the running times of loop-based programs. In this new approach, we represent the running time in the form of summations, allowing for a more generalized and efficient method of analysis.

Note that the inner loop in Algorithm 7.15 of Example 7.9 is the part that contributed most to the running time. In fact, there is an informal rule called the 90-10 complexity rule which states that 90% of the running time is spent in 10% of the code. However, the exact percentage varies from one program to another. Instead of analyzing every statement, we can save time and effort by focusing on the statement that contributes most to the running time.

Example 7.14 Consider the code in Algorithm 7.20. It does not really matter what the code does, but what really matters is finding the running time of its execution. Instead of analyzing very statement, we can save time and effort by focusing on the statement in line (4).

Assuming n is a constant, say 3, how many times would the statement in line (4) be executed? It is clear that the answer here is $3 \times 3 = 3^2 = 9$ times. Generalizing this for any n, we find that the statement in line (4) would be executed n^2 times.

Let c be the execution cost of the statement in line (4). Then we can represent the running time using summations and obtain $\sum_{i=1}^{n} \sum_{j=1}^{n} c = c \sum_{i=1}^{n} n = cn^2$.

Thus, the running time of the code in Algorithm 7.20 is cn^2. ■

When endeavoring to determine the running time of an algorithm, there are some noteworthy considerations to bear in mind:

- One can analyze algorithms by focusing on the part(s) where most of the execution occurs.
- The running time is significantly influenced by the input data provided to the algorithm. Different inputs can lead to variations in the running time of the algorithm.

Example 7.15 Find the running time of the code in Algorithm 7.21 without doing a line-by-line analysis.

Algorithm 7.20: The algorithm of Example 7.14

```
1: x = 0
2: for (i = 1; i ≤ n; i++) do
3:     for (j = 1; j ≤ n; j++) do
4:         x = x + (i − j)            // Execution cost is c
5:     end
6: end
7: return x
```

Algorithm 7.21: The algorithm of Example 7.15

```
x = 0
for (i = 1; i ≤ n; i++) do
    for (j = 1; j ≤ i; j++) do
        x = x + (i − j)            // Execution cost is c
    end
end
return x
```

Solution To save time and effort, we focus on the statement in line (4). Assume that n is a constant, say 3. We determine the number of times the statement in line (4) is executed by tracing the code and find that

i	j = 1 to i
1	1 to 1 = 1
2	1 to 2 = 2
3	1 to 3 = 3

So, the number of times the statement in line (4) is executed is $1 + 2 + 3 = 6$ times.

Now, generally speaking, for any n, we determine the number of times the statement in line (4) is executed by tracing the code and find that

i	j = 1 to i
1	1 to 1 = 1
2	1 to 2 = 2
⋮	⋮
n	1 to n = n

Therefore, the number of times the statement in line (4) is executed is

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ times,}$$

where we used the arithmetic series to obtain the last equality.

Let c be the execution cost of the statement in line (4). Then, the running time of the code in Algorithm 7.21 is $cn(n+1)/2 \approx cn^2$. ■

Example 7.16 Find the running time of the code in Algorithm 7.22 without doing a line-by-line analysis.

Algorithm 7.22: The algorithm of Example 7.16

```
1: x = 0
2: for (i = 1; i ≤ n; i++) do
3:     for (j = i; j ≤ n; j++) do
4:         x = x + (i − j)                    // Execution cost is c
5:     end
6: end
7: return x
```

Solution To save time and effort, we focus on the statement in line (4). Assume that n is a constant, say 3. We determine the number of times the statement in line (4) is executed by tracing the code and find that

i	j = i to n
1	1 to 3 = 3
2	2 to 3 = 2
3	3 to 3 = 1

So, the number of times the statement in line (4) is executed is $3 + 2 + 1 = 6$ times.

In general, for any n, we have

i	j = i to n
1	1 to n = n
2	2 to n = n-1
3	3 to n = n-2
⋮	⋮
n-1	n-1 to n = 2
n	n to n = 1

Therefore, the number of times the statement in line (4) is executed is

$$n + (n-1) + (n-2) + \cdots + 2 + 1 = \sum_{i=1}^{n}(n-i+1) = \frac{n(n+1)}{2} \text{ times.}$$

Let c be the execution cost of the statement in line (4). Then, the running time of the code in Algorithm 7.21 is $cn(n+1)/2 \approx cn^2$. ■

Other examples of (double and triple) summation representations for looping are the matrix-vector multiplication and the matrix-matrix multiplication.

Upper and lower bounds for running time

Some summations are difficult to work with. In this part, we find upper and lower bounds for running time. This allows us to simplify the summations.

In adopting this approach, we require that the upper and lower bounds must be the same function, only differing by a constant. The constant for the upper bound must be larger than or equal to the constant for the lower bound.

To obtain an upper bound in this approach, we remove terms of expression being subtracted, if helpful, and substitute terms (usually, but not necessarily, the upper/top bound of the summation) into expression. To obtain a lower bound, we split summations to reduce size, if helpful, and substitute terms (usually, but not necessarily, the lower/bottom bound of the summation) into expression.

Example 7.17 (Example 7.15 revisited) We find the upper and lower bounds for the running time of the code in Algorithm 7.21. Let c be the execution cost of the statement in line (4). Then we can represent the running time using summations and obtain

$$\sum_{i=1}^{n}\sum_{j=i}^{n} c = \sum_{i=1}^{n} c(n-i+1).$$

Finding an upper bound is straightforward:

$$\sum_{i=1}^{n} c(n-i+1) \leq \sum_{i=1}^{n} cn = cn^2.$$

We now find a lower bound:

$$\sum_{i=1}^{n}\sum_{j=i}^{n} c \geq \sum_{i=1}^{n} c(n-i) \geq \sum_{i=\frac{n}{2}}^{n} c(n-i). \tag{7.1}$$

Our goal is to make the right-hand summation in (7.1) at least as large as a quadratic function. Let us try to substitute $\frac{n}{2}$ or n for i and see what happens.

Substituting $\frac{n}{2}$ for i in (7.1), we get

$$\sum_{i=\frac{n}{2}}^{n} c(n-i) \leq \sum_{i=\frac{n}{2}}^{n} c\left(n-\frac{n}{2}\right),$$

but this does not work because it makes the lower bound larger instead of smaller (for lower bounds, we should not make the lower bound larger).

Substituting n for i in (7.1), we get

$$\sum_{i=\frac{n}{2}}^{n} c(n-i) \geq \sum_{i=\frac{n}{2}}^{n} c(n-n) = 0,$$

but this does not work either because it makes the lower bound larger very small, it is zero.

Splitting the summation differently and using the lower split, we get

$$\sum_{i=1}^{n} c(n-i) \geq \sum_{i=1}^{\frac{n}{2}} c(n-i) \geq \sum_{i=1}^{\frac{n}{2}} c\left(n-\frac{n}{2}\right) = c\left(\frac{n}{2}\right)\left(\frac{n}{2}\right) = \left(\frac{c}{4}\right)n^2.$$

Thus, the running time is $\bar{c}n^2$ where $c/4 \leq \bar{c} \leq c$. ■

Example 7.18 The running time of an algorithm is represented by the summation

$$\sum_{i=1}^{n}\sum_{j=i}^{n^2} c, \tag{7.2}$$

where n is the number of times the outer loop is iterated and c is the execution cost of the statement that contributed most to the running time of the algorithm. Find upper and lower bounds for the running time of the algorithm. Can your upper/lower bound(s) be sharpened to obtain a tight bound for the running time?

Solution Finding an upper bound is straightforward:

$$\sum_{i=1}^{n}\sum_{j=i}^{n^2} c = c\sum_{i=1}^{n}(n^2 - i + 1) \le c\sum_{i=1}^{n} n^2 = cn^3.$$

For a lower bound, we need to make the summation representation in (7.2) at least as large as a cubic function. This is immediately obtained by noting that

$$\sum_{i=1}^{n}\sum_{j=i}^{n^2} c \ge c\sum_{i=1}^{n}(n^2 - i) \ge c\sum_{i=\frac{n}{2}}^{n}(n^2 - i) \ge c\sum_{i=\frac{n}{2}}^{n}(n^2 - n) = \frac{c}{2}(n^3 - n^2) = \left(\frac{c}{2}\right)n^3. \tag{7.3}$$

Although a lower bound was successfully obtained, the splitting was not helpful in (7.3). In fact, the above lower bound can be sharpened as follows.

$$\sum_{i=1}^{n}\sum_{j=i}^{n^2} c \ge c\sum_{i=1}^{n}(n^2 - i) \ge c\sum_{i=1}^{n}(n^2 - n) = c(n^3 - n^2) \ge cn^3. \tag{7.4}$$

Therefore, a tight bound for the running time of the algorithm represented by the summation

$$\sum_{i=1}^{n}\sum_{j=i}^{n^2} c$$

is cn^3. This gives us the desired bound. ■

Note that, in obtaining lower bounds in Example 7.18 (where second inequality in (7.3) and the first inequality in (7.4) follow), we substituted n for i and did not get 0. This is the difference between Examples 7.17 and 7.18.

Example 7.19 Find upper and lower bounds for the running time of the algorithm represented by the summation

$$\sum_{i=1}^{n}\sum_{j=1}^{i^2}\sum_{k=j}^{i^2} c, \tag{7.5}$$

where n is the number of times the outer loop iterated and c is the execution cost of the statement that contributed most to the running time of the algorithm.

Solution For an upper bound, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{i^2}\sum_{k=j}^{i^2} c = c\sum_{i=1}^{n}\sum_{j=1}^{i^2}(i^2 - j + 1) \le c\sum_{i=1}^{n}\sum_{j=1}^{i^2} i^2 = c\sum_{i=1}^{n} i^4 \le c\sum_{i=1}^{n} n^4 = cn^5.$$

For a lower bound, we need to make the summation representation in (7.5) at least as large as a quintic function. This is immediately obtained by noting that

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{i^2}\sum_{k=j}^{i^2} c &\geq c\sum_{i=1}^{n}\sum_{j=1}^{i^2}(i^2-j)\\
&\geq c\sum_{i=1}^{n}\sum_{j=1}^{i^2/2}(i^2-j)\\
&\geq c\sum_{i=1}^{n}\sum_{j=1}^{i^2/2}\left(i^2-\frac{i^2}{2}\right)\\
&= c\sum_{i=1}^{n}\left(\frac{i^2}{2}\right)\left(i^2-\frac{i^2}{2}\right)\\
&= c\sum_{i=1}^{n}\frac{i^4}{4}\\
&\geq c\sum_{i=\frac{n}{2}}^{n}\frac{i^4}{4}\\
&\geq c\sum_{i=\frac{n}{2}}^{n}\frac{(n/2)^4}{4}\\
&\geq c\left(\frac{n}{2}\right)\left(\frac{n^4/16}{4}\right)=\left(\frac{c}{128}\right)n^5.
\end{aligned}$$

Therefore, the running time of the algorithm represented by the summation in (7.5) is $\bar{c}n^5$, where $\frac{c}{128} \leq \bar{c} \leq c$. ■

7.3 Asymptotic notation

The asymptotic notation allows us to express the behavior of a function as the input approach infinity. In other words, it is connected about what happens to a function $f(n)$ as n gets larger, and is not concerned about the value of $f(n)$ for small values of n. In this section, we introduce the notations, present their properties and criteria of selection, characterize the definitions of the notations using limits, and give a complexity classification of algorithms based on these notations.

The notations

We present the definitions of Big-Oh, Big-Omega, and Big-Theta which are all asymptotic notations to describe the running time of algorithms. Throughout this section, unless it is stated explicitly otherwise, we assume that f and g are two asymptotically nonnegative functions on $\mathbb{R}$, that is, $f(n)$ and $g(n)$ are nonnegative whenever n is sufficiently large.

Definition 7.2 (Big-O) *We say that $f(n)$ is Big-O of $g(n)$, written as $f(n) = O(g(n))$, if there are positive constants c and n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$.*

According to Definition 7.2, $f(n) = O(g(n))$ means that $f(n)$ grows no faster than $g(n)$. See Figure 7.4.

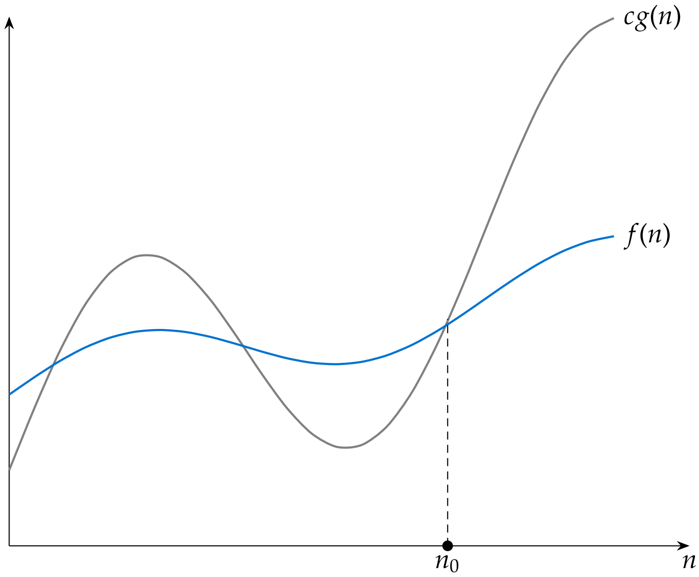


Figure 7.4: $f(n) = O(g(n))$.

Remark 7.1 *The "=" in the statement "$f(n) = O(g(n))$" should be read and thought of as "is", not "equals". The reader can think of it as a one-way equals. An alternative notation is to write $f(n) \in O(g(n))$ instead of $f(n) = O(g(n))$.*

Bearing in mind Remark 7.1, Definition 7.2 can be rewritten as follows.

Definition 7.3 (Big-O revisited) *Big-Oh of $g(n)$, written as $O(g(n))$, is the set of functions with smaller or same order of growth as $g(n)$. More specifically,*

$$O(g(n)) \triangleq \{f(n) : \exists \text{ positive constants } c \text{ and } n_0 \text{ s.t. } 0 \le f(n) \le cg(n)\ \forall n \ge n_0\}.$$

Here $g(n)$ is called an asymptotic upper bound for $f(n)$.

The following example illustrates Definitions 7.2 and 7.3.

Example 7.20 Prove the following asymptotic statements.

(*a*) $n^2 + n = O(n^2)$.

(*b*) $3n^3 - 2n^2 + 7n - 9 = O(n^3)$.

(*c*) $n^{\frac{7}{2}} + n^3 \log n = O(n^4)$.[1]

Solution (*a*) If $n \ge 1$, we have $n \le n^2$. It follows that $n^2 + n \le n^2 + n^2 = 2n^2$ for all $n \ge 1$. Therefore, according to Definition 7.2, we have $n^2 + n = O(n^2)$ with $n_0 = 1$ and $c = 2$.

(*b*) If $n \ge 1$, we have that $3n^3 - 2n^2 + 7n - 9 \le 3n^3 + 7n \le 3n^3 + 7n^3 = 10n^3$ for all $n \ge 1$. Thus $3n^3 - 2n^2 + 7n - 9 = O(n^3)$ with $n_0 = 1$ and $c = 10$.

[1] Computer scientists generally think of "$\log n$" as meaning $\log_2 n$ rather than $\log_{10} n$ and $\log_e n$.

(*c*) If $n \geq 1$, we have $n^{\frac{7}{2}} \leq n^4$ and $\log n \leq n$, which implies that

$$n^{\frac{7}{2}} + n^3 \log n \leq n^4 + (n^3)(n) = 2n^4.$$

Thus, with $n_0 = 1$ and $c = 2$, we have $n^{\frac{7}{2}} + n^3 \log n = O(n^4)$.

■

Below are some relationships among the growth rates of some common functions:

$$c \ll \log n \ll \log^2 n \ll \sqrt{n} \ll n \ll n \log n \ll n^{1.1} \ll n^2 \ll 2^n \ll 3^n \ll n! \ll n^n.$$

For instance, $n^3 \ll 2^n$ means that n^3 is asymptotically much less than 2^n. In fact, when $n = 10$, we have $2^{10} = 1024$ and $10^3 = 1000$. Now, each time we add 1 to n, 2^n doubles, while n^3 is multiplied by the quantity $(\frac{n+1}{n})^3$ which is less than 2 when $n \geq 10$.

Example 7.21 Prove the following asymptotic statements.

(*a*) $2^n + n^3 = O(2^n)$. (*b*) $\log(n!) = O(n \log n)$. (*c*) $(\sqrt[3]{2})^{\log n} = O(\sqrt[3]{n})$.

Solution (*a*) Note that for $n \geq 10$, we have $n^3 \leq 2^n$. It follows that, for $n \geq 10$, we have $2^n + n^3 \leq 2^n + 2^n = 2(2^n)$. Thus, $2^n + n^3 = O(2^n)$ with $n_0 = 10$ and $c = 2$.

(*b*) If $n \geq 1$, we have $n! = n(n-1)\cdots 2 \cdot 1 \leq \overbrace{n \cdot n \cdots n \cdot n}^{n \text{ times}} = n^n$, and hence $\log(n!) \leq \log n^n = n \log n$. Thus, $\log(n!) = O(n \log n)$ with $n_0 = 1$ and $c = 1$.

(*c*) Note that

$$\left(\sqrt[3]{2}\right)^{\log n} = \left(2^{\frac{1}{3}}\right)^{\log n} = 2^{(\frac{1}{3})(\log n)} = 2^{\log n^{\frac{1}{3}}} = n^{\frac{1}{3}} = \sqrt[3]{n}.$$

Then it is clear that $\left(\sqrt[3]{2}\right)^{\log n} = O(\sqrt[3]{n})$.

■

Definition 7.4 (Big-Ω) *Big-Omega of $g(n)$, written as $\Omega(g(n))$, is the set of functions with larger or same order of growth as $g(n)$. More specifically,*

$$\Omega(g(n)) \triangleq \{f(n) : \exists \text{ positive constants } c \text{ and } n_0 \text{ such that } 0 \leq cg(n) \leq f(n)\ \forall n \geq n_0\}.$$

Here $g(n)$ is called an asymptotic lower bound for $f(n)$.

Figure 7.5 illustrates Definition 7.4. We also present Example 7.22.

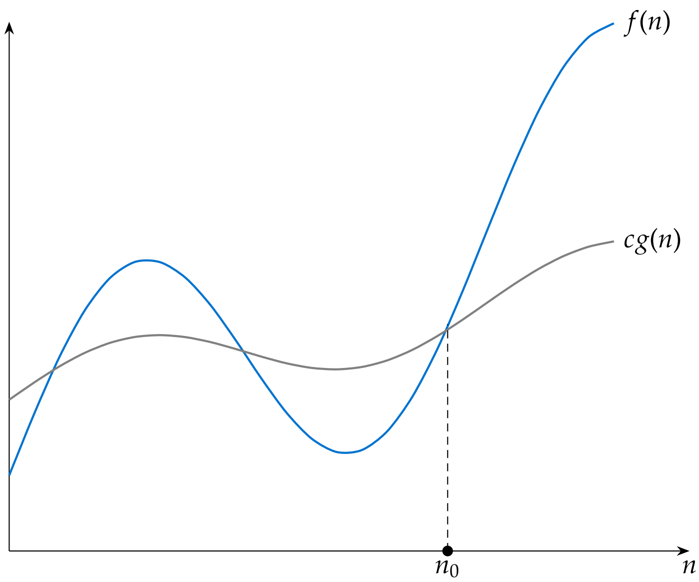


Figure 7.5: $f(n) = \Omega(g(n))$.

Example 7.22 Prove the following asymptotic statements.

(*a*) $n^3 + 5n^2 = \Omega(n^2)$.

(*b*) $n \log n - n + \log 9 = \Omega(n \log n)$.

Solution (*a*) If $n \geq 1$, we have $n^2 \leq n^3 \leq n^3 + 5n^2$. Thus, according to Definition 7.4, we conclude that $n^3 + 5n^2 = \Omega(n^2)$ with $n_0 = 1$ and $c = 1$.

(*b*) We need to show that there exist positive constants c and n_0 such that $cn \log n \leq n \log n - n + \log 9$ for all $n \geq n_0$. Since $n \log n - n \leq n \log n - n + \log 9$, we will instead show that

$$cn \log n \leq n \log n - n, \quad \text{or equivalently} \quad c \leq 1 - \frac{1}{\log n}.$$

If $n \geq 4$, then $\frac{1}{\log n} \leq \frac{1}{2}$ (as $\log 4 = 2$), or equivalently $\frac{1}{2} \leq 1 - \frac{1}{\log n}$. So, picking $c = \frac{1}{2}$ suffices. In other words, we have just shown that if $n \geq 4$,

$$\frac{1}{2} n \log n \leq n \log n - n.$$

Thus, if $c = \frac{1}{2}$ and $n_0 = 4$, then for all $n \geq n_0$, we have

$$cn \log n \leq n \log n - n \leq n \log n - n + \log 9.$$

Therefore, $n \log n - n + \log 9 = \Omega(n \log n)$. ■

Now, we shift our focus to the Big-Theta notation. Early works by Donald Knuth and Robert Floyd in the 1960s and 1970s (see Knuth [1997], Floyd [1993]) significantly contributed to the formalization and widespread use of Big-Theta notation in the analysis of algorithms. Like Big-Oh notation, the Big-Theta notation is also a fundamental tool in the field of computer science for describing the asymptotic behavior and efficiency of algorithms.

Definition 7.5 (Big-Θ) *Big-Theta of $g(n)$, written as $\Theta(g(n))$, is the set of functions with the same order of growth as $g(n)$. More specifically,*

$$\begin{aligned}\Theta(g(n)) \triangleq \{f(n) : &\exists \text{ positive constants } c_1, c_2 \text{ and } n_0 \text{ such that} \\ &0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \ \ \forall n \geq n_0\}.\end{aligned}$$

Here $g(n)$ is called an asymptotically tight bound for $f(n)$.

Figure 7.6 illustrates Definition 7.5. We also present the following example.

Example 7.23 Prove that $n^2 + 3n + 6 = \Theta(n^2)$.

Solution When $n \geq 1$, we have $n^2 \leq n^2 + 3n + 6 \leq n^2 + 3n^2 + 6n^2 \leq 10n^2$. Thus, according to Definition 7.4, we conclude that $n^2 + 3n + 6 = \Theta(n^2)$ with $n_0 = 1$ and $c_1 = 1$ and $c_2 = 10$. ■

What if some function is not Big-Oh, not Big-Omega, or not Big-Theta of some other function? The method of proving this is to assume that witnesses n_0 and c exist, and derive a contradiction. We have the following example.

Example 7.24 Show that n^2 is not $O(n)$.

Solution Suppose that $n^2 = O(n)$. Then there exists n_0 and c such that $n^2 \leq cn$ for all $n \geq n_0$. Picking $n_1 = \max\{n_0, 2c\}$, we have $n_1^2 \leq cn_1$, and dividing both sides by n_1, we get $n_1 \leq c$. But we chose n_1 to be at least $2c$, a contradiction. This concludes that n^2 is not $O(n)$. ■

Properties of the notations

There are a lot of properties for Big-Oh, Big-Omega, and Big-Theta notations. In this part, we present a few of the most important ones.

The following property follows almost immediately from Definitions 7.3, 7.4 and 7.5, and therefore its proof is left as an exercise for the reader.

Property 7.1 *$f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.*

Figure 7.7 shows a conceptual relationship among the notations.

Example 7.25 Prove the following asymptotic statements. Do not prove by a direct use of Big-Theta definition (Definition 7.5).

(*a*) $\frac{1}{4}n^2 + 4n = \Theta(n^2)$.

(*b*) $n^8 + 5n^7 - 13n^5 - 6n^4 + 4n^2 - 21 = \Theta(n^8)$.

Solution (*a*) If $n \geq 1$, we have $\frac{1}{4}n^2 + 4n \leq \frac{1}{4}n^2 + 4n^2 = 4.25n^2$. Therefore, $\frac{1}{4}n^2 + 4n^2 = O(n^2)$. On the other hand, when $n \geq 1$, we have $\frac{1}{4}n^2 \leq \frac{1}{4}n^2 + 4n$. Therefore, $\frac{1}{4}n^2 + 4n^2 = \Omega(n^2)$. Consequently, by Theorem 7.1, we conclude that $\frac{1}{4}n^2 + 4n = \Theta(n^2)$.

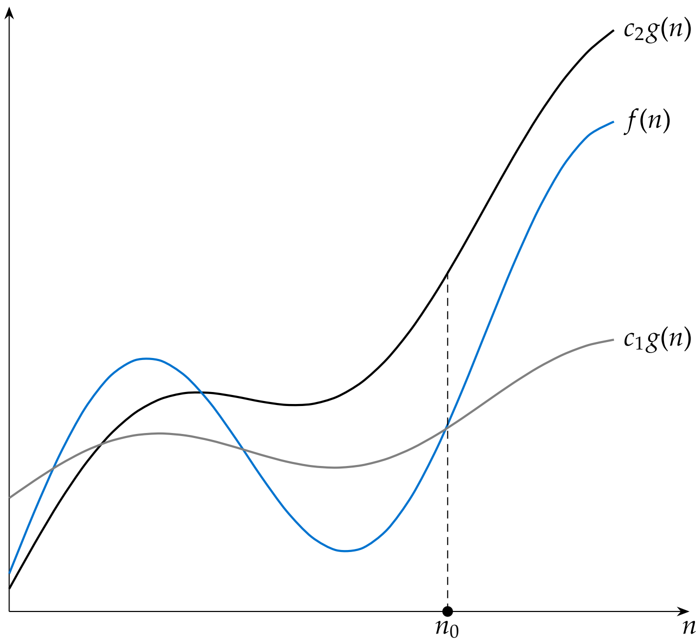


Figure 7.6: $f(n) = \Theta(g(n))$.

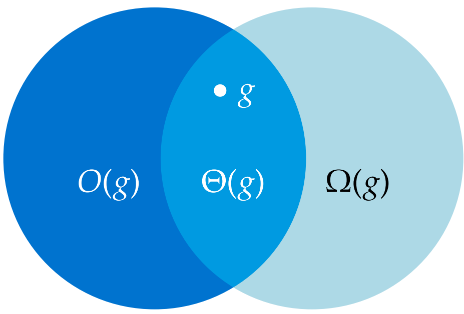


Figure 7.7: Relationships among Big-Oh, Big-Omega, and Big-Theta notations.

(*b*) Let $f(n) = n^8 + 5n^7 - 13n^5 - 6n^4 + 4n^2 - 21$. To prove that $f(n) = \Theta(n^8)$, we show that $f(n) = O(n^8)$ and $f(n) = \Omega(n^8)$.

It is clear that when $n \geq 1$, we have

$$f(n) \leq n^8 + 5n^7 + 4n^2 \leq n^8 + 5n^8 + 4n^8 = 10n^8.$$

Thus, $f(n) = O(n^8)$ as desired.

Next, we prove that $f(n) = \Omega(n^8)$. That is, we need to show that there exist positive constants n_0 and c such that $cn^8 \leq f(n)$ for all $n \geq n_0$.

Now, because

$$\begin{aligned} f(n) &= n^8 + 5n^7 - 13n^5 - 6n^4 + 4n^2 - 21 \\ &\geq n^8 - 13n^5 - 6n^4 - 21 \\ &\geq n^8 - 13n^7 - 6n^7 - 21n^7 = n^8 - 40n^7, \end{aligned}$$

we will instead show that $cn^8 \leq n^8 - 40n^7$ for some $c > 0$ and for all $n \geq n_0$.

Note that

$$cn^8 \leq n^8 - 40n^7 \iff (1-c)n^8 \geq 40n^7 \iff c \leq 1 - \frac{40}{n}.$$

So, if $n \geq 80$, then $c = \frac{1}{2}$ suffices.

Thus, we have shown that if $n \geq 80$, then $f(n) \geq \frac{1}{2}n^8$. Thus, $f(n) = \Omega(n^8)$. This completes the proof.

■

The following property follows almost immediately from Definition 7.3, and therefore its proof is also left as an exercise for the reader.

Property 7.2 (Transitivity) *If we have $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$. Same for Big-Omega and Big-Theta.*

Example 7.26 It is easy to see that $4n^2 + 3n + 17 = O(n^3)$ and that $n^3 = O(n^4)$. Based on Property 7.2, we conclude that $4n^2 + 3n + 17 = O(n^4)$. ■

Property 7.3 (Scaling by a constant) *If $f(n) = O(g(n))$, then $kf(n) = O(g(n))$ for any $k > 0$. Same for Big-Omega and Big-Theta.*

Proof Assume that $f(n) = O(g(n))$, then from Definition 7.3, there are positive constants c and n_0 such that $f(n) \leq cg(n)$ for $n \geq n_0$. Multiplying with $k > 0$, we get $kf(n) \leq \bar{c}g(n)$, for $n \geq n_0$, where $\bar{c} = kc$. The result is established for Big-O. Similar arguments can be made for Big-Omega and Big-Theta. ■

Example 7.27 Let a and b be positive numbers that are different from 1. Bearing in mind that $\log_a n = \log_b n / \log_b a$. Then, according to Property 7.3, it follows that $\log_a n = O(\log_b n)$. ■

Property 7.4 (Sums) *If we have $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n) + f_2(n) = O(g_1(n) + g_2(n)) = O(\max\{g_1(n), g_2(n)\})$. Same for Big-Omega and Big-Theta.*

Proof We prove the result for Big-O, and similar arguments can be made for Big-Omega and Big-Theta.

Let $f_i(n) = O(g_i(n))$ for $i = 1, 2$, then from Definition 7.3, there are positive constants c_i and n_i such that $f_i(n) \leq c_i g_i(n)$ for $n \geq n_i$.

Now, let $n_0 = \max\{n_1, n_2\}$, then for all $n \geq n_0$, we have

$$\begin{aligned} f_1(n) + f_2(n) &\leq c_1 g_1(n) + c_2 g_2(n) \\ &\leq c_1 \max\{g_1(n), g_2(n)\} + c_2 \max\{g_1(n), g_2(n)\} \\ &= (c_1 + c_2) \max\{g_1(n), g_2(n)\} \\ &\leq \bar{c} \max\{g_1(n), g_2(n)\} \leq \bar{c}(g_1(n) + g_2(n)), \end{aligned}$$

where $\bar{c} = 2\max\{c_1, c_2\}$. The proof is complete. ∎

We leave the proof of the following property as an exercise for the reader.

Property 7.5 (Products) *If we have $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n)f_2(n) = O(g_1(n)g_2(n))$. Same for Big-Omega and Big-Theta.*

Example 7.28 In Example 7.20, we showed that

$$n^2 + n = O(n^2) \quad \text{and} \quad n^{\frac{7}{2}} + n^3 \log n = O(n^4).$$

Applying Property 7.4, we conclude that

$$n^{\frac{7}{2}} + n^3 \log n + n^2 + n = O\left(n^4 + n^2\right) = O(n^4).$$

Applying Property 7.5, we conclude that

$$\left(n^{\frac{7}{2}} + n^3 \log n\right)(n^2 + n) = O\left(\left(n^4\right)\left(n^2\right)\right) = O(n^6).$$

∎

We also leave the proofs of Properties 7.6 and 7.7 as exercises for the reader.

Property 7.6 (Symmetry) *We have $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$. We also have $f(n) = O(g(n))$ iff $g(n) = \Omega(f(n))$.*

Property 7.7 (Reflexivity) *$f(n) = O(f(n))$. Same for Big-Omega and Big-Theta.*

Property 7.8 *If $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then $f(n) = \Theta(g(n))$.*

Proof Let $f(n) = O(g(n))$, then there are positive constants c_1 and n_1 such that

$$f(n) \leq c_1 g(n), \forall n \geq n_1. \tag{7.6}$$

Let also $f(n) = O(g(n))$, then there are positive constants c_2 and n_2 such that

$$g(n) \leq c_2 f(n), \forall n \geq n_2. \tag{7.7}$$

Combining (7.6) and (7.7), we obtain

$$\frac{1}{c_2} g(n) \leq f(n) \leq c_1 g(n), \forall n \geq n_0,$$

where $n_0 = \max\{n_1, n_2\}$. The proof is complete. ■

The following remark follows immediately from Properties 7.2, 7.6 and 7.7.

Remark 7.2 *The asymptotic notation Θ defines an equivalence relation over the set of nonnegative functions on $\mathbb{N}$.*

From Theorem 2.1, Θ defines a partition on these functions, with two functions being in the same partition (or the same equivalence class) if and only if they have the same growth rate. For instance, the functions $n^2, n^2 + \log n$ and $3n^2 + n + 1$ are all $\Theta(n^2)$. So, they all belong to the same equivalence class.

The notations in terms of limits

In this part, we characterize Big-Oh, Big-Omega and Big-Theta notations in terms of limits. The limits are used as another technique to prove asymptotic statements, which is often much easier than that of definitions. It is not hard to prove the following theorem.

Theorem 7.1 (Limit characterization of notations) *Let $f(n)$ and $g(n)$ be two asymptotically nonnegative functions on $\mathbb{R}$ such that $\lim_{n\to\infty} \dfrac{f(n)}{g(n)} = L$. Then*

(i) *If $L = 0$, then $f(n) = O(g(n))$.*

(ii) *If $L = \infty$, then $f(n) = \Omega(g(n))$.*

(iii) *If $0 < L < \infty$, then $f(n) = \Theta(g(n))$.*

If the limit in Theorem 7.1 does not exist, we need to resort using definitions or some other technique.

Example 7.29 Use limits to prove the following asymptotic statements.

(a) $n^2 = O(n^3)$.

(b) $n^2 = \Omega(n)$.

(c) $7n^2 = \Theta(n^2)$.

(d) $n^4 - 14n^3 + 8n^2 + 27n - 9 = \Theta(n^4)$.

(e) $\frac{n(n+1)}{3} = O(n^3)$.

(f) $\log n = O(n)$.

(g) $n^3 = O(2^n)$.

(h) $\sqrt{7n^2 - 3n + 2} = \Theta(n)$.

Solution We simply apply Theorem 7.1.

(a) $\lim_{n\to\infty} \dfrac{n^2}{n^3} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. By Theorem 7.1(*i*), we have $n^2 = O(n^3)$.

(b) $\lim_{n\to\infty} \dfrac{n^2}{n} = \lim\limits_{n\to\infty} n = \infty$. By Theorem 7.1(*ii*), we have $n^2 = \Omega(n)$.

(c) $\lim_{n\to\infty} \dfrac{7n^2}{n^2} = \lim\limits_{n\to\infty} 7 = 7 > 0$. By Theorem 7.1(*iii*), we have $7n^2 = \Theta(n^2)$.

(*d*) We have

$$\begin{aligned}\lim_{n\to\infty}\frac{n^4-14n^3+8n^2+27n-9}{n^4} &= \lim_{n\to\infty}\left(1-\frac{14}{n}+\frac{8}{n^2}+\frac{27}{n^3}-\frac{9}{n^4}\right)\\ &= 1-0+0+0-0 \;=\; 1.\end{aligned}$$

Thus, by Theorem 7.1(*iii*), we have $n^4-14n^3+8n^2+27n-9=\Theta(n^4)$.

(*e*) We have

$$\lim_{n\to\infty}\frac{n(n+1)/3}{n^3}=\lim_{n\to\infty}\frac{n^2+n}{3n^3}=\lim_{n\to\infty}\frac{2n+1}{9n^2}=\lim_{n\to\infty}\frac{2}{18n}=0,$$

where we used L'Hospital's rule twice to find the limit. Therefore, by Theorem 7.1(*i*), we have $\frac{n(n+1)}{3}=O(n^3)$.

(*f*) We have

$$\lim_{n\to\infty}\frac{\log n}{n}=\lim_{n\to\infty}\frac{1/n}{1}=\lim_{n\to\infty}\frac{1}{n}=0,$$

where we used L'Hospital's rule to find the limit. Therefore, by Theorem 7.1(*i*), we have $\log n=O(n)$.

(*g*) We have

$$\lim_{n\to\infty}\frac{n^3}{2^n}=\lim_{n\to\infty}\frac{3n^2}{2^n\ln 2}=\lim_{n\to\infty}\frac{6n}{2^n(\ln 2)^2}=\lim_{n\to\infty}\frac{6}{2^n(\ln 2)^3}=0,$$

where we used the L'Hospital's rule three times to find the limit. Therefore, by Theorem 7.1(*i*), we have $n^3=O(2^n)$.

(*h*) We have

$$\lim_{n\to\infty}\frac{\sqrt{7n^2-3n+2}}{n}=\lim_{n\to\infty}\sqrt{\frac{7n^2-3n+2}{n^2}}=\sqrt{\lim_{n\to\infty}\left(7-\frac{3}{n}+\frac{2}{n^2}\right)}=\sqrt{7}.$$

Thus, by Theorem 7.1(*iii*), we have $\sqrt{7n^2-3n+2}=\Theta(n)$.

The desired proofs are obtained. ■

Complexity classification of algorithms

In this part, we give the computational complexity classification of algorithms based on these notations. First, we discuss the properties that we should consider to choose the Big-Oh of algorithms.

Choosing Big-Oh for algorithms For any algorithm, our Big-Oh choice will have two major properties: simplicity and tightness.

- Simplicity: Our choice of a Big-Oh bound is simplicity in the expression of the function. We have the following definition.

Definition 7.6 *Let $g(n)$ be a Big-Oh bound on $f(n)$, i.e., $f(n) = O(g(n))$. The function $g(n)$ is said to be simple if the following two conditions hold:*

(*i*) *It is a single term.* (*ii*) *The coefficient of that term is one.*

The following example illustrates Definition 7.6.

Example 7.30 In Example 7.9, we showed that the running time of the code in Algorithm 7.15 is

$$f(n) = \bar{c}n^2 + \hat{c}n + \tilde{c},$$

where $\bar{c} = c_2 + c_3, \hat{c} = 2c_2, \tilde{c} = c_1 + c_2$ and c_i is the execution cost of line (i) in Algorithm 7.15 for $i = 1, \ldots, 4$.

Define

$$g_1(n) = n^2, \; g_2(n) = \bar{c}n^2, \; \text{ and } \; g_3(n) = \bar{c}n^2 + \hat{c}n.$$

Then, we can show that $f(n) = O(g_i(n))$ for each $i = 1, 2, 3$. Note that g_1 is simple, but g_2 and g_3 are not simple functions. So, choosing $g_1(n)$ as a Big-Oh bound on $f(n)$ is better than choosing $g_2(n)$ and $g_3(n)$. ■

- Tightness: We generally want the "tightest" Big-Oh upper bound we can prove. We have the following definition.

Definition 7.7 *Let $g(n)$ be a Big-Oh bound on $f(n)$, i.e., $f(n) = O(g(n))$. The function $g(n)$ is said to be a tight bound on $f(n)$ if the following condition holds:*

$$\big(\exists h(n), f(n) = O(h(n))\big) \longrightarrow \big(g(n) = O(h(n))\big). \tag{7.8}$$

The implication in (7.8) means that if we can find a function $h(n)$ that satisfies the statement that $f(n) = O(h(n))$, then the statement $g(n) = O(h(n))$ is also satisfied. In other words, according to Definition 7.7, $g(n)$ be a tight Big-Oh bound on $f(n)$ if $g(n)$ is a Big-Oh bound on $f(n)$ and we cannot find a function that grows at least as fast as $f(n)$ but grows slower than $g(n)$. The following example illustrates Definition 7.7.

Example 7.31 (Example 7.30 revisited) In Example 7.30, we found that the function n^2 is a simple Big-Oh bound on $f(n) = \bar{c}n^2 + \hat{c}n + \tilde{c}$. In this example, we use Definition 7.7 to show that n^2 is a tight bound on $f(n)$, while n^3 is not a tight bound on $f(n)$.

To show that n^2 is a tight bound on $f(n)$, suppose that there exists a function $h(n)$ such that $f(n) = O(h(n))$. Then there are positive constants c and n_0 such that $f(n) = \bar{c}n^2 + \hat{c}n + \tilde{c} \leq ch(n)$ for all $n \geq n_0$. Then $h(n) \geq (\frac{\bar{c}}{c})n^2$ for all $n \geq n_0$, which in turn implies that $n^2 \geq (\frac{c}{\bar{c}})h(n)$ for all $n \geq n_0$. This means that $n^2 = O(h(n))$. According to Definition 7.7, we conclude that n^2 is a tight Big-Oh bound on $f(n)$.

To see why the function n^3 is not bound on $f(n)$, we pick $h(n) = n^2$. We have see that $f(n) = O(h(n))$, but it is clear that n^3 is not $O(h(n))$. ■

One can prove the following theorem.

Theorem 7.2 *If $f(n) = \Theta(g(n))$, then $g(n)$ is a tight Big-Oh bound on $f(n)$.*

Example 7.32 In Example 7.25, we proved that $\frac{1}{4}n^2 + 4n = \Theta(n^2)$. By Theorem 7.2, we conclude that n^2 is a tight Big-Oh bound on $\frac{1}{4}n^2 + 4n$. ■

Based on the asymptotic notations introduced in this section, we can now express the worst-, average- and best-case time complexities (i.e., running times) of algorithms using Big-Oh, Big-Theta and Big-Omega notations. In the following chapters, we will analyze different programs by expressing their running times using the asymptotic notations.

Classification of algorithms based on the notations Time complexity in algorithm analysis represents the overall time needed for a program to execute until its completion. Deriving an exact formula for the total runtime function, denoted as $f(n)$, can be a complex or even impractical endeavor. Often, we can significantly simplify this task by employing Big-Oh notation. In essence, algorithm time complexities are typically denoted using the Big-Oh notation, expressed as $O(g(n))$, which serves as an upper bound on $f(n)$. These complexities are categorized based on the type of function found in the asymptotic notation. In this part, we classify the algorithms according to this.

Definition 7.8 *Algorithms with running time:*

(*i*) $\Theta(n)$ *are said to have linear complexity and they are called linear-time algorithms.*

(*ii*) $\Theta(n^2)$ *are said to have quadratic complexity and they are called quadratic-time algorithms.*

(*iii*) $\Theta(n^k)$, *for some constant k, are said to have polynomial complexity and they are called polynomial-time algorithms.*

Note that, as n doubles, the run time doubles in the linear-time algorithms and quadruples in the quadratics-time algorithms. In Example 7.13, we found that, when the best-case analysis is considered, the algorithm has linear-time, while when the worst-case analysis is considered, the algorithm has linearithmic-time. Note also that linear and quadratic-time algorithms are special cases of polynomial-time algorithms. The following definition identifies precisely the efficiency of an algorithm.

Definition 7.9 *An algorithm is called efficient if it runs in polynomial time.*

Therefore, when we assert the existence of an efficient algorithm to address a problem, we usually refer to a polynomial-time algorithm. In Table 7.1, we list some of the more common running times for programs and their names.

The set of all decision problems that can be solved with worst-case polynomial time-complexity is said to be the complexity class P. There are other complexity classes to be introduced. One of the most important among them is so-called NP-complete class, which will be introduced in Section 7.7.

Running time	Name
$O(1)$	Constant time
$O(\log n)$	Logarithmic time
$O(\log^k n)$	Polylogarithmic time
$O(n)$	Linear time
$O(n \log n)$	Linearithmic time
$O(n^2)$	Quadratic time
$O(n^3)$	Cubic time
$O(n^k)$	Polynomial time
$O(2^n)$	Exponential time
$O(n!)$	Factorial time
$O(n^n)$	Super-exponential time

Table 7.1: Some common time complexities.

7.4 Analyzing decision making statements

In this section, we analyze and identify the time complexity of the decision making statements given in Section 7.1. We consider only basic statements such as if-statement, for-statement, while-statement and others.

Simple statements: The time complexity for a simple statement, which includes assignments, reads, writes, or jump statements, is $O(1)$.

If-statement: Upon initial examination of Figure 7.8, it becomes evident that an upper limit on the running time of an if-statement can be expressed as "1+" factor corresponds to the testing process. Since both $f_1(n)$ and $f_2(n)$ are positive for all values of n, we can omit the "1+" and conclude that the running time of the if-statement is $O(\max f_1(n), f_2(n))$, as presented in Figure 7.8.

For-statement: Upon initial examination of Figure 7.9, it becomes evident that an upper limit on the running time of a for loop can be expressed as $O(1 + (f(n) + 1)g(n))$. The factor $f(n) + 1$ signifies the cost of going around once, encompassing the body, the test, and the reinitialization. The "1+" at the beginning accounts for the initial initialization and the possibility of the first test being negative, resulting in zero iterations of the loop. In the common scenario where both $f(n)$ and $g(n)$ are positive, the running time of the for-statement simplifies to $O(g(n)f(n))$, as outlined in Figure 7.9.

While-statement: Upon initial examination of Figure 7.10, it becomes evident that an upper limit on the running time of a while loop can be expressed as $O(1+(f(n)+1)g(n))$. The term $O(f(n) + 1)$ serves as an upper bound on the running time of both the body and the test after the body. Additionally, the "1" at the beginning of the formula accounts for the test that occurs before entering the loop. In the typical scenario where both $f(n)$ and $g(n)$ are positive, the running time of the while statement simplifies to $O(g(n)f(n))$, as outlined in Figure 7.10.

Do-while-statement: After a first look at Figure 7.11, it becomes evident that an upper limit on the running time of a do-while loop can be expressed as $O((f(n)+1)g(n))$. The "+1" term signifies the time required to compute and test the condition at the end of each iteration of the loop. It is worth noting that in this context, $g(n)$ is always positive. In the common case, where $f(n)$ is positive for all values of n, the running time of the do-while statement simplifies to $O(g(n)f(n))$, as indicated in Figure 7.11.

Block: A first look at Figure 7.12 indicates that an upper bound on the running time of a lock is $O\big(f(n)+\cdots+f_k(n)\big)$. Using the Big-Oh rules, this can be simplified as the one stated in Figure 7.12.

In summary, below are the major points we should take from above:

- The bound for a simple statement is $O(1)$.

- The running time of a selection statement is the time required to make a decision regarding which branch to pursue, in addition to the greater of the running times of the available branches.

- The running time of a loop is determined by calculating the time it takes to execute the loop's body, including any control steps like reinitializing the loop index and comparing it to the limit. This running time is then multiplied by an upper bound on the number of iterations the loop can undergo. Additionally, any actions performed just once, such as initialization or the initial termination test (if the loop may not iterate at all), are added to this total if the loop could potentially have zero iterations.

- The running time of a sequence of statements can be calculated by summing the running times of each individual statement. Frequently, one of these statements will have a significantly greater running time compared to the others, and according to the summation rule, the overall running time of the sequence is essentially determined by the dominant statement.

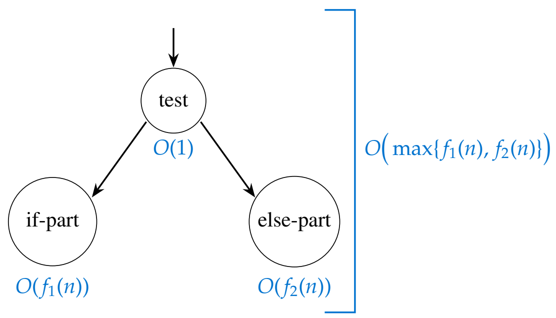


Figure 7.8: The running time complexity of an if-statement.

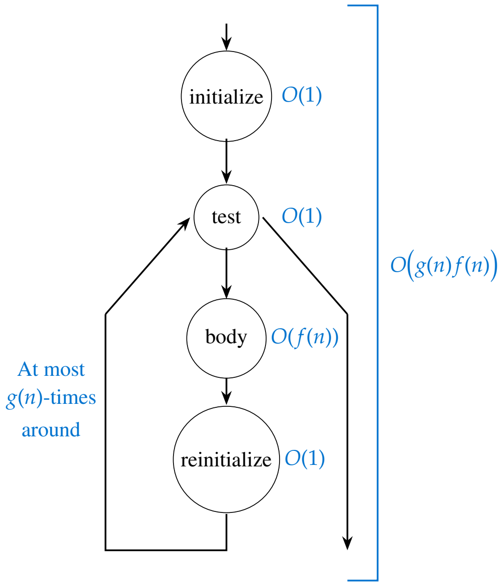


Figure 7.9: The running time complexity of a for-statement.

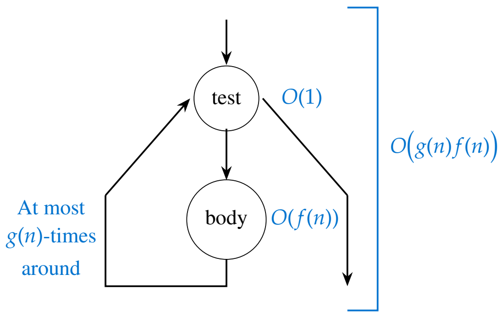


Figure 7.10: The running time complexity of a while statement.

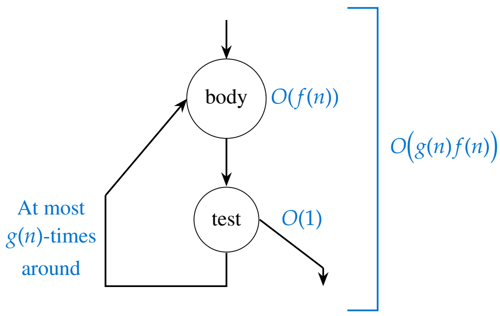


Figure 7.11: The running time complexity of a do-while statement.

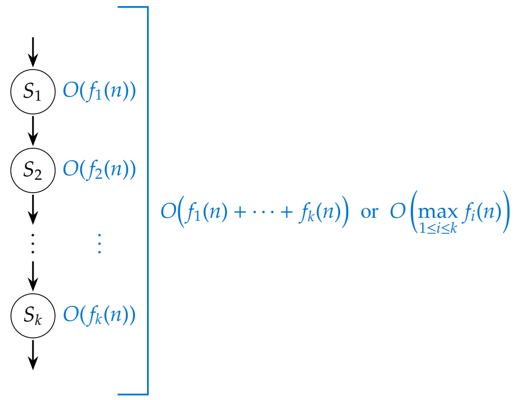


Figure 7.12: The running time complexity of a block.

7.5 Analyzing programs without function calls

In this section, we analyze the running time of programs that do not contain function calls (other than library functions such as "printf"), leaving the matter of analyzing programs with function calls to the next section.

We use a graph- or tree-based running time analysis to derive Big-Oh upper bounds on the running time of programs without function calls (also known as sequential programs).

A graph structure, or a flowchart, is a graphical representation of the separate steps of a program in sequential order. A tree structure is a graph structure where loops are clustered as nodes instead of cycles. In this section, we create and use graph and tree structures to analyze and identify the running time of sequential programs (i.e., programs without function calls). We have the following examples.

Example 7.33 Draw the graph structure for the code in Algorithm 7.23. Give a Big-Oh upper bound on the running time of each compound statement in Algorithm 7.23, as a function of n. What is the total time it takes for the function to run?

Algorithm 7.23: The algorithm of Example 7.33

$i = n$
while $(i < n^2)$ **do**
    $i = i + \sqrt{n}$
end
if $(i < 3n)$ **then**
    `printf`("The exam is multiple choice.")
end
else
    `printf`("The exam is essay.")
end

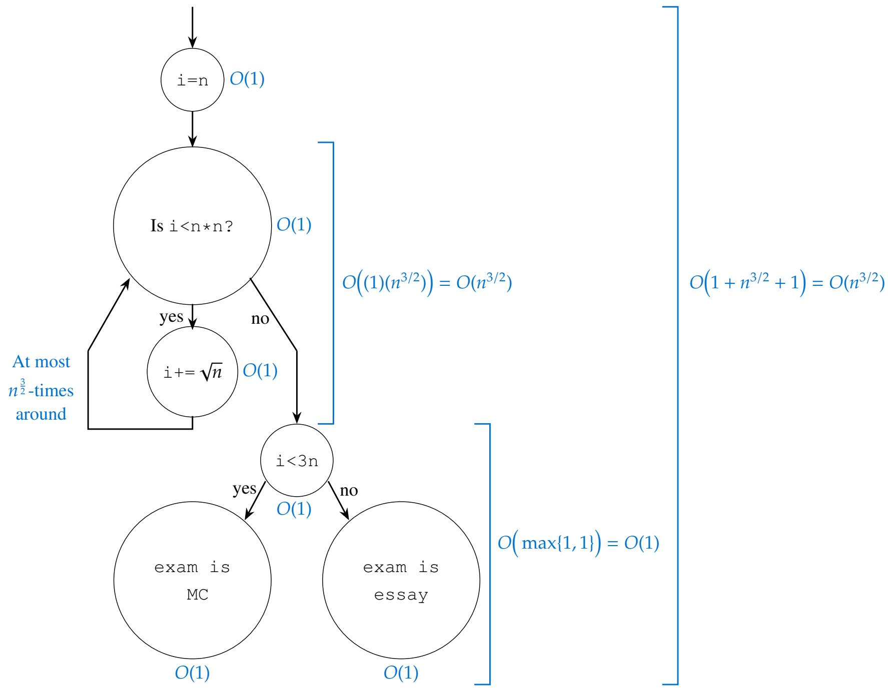


Figure 7.13: The graph structure with time complexity for the code in Algorithm 7.23.

Solution The graph structure is shown in Figure 7.13. The while loop of lines (2) and (3) in Algorithm 7.23 may be executed as many as $n^{3/2}$-times, but no more. To see this, note that the initial value of i is n, and that in each iteration we increment i by $\sqrt{n}$. We keep executing the while loop until i is no longer smaller than n^2. Therefore, the number of times we go around the while loop is k where k is the largest positive integer so that

$$n + \underbrace{\sqrt{n} + \sqrt{n} + \cdots + \sqrt{n}}_{k-\text{time}} = n + k\sqrt{n} < n^2.$$

Hence $k < (n^2 - n)/\sqrt{n} = O(n^{3/2})$. Now, according to the summation rule, we can determine that the running time of Algorithm 7.23 is bounded by $O(n^{3/2})$. This bound is established by considering the maximum running time among various parts of the algorithm, including the assignment in line (1), the while loop in lines (2) and (3), and the selection statement in lines (5) through (10). The running time found in this example is the worst-case performance of the code in Algorithm 7.23. ■

Example 7.34 The function in this example determines whether a given integer `n` is prime or not prime. A positive integer `n` is said to be prime if it is divisible by only 1 and itself. For example, 7 is a prime number. So, if `n` is not a prime, then it is divisible evenly by some integer `i` between 2 and $\sqrt{\texttt{n}}$. The primality test function is shown in Algorithm 7.24.

Draw the tree structure for this algorithm. What is the upper bound on the running time, expressed using Big-Oh notation, for each compound statement within the algorithm, in terms of n? Additionally, what is the overall running time of the entire function?

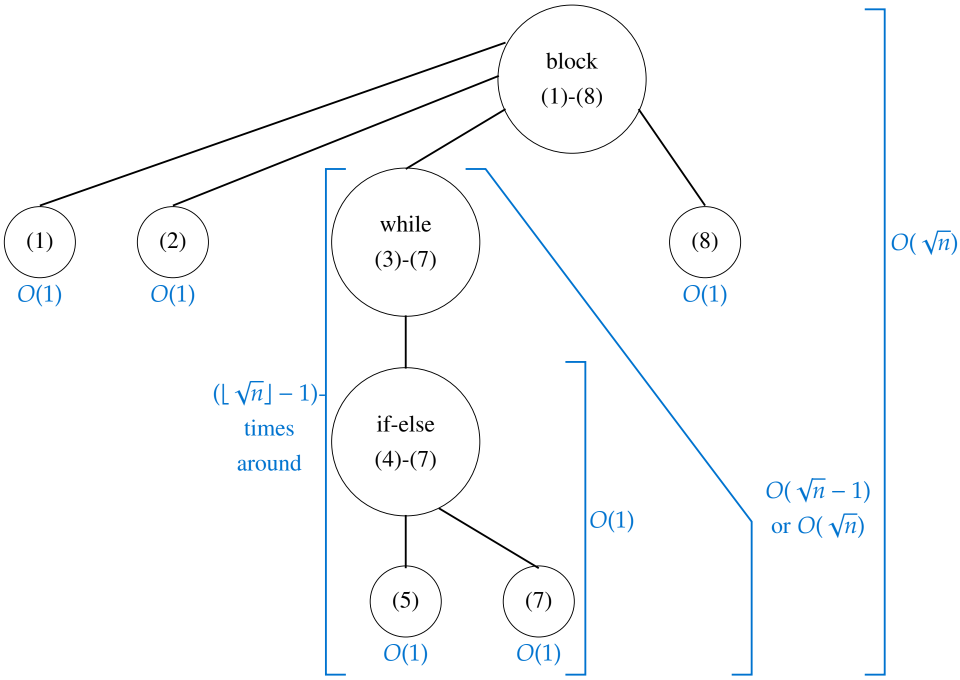


Figure 7.14: The tree structure for Algorithm 7.24 with running time complexity.

Algorithm 7.24: Primality test function: `prime`(int n)

Input: A positive integer n
Output: A TRUE/FALSE answer to Question "Is n a prime number?"

int i
$i = 2$
while $(i^2 \leq n)$ **do**
  if $(n\%i == 0)$ **then**
    return FALSE
  end
  else
    $i++$
  end
end
return TRUE

Solution The illustration of the tree structure can be observed in Figure 7.14. As delineated within Figure 7.14, the time taken by the if-else statement amounts to $O(1)$, thereby establishing the runtime of the while-statement as $O(\sqrt{n})$. Note that each of the remaining lines constitutes singular statements and consequently operates in $O(1)$ time. This realization culminates in an understanding that the collective runtime performance of the entire function converges and is encapsulated by the notation $O(\sqrt{n})$. ■

Example 7.35 Algorithm 7.25 is an example of the so-called interior-point algorithm and is used for solving linear (and nonlinear) programs.[2]

(*a*) No matter how the algorithm works and how the mathematical notations therein look, the graph structure (flowchart) of the algorithm can be visualized. Draw the graph structure for Algorithm 7.25.

(*b*) Let N_{in} be the number of times we go around the inner while loop of line (8). In view of the graph structure of the algorithm drawn in item (*a*), determine a good asymptotic upper bound on the number of iterations needed to obtain a desired solution using Algorithm 7.25 in each of the following cases.

(*i*) $N_{\text{in}} = O(nm)$ and $\gamma \in (0,1)$ is an arbitrarily chosen constant.

(*ii*) $N_{\text{in}} = O(1)$ and $\gamma = 1 - \sigma/\sqrt{nm}$, where $\sigma > 0$.

In each case, give your answer as a function of $n, m, \mu^{(0)}, \epsilon$ using Big-oh, and justify your determination.

Algorithm 7.25: Interior-point linear optimization algorithm

Input: Linear program with an $n \times m$ matrix A and a barrier function f
Output: An ϵ-optimal solution to the linear program

1: initialize $\epsilon > 0, \gamma \in (0,1), \theta > 0, \beta > 0, \boldsymbol{x}^{(0)}, \mu^{(0)} > 0, \boldsymbol{\lambda}^{(0)}$
2: set $\boldsymbol{x} \triangleq \boldsymbol{x}^{(0)}, \mu \triangleq \mu^{(0)}, \boldsymbol{\lambda} = \boldsymbol{\lambda}^{(0)}$
3: **while** $(\mu \geq \epsilon)$ **do**
4:     compute $\boldsymbol{g} = \nabla f(\mu, \boldsymbol{x}) - A^\mathsf{T}\boldsymbol{\lambda}$
5:     compute$\Delta \boldsymbol{x} = -\left((\nabla^2 f)^{-1} - (\nabla^2 f)^{-1} A^\mathsf{T}\left(A\,(\nabla^2 f)^{-1} A^\mathsf{T}\right)^{-1} A\,(\nabla^2 f)^{-1}\right)(\boldsymbol{g})$
6:     compute $\Delta\boldsymbol{\lambda} = \left(A\left(\nabla^2 f(\mu,\boldsymbol{x})\right)^{-1} A^\mathsf{T}\right)^{-1} A\left(\nabla^2 f(\mu,\boldsymbol{x})\right)^{-1}(\boldsymbol{g})$
7:     compute $\delta(\mu,\boldsymbol{x}) = \sqrt{\frac{1}{\mu}(\Delta\boldsymbol{x})^\mathsf{T}\nabla^2 f(\mu,\boldsymbol{x})(\Delta\boldsymbol{x})}$
8:     **while** $(\delta > \beta)$ **do**
9:         set $\boldsymbol{x} \triangleq \boldsymbol{x} + \theta\Delta\boldsymbol{x}$
10:         set $\boldsymbol{\lambda} \triangleq \boldsymbol{\lambda} + \theta\Delta\boldsymbol{\lambda}$
11:         compute $\boldsymbol{g} = \nabla f(\mu, \boldsymbol{x}) - A^\mathsf{T}\boldsymbol{\lambda}$
12:         compute $\Delta \boldsymbol{x} = -\left(\left(\nabla^2 f\right)^{-1} - \left(\nabla^2 f\right)^{-1} A^\mathsf{T}\left(A\left(\nabla^2 f\right)^{-1} A^\mathsf{T}\right)^{-1} A\left(\nabla^2 f\right)^{-1}\right)(\boldsymbol{g})$
13:         compute $\Delta\boldsymbol{\lambda} = \left(A\left(\nabla^2 f(\mu,\boldsymbol{x})\right)^{-1} A^\mathsf{T}\right)^{-1} A\left(\nabla^2 f(\mu,\boldsymbol{x})\right)^{-1}(\boldsymbol{g})$
14:         compute $\delta(\mu,\boldsymbol{x}) = \sqrt{\frac{1}{\mu}(\Delta\boldsymbol{x})^\mathsf{T}\nabla^2 f(\mu,\boldsymbol{x})(\Delta\boldsymbol{x})}$
15:     **end**
16:     set $\mu \triangleq \gamma\mu$
17: **end**

---

[2]Linear programming (also-called linear optimization) is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships (see Chapter 10 for more detail).

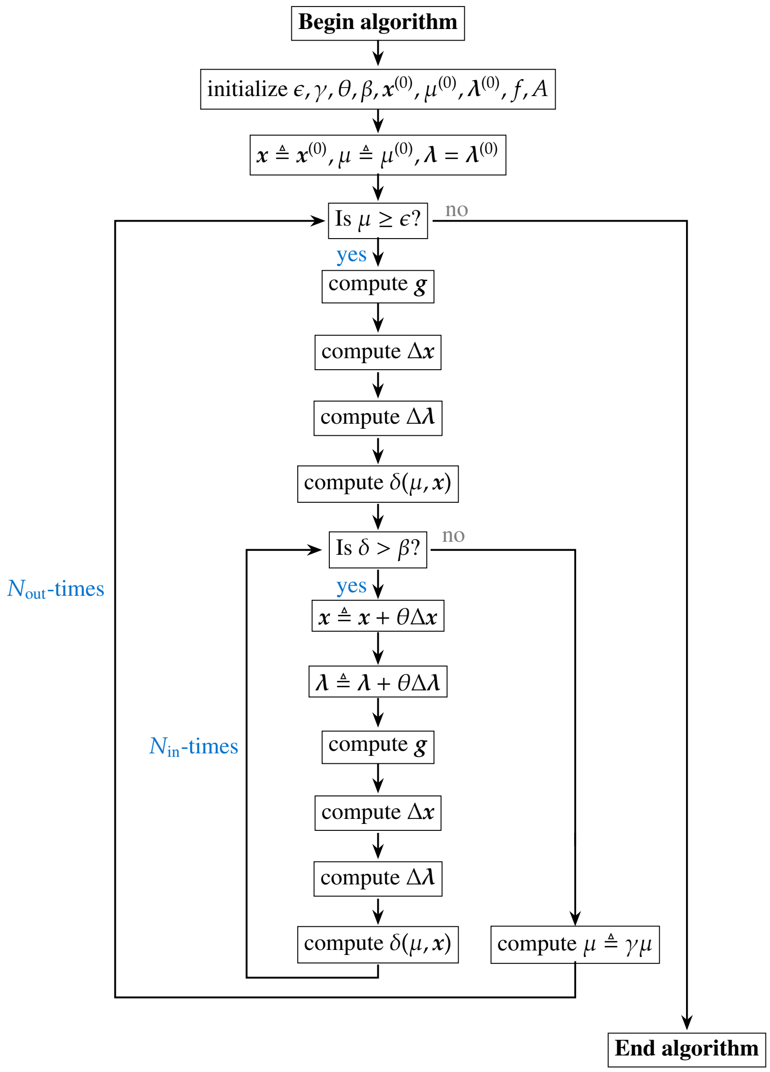


Figure 7.15: The graph structure of Algorithm 7.25.

Solution (*a*) The graph structure of Algorithm 7.25 is shown in Figure 7.15.

(*b*) Let N_{out} (respectively, N_{in}) be the number of times we go around the outer (respectively, inner) while loop, and N be the number of iterations needed to obtain a desired solution using Algorithm 7.25. Then, in view of the graph structure of the algorithm drawn in item (*a*), we have

$$N = N_{\text{in}}\, N_{\text{out}}.$$

We now estimate N_{out}. Let $\mu^{(i)}$ be the parameter at the i^{th} iteration. Then we have

$$\mu^{(i)} = \gamma\mu^{(i-1)} = \gamma^2\mu^{(i-2)} = \cdots = \gamma^i\mu^{(0)}.$$

Thus $\mu^{(i)}$ will be less than the given $\epsilon > 0$ if $\gamma^i \mu^{(0)} < \epsilon$. Note that

$$\gamma^i \mu^{(0)} < \epsilon \iff \gamma^i < \epsilon/\mu^{(0)} \iff i \log\gamma = \log(\gamma^i) < \log(\epsilon/\mu^{(0)}).$$

Since $\gamma \in (0, 1)$, we have

$$i > \frac{\log(\epsilon/\mu^{(0)})}{\log\gamma} = \frac{\log(\mu^{(0)}/\epsilon)}{-\log\gamma}.$$

Hence,

$$N_{\text{out}} \leq \frac{\log(\mu^{(0)}/\epsilon)}{-\log\gamma}.$$

(i) It is given that $N_{\text{in}} = O(nm)$. Now, if $\gamma \in (0, 1)$ is an arbitrarily chosen constant, then $\gamma = O(1)$, and hence

$$N_{\text{out}} \leq \log\left(\frac{\mu^{(0)}}{\epsilon}\right) O(1).$$

Thus, the number of iterations needed to obtain a desired solution is

$$N = N_{\text{in}} N_{\text{out}} = O\left(nm \log\left(\frac{\mu^{(0)}}{\epsilon}\right)\right).$$

(ii) It is given that $N_{\text{in}} = O(1)$. Now, if $\gamma = 1 - \sigma/\sqrt{nm}$ $(\sigma > 0)$, then[3]

$$\log\gamma = \log\left(1 - \sigma/\sqrt{nm}\right) \approx -\sigma/\sqrt{nm},$$

and hence

$$N_{\text{out}} \leq \frac{\log(\mu^{(0)}/\epsilon)}{-\log\gamma} \approx \frac{\log(\mu^{(0)}/\epsilon)}{\sigma/\sqrt{nm}} = \sqrt{nm}\log\left(\frac{\mu^{(0)}}{\epsilon}\right) O(1).$$

Thus, the number of iterations needed to obtain a desired solution is

$$N = N_{\text{in}} N_{\text{out}} = O\left(\sqrt{nm}\log\left(\frac{\mu^{(0)}}{\epsilon}\right)\right).$$

■

In Example 7.35, having a good asymptotic bound on the number of iterations required to obtain a desired solution using Algorithm 7.15, we can find a good asymptotic time complexity of the algorithm if the running times of lines (1)-(17) in Algorithm 7.15 are provided.

Other examples of sequential programs are Linear Search and Selection Sort which will be introduced and analyzed in Chapter 8. More specifically, a graph structure will be drawn to analyze Linear Search, and a tree structure will be drawn to analyze Selection Sort.

[3]For small positive values of x, we have $\log(1 + x) \approx (1 + x) - 1 = x$.

7.6 Analyzing programs with function calls

In this section, we analyze the running time of programs or program fragments that contain function calls. The function call can be recursive or nonrecursive. We have the following definition.

> **Definition 7.10** *A function that calls itself recursively is called a recursive function. A function that does not call itself but calls other functions is called a nonrecursive function.*

We start with analyzing nonrecursive programs.

Analyzing nonrecursive programs

A nonrecursive program is the one that contains nonrecursive functions. The following example analyzes a nonrecursive program that contains a nonrecursive function.

Example 7.36 Consider the fragment shown in Algorithm 7.26. Here `f(n)` is a function call. Give a simple and tight Big-Oh upper bound on the running time of Algorithm 7.26, as a function of `n`, on the assumption that:

(a) The running time of `f(n)` is $O(1)$, and the value of `f(n)` is `0`.

(b) The running time of `f(n)` is $O(\mathrm{n}^2)$, and the value of `f(n)` is `n`.

Solution The first statement is an assignment statement with constant time. The inside of the for loop is also a simple statement with constant time. So, the part that will affect overall runtime is the runtime and value of $f(n)$.

In item (a), the value of $f(n)$ is 0, so the loop will never execute the body. It will only evaluate the condition once. Since the runtime of $f(n)$ (and thus the runtime of checking the condition) is $O(1)$, the overall runtime is $O(1)$.

For item (b), Figure 7.9 shows how the runtime of for loops is calculated. We add the initialization runtime (usually $O(1)$) plus the cost of going around the loop once multiplied by the number of times we go around the loop, represented $O(1 + (f(n) + 1)g(n))$. We keep in mind “the cost of going around the loop once" is represented in Figure 7.9 as the runtimes of “test” plus “body” plus “reinitialize”, with “test” being the condition of the for loop and “body” obviously being the body. Now, we can apply this to our problem. The“test” runtime in the for loop is $O(n^2)$, since that is how long it takes $f(n)$ to run. The “body” of the for loop takes $O(1)$, as it is just a simple statement. The “reinitialization” runtime still takes $O(1)$. We perform the loop $O(n)$ times, since the value of $f(n)$ is n. Thus, putting it all together, we have $O(1 + (n^2 + 1 + 1)n) = O(n^3)$. ■

Algorithm 7.26: The algorithm of Example 7.36

```
sum = 0
for (i = 1; i ≤ f(n); i++) do
|   sum += i
end
```

When determining the time complexity or running time of a program that comprises nonrecursive functions, a systematic process is employed. This process involves a sequential set of steps designed to ascertain the individual running times of these functions. The procedure given in the following workflow, followed by Example 7.37, will teach us to determine the runtime of a program consisting of nonrecursive functions.

Workflow 7.1 *There are three steps involved in the process of determining the runtime of a program consisting of nonrecursive functions:*

(*i*) *Calculate the running times of functions that do not call any other functions.*

(*ii*) *Calculate the running times of functions that call only functions for which you have already determined the running times.*

(*iii*) *Proceed in this manner until you have computed the running time for all functions.*

To further clarify the steps provided in Workflow 7.1, we present the following example, which is a restatement of Example 3.23 in Aho and Ullman [1994].

Example 7.37 Calculate the time complexity of the program in Algorithm 7.27.

Algorithm 7.27: The algorithm of Example 7.37

```
# include <stdio. h >
int cat(int x, int n)
int cow(int x, int n)
// This line is left intentionally blank
main( )
int a, n
scanf("%d", & n)
a=cow(0, n);
printf("%d n", cat(a, n))
// This line is left intentionally blank
int cat(int x, int n)
int i
for (i = 1; i ≤ n; i + +) do
|   x+ = i;
end
return x
// This line is left intentionally blank
int cow(int x, int n)
int i
for (i = 1; i ≤ n; i + +) do
|   x+ = cat(i, n);
end
return x
```

The function `main` calls both the functions `cow` and `cat`, the function `cow` calls the function `cat`, and the function `cat` does not call any function. We first analyze the function `cat`, then to analyze the function `cow`, and finally to analyze the function `main`.

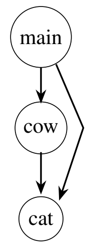


Figure 7.16: A basic scheme of the program shown in Algorithm 7.27.

Solution Note that this program is nonrecursive. Figure 7.16 provides a basic overview of the program outlined in Algorithm 7.27. Considering Figure 7.16, we first analyze the function `cat`, which does not call any other function. Next, we analyze the function `cow`, which only calls the function `cat`. Finally, we analyze the function `main`, which calls both the functions `cow` and `cat`.

It is evident that the function `cat` has a time complexity of $O(n)$. Now, if a function call is within the body of a for loop, we add its cost to the time complexity for each iteration. Consequently, the running time of a call to `cow` is $O(n^2)$. Furthermore, when the function call is within a simple statement, we add its cost to the cost of that statement. Thus, the function `main` takes $O(n^2)$ time. Consequently, the overall time complexity of this program is $O(n^2)$. ∎

Analyzing recursive programs

A recursive program is the one that contains recursive functions. Analyzing the running time of a function that employs recursive calls demands more effort than evaluating nonrecursive functions. Running time of a recursive program is represented by a recurrence.

Remember that a recurrence is a mathematical equation or inequality used to express a function's behavior based on its values for smaller inputs.

For example, the following recurrence

$$T(1) = 1, \quad T(n) = 3T(n-1) + 4, \quad n = 1, 2, 3, \ldots, \tag{7.9}$$

describes the function (see Exercise 5.2 (b))

$$T(n) = 3^n - 2, \quad n = 0, 1, 2, \ldots. \tag{7.10}$$

The following are some examples of recurrence formulas commonly found in recursive algorithms:

- A recursive algorithm that iteratively processes the input by removing one item at each step. For example:

$$T(n) = T(n-1) + 1, \; T(n) = T(n-1) + n, \text{ etc.}$$

- A recursive algorithm that halves the input size. For example:

$$T(n) = T(n/2) + 1, \; T(n) = T(n/2) + n, \text{ etc.}$$

- A recursive algorithm that divides the input into two equal halves. For example:

$$T(n) = 2T(n/2) + 1, \text{ etc.}$$

The question that naturally arises now is: how do we determine the actual running time of a recursive program? The answer to this question is by following three steps:

- First, derive a recurrence relation or formula from the given recursive program. This relation should express the time complexity of the problem size in terms of the time complexity of smaller instances of the same problem.

- Next, solve the recurrence formula to find an explicit expression of the recurrence. In this section, we present and use the so-called **iteration method** to solve recurrences. In the iteration method, we decompose the recurrence into a series of terms, and derive the nth expression from the previous ones. Other methods for solving recurrences will be the substance of the next section.

- Last, bound the recurrence explicit formula by an asymptotic expression that involves n. For instance, from (7.10), it is clear that the running time for the recursive program of the recurrence formula (7.9) is $O(3^n)$.

In the following examples, we introduce three recursive programs, derive their recurrence formulas, and analyze their running times.

Example 7.38 (Factorial) The recursive program in Algorithm 7.28 computes the factorial function $n!$. Assuming the cost times of the simple statements `return 1` in line (3) and `return n` in line (6) are c_1 and c_2, respectively, and observing that the function `fact(n-1)` that the program calls in line (6) is the same problem but with size $(n-1)$, we conclude that the cost time of executing line (6) is the constant time c_2 plus the time taken by the function `fact`$(n-1)$. Let $T(n)$ be the running time of Algorithm 7.28 on an integer n. It follows that the recurrence formula for this recursive program is

$$\begin{aligned} T(1) &= c_1, \\ T(n) &= c_2 + T(n-1), \quad n > 1. \end{aligned} \tag{7.11}$$

From Example 5.3, we have

$$T(n) = (n-1)c_2 + T(1) = (n-1)c_2 + c_1.$$

Algorithm 7.28: Factorial function: `fact`(int n)

Input: A positive integer n
Output: The factorial $n!$

```
1: fact(int n)
2: if (n ≤ 1) then
3:     return 1
4: end
5: else
6:     return n*fact(n − 1)
7: end
```

Thus, the recurrence formula (7.11) of Algorithm 7.28 describes the linear function $T(n) = c_2 n + (c_1 - c_2)$ for $n \geq 1$. As a result, the factorial function algorithm is linear, running in $O(n)$ time. ■

Example 7.39 (Integer Power) The recursive program in Algorithm 7.29 computes x^n. We assume that the cost time of the simple statement `return 1` in line (3) is c_1, and that the cost time of each of the simple statements `return p*p` in line (8) and `return x*p*p` in line (11) is c_2. Note that the function `pow(x,n/2)` that the program calls in line (7) is the same problem but with size n/2, and that the function `pow(x,(n-1)/2)` that the program calls in line (10) is the same problem but with size (n-1)/2. Let $T(n)$ be the running time of Algorithm 7.29 on an integer n, then the recurrence formula for this recursive program is

$$T(n) = \begin{cases} c_1, & \text{if } n = 1; \\ T(n/2) + c_2, & \text{if } n > 1 \text{ and is even}; \\ T((n-1)/2) + c_2, & \text{if } n > 1 \text{ and is odd}. \end{cases} \tag{7.12}$$

Since the function T is monotonically increasing, the recurrence formula (7.12) can be simplified to

$$T(n) = T(\lfloor n/2 \rfloor) + c_2 \ \text{ if } \ n > 1,$$

and hence

$$T(n) \leq T(n/2) + c_2.$$

The same method that was used in Example 5.4 can be used to show that

$$T(n) \leq kc_2 + T(n/2^k) \leq c_2 \log n + T(1) = c_2 \log n + c_1.$$

Thus, the recurrence formula (7.12) of Algorithm 7.29 describes the logarithmic function $T(n) = c_2 \log n + c_1$. As a result, the integer power algorithm is logarithmic, running in $O(\log n)$ time. ■

Algorithm 7.29: Integer power function: `pow`(int x,int n)

Input: Positive integers x and n

Output: The nth power of x

```
1:  pow(int x, int n)
2:  if (n == 0) then
3:  |   return 1
4:  end
5:  else
6:  |   if (n%2 == 0) then                         // Checking if n is even
7:  |   |   int p = pow(x, n/2)
8:  |   |   return p * p
9:  |   |   else
10: |   |   |   int p = pow(x, (n-1)/2)
11: |   |   |   return x * p * p
12: |   |   end
13: |   end
14: end
```

Other examples of recursive programs are Binary Search and Merge Sort which will be studied in Chapter 8. At the end of this section, we note that sometimes it is convenient to distinguish common recursive algorithms as in Table 7.2.

Divide and conquer algorithms $T(n) = aT(n/b) + f(n)$, $(a \geq 1$ and $b > 1)$	Running complexity time
$T(n) = T(n/2) + c$	$\Theta(\log n)$
$T(n) = T(n/3) + c$	$\Theta(\log n)$
$T(n) = T(n/2) + cn$	$\Theta(n)$
$T(n) = T(n/3) + cn$	$\Theta(n)$
$T(n) = 2T(n/2) + cn$	$\Theta(n \log n)$
$T(n) = 3T(n/3) + cn$	$\Theta(n \log n)$
$T(n) = 3T(n/2) + cn$	$\Theta(n^{\log 2(3)})$
$T(n) = 4T(n/2) + cn$	$\Theta(n^2)$
$T(n) = 2T(n/2) + cn^2$	$\Theta(n^2)$
$T(n) = 4T(n/2) + cn^2$	$\Theta(n^2 \log n)$

Chip and conquer algorithms $T(n) = T(n-a) + f(n)$, $(a \geq 1)$	Running complexity time
$T(n) = T(n-1) + c$	$\Theta(n)$
$T(n) = T(n-1) + cn$	$\Theta(n^2)$
$T(n) = T(n-1) + cn^2$	$\Theta(n^3)$

Exponential algorithms	Running complexity time
$T(n) = 2T(n-1) + f(n)$	$\Omega(2^n)$
$T(n) = 3T(n-1) + f(n)$	$\Omega(3^n)$
$T(n) = 4T(n-1) + f(n)$	$\Omega(4^n)$
$T(n) = 2T(n-2) + f(n)$	$\Omega(2^{n/2})$
$T(n) = 2T(n-3) + f(n)$	$\Omega(2^{n/3})$
$T(n) = T(n-1) + T(n-2) + f(n)$	$\Omega(2^{n/2})$
$T(n) = T(n-1) + T(n-2) + T(n-3) + f(n)$	$\Omega(2^{n/2})$
$T(n) = \sum_{i=1}^{n-1} T(i) + f(n)$	$\Omega(2^{n/2})$

Table 7.2: Some common recurrence formulas and their complexities.

7.7 The complexity class NP-complete

In 1971, Leonid Levin independently, along with Stephen Cook, made a significant discovery in the field of computational complexity theory. They introduced the concept of NP-complete problems by demonstrating that the satisfiability problem[4] falls into the category of NP-complete problems. NP-complete problems play a crucial role in computational complexity theory, specifically in the domain of decision problems. Therefore, in this section, we redirect our attention from "algorithms" to "problems". We have the following definitions.

Definition 7.11 *An abstract problem is a binary relation on a set of problem instances and a set of problem solutions.*

For instance, consider the shortest path problem[5]. In this problem, we are given an undirected graph, denoted as G, along with two specific vertices, say u and v, within that graph. The objective is to determine a path between u and v that employs the fewest possible edges. The problem itself represents a binary relation. It associates each instance consisting of a graph and two vertices with a shortest path within that graph, connecting the specified pair of vertices. It is important to note that shortest paths may not be unique, meaning that a given problem instance might have more than one valid solution.

Definition 7.12 *A decision problem is any problem to which the answer is simply yes or no (or, more formally, 1 or 0).*

Examples of decision problems are: Is this a satisfiable proposition? Is this a Hamiltonian graph? Is this an Eulerian graph?[6] etc.

Numerous problems fall into the category of optimization rather than decision problems. These optimization problems[7] involve identifying the optimal solution, which could be either the maximum or minimum value, from a set of feasible solutions. Feasible solutions are essentially sets of values for the decision variables that satisfy all the constraints laid out in the optimization problem. For instance, consider the shortest path problem, where the goal is to discover a feasible path with the fewest edges possible. Notably, NP-completeness pertains directly to decision problems, not optimization problems. Nevertheless, it is typically possible to reformulate an optimization problem as an equivalent decision problem by introducing a constraint on the optimized value. To illustrate, we can frame a decision problem associated with the shortest path problem as follows: Given an undirected graph G, two vertices u and v within G, and a positive integer k, does there exist a path between u and v that comprises at most k edges?

In order to introduce the NP-complete problems, we must first define the class NP. Before that, we define the following notions and terminology in the computational complexity theory. Recall that an efficient algorithm the one that can run in polynomial time. We also have the following definitions.

[4]The satisfiability problem was studied in Section 1.5.
[5]The shortest path problem will be studied in Section 9.3.
[6]Eulerian and Hamiltonian graphs were defined in Section 4.3.
[7]Optimization problems will be formally studied in Part IV.

Definition 7.13 *A problem is said to be tractable if there exists an efficient algorithm that solves all instances of it, and is intractable otherwise.*

Definition 7.14 *A certificate of a solution to a decision problem is a scheme that is used to check and certify whether a decision problem gives the answer yes or no.*

Definition 7.15 *A decision problem is called verifiable if a certificate of a solution to the problem exists.*

In particular, a decision problem is verifiable in polynomial time if for every instance with the answer yes, there is a solution S which we can use to check in polynomial time that the answer is yes.

The notion of reduction from one problem to another explains to us how problems relate.

Definition 7.16 *A reduction of a decision problem Q_1 to a decision problem Q_2 is a mapping of every instance q_1 of problem Q_1 to an instance q_2 of problem Q_2 such that q_1 is yes if and only q_2 is yes.*

A reduction provides us with a way to state, "If I can find a solution for problem Q_1, then I can also find a solution for problem Q_2". In such a scenario, if, for instance, Q_1 is solvable in polynomial time, it implies that Q_2 can also be solved within polynomial time. This insight helps us gauge the relative difficulty of problem Q_1. A notable property of reductions is their transitive nature. Specifically, if we can demonstrate that Q_1 reduces to Q_2 in polynomial time, and Q_2 reduces to Q_3 in polynomial time, then we can infer that Q_1 reduces to Q_3 in polynomial time.

Definition 7.17 *A problem is classified as P if it can be solved in polynomial time.*

All of the actual problems we have thus far solved in this book have been in P. Additionally, many problems in the book, including all of the problems we will study in Chapters 8-12, are in P.

Because P stands for polynomial, many people assume that NP must stand for "non-polynomial". This is incorrect. The set NP refers to "non-deterministic polynomial". We need some more definitions.

Definition 7.18 *A problem is classified as NP if it can be verified in polynomial time.*

Note that P $\subseteq$ NP. In fact, being able to solve a problem in polynomial ensures that we can also verify the problem by simply solving it. Figure 7.17 shows a Venn diagram of the different classes of problems.

Definition 7.19 *A problem is classified as NP-hard if any NP problem can be reduced to this problem in polynomial time.*

In other words, a problem is classified as NP-hard if an algorithm for solving it can be translated to solve any NP problem. In light of Definition 7.19, an NP-hard problem is at least as hard as any NP problem (it could be even much harder).

NP-complete problems are problems that reside in both the NP and NP-hard classes.

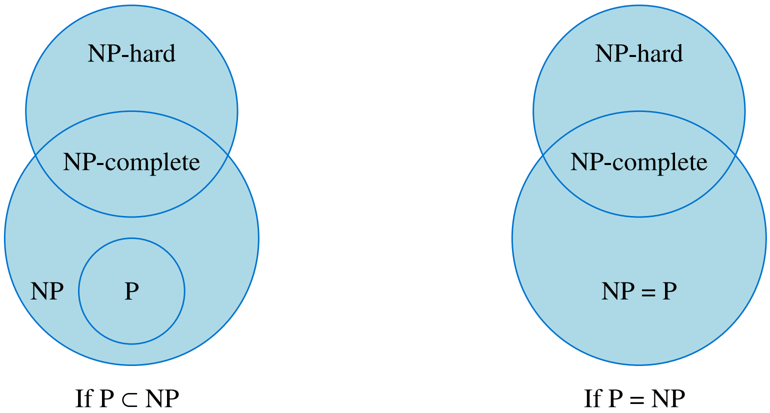


Figure 7.17: Graphical relationships among the complexity classes P, NP, NP-hard, and NP-complete.

Definition 7.20 *A problem is classified as NP-complete if it can be verified in polynomial time and that any NP problem can be reduced to this problem in polynomial time.*

Examples of NP-Complete problems include the Hamiltonian path problem, the Boolean satisfiability problem, and the integer programming problem.

It is a natural question to ask how these sets relate to each other. A million-dollar question is: Does P=NP? This is a well-known unsolved problem in mathematics and theoretical computer science.

Exercises

7.1 Choose the correct answer for each of the following multiple-choice questions/items.

(*a*) Which loop is guaranteed to execute at least one time?

(*i*) For loop. (*ii*) While loop. (*iii*) Do-while loop. (*iv*) Not any listed.

(*b*) What is the use of for loop?

(*i*) To repeat the statement a finite number of times.
(*ii*) To repeat the statement until any condition holds true.
(*iii*) To repeat the statements for infinite time.
(*iv*) To repeat statements inside until any condition is false.

(*c*) We have three algorithms to solve a problem, Algorithms 1, 2, and 3. The running time function of Algorithm 2 is the function $g(n) = n^2$, that of Algorithm 1 is the function $f(n) = \frac{999999}{n} g(n)$, and that of Algorithm 3 is the function $h(n) = \frac{n}{999999} g(n)$. Here, n is the input size. Which algorithm is asymptotically the fastest?

(*i*) Algorithm 1. (*ii*) Algorithm 2. (*iii*) Algorithm 3.

(*iv*) All algorithms have the same performance.

(*d*) We have three algorithms to solve a problem. The running time function of Algorithm 1 is $f(n) = 1000n$, that of Algorithm 2 is $g(n) = n^2$, and that of Algorithm 3 is $h(n) = \frac{1}{1000}n^3$, where n is the input size. Which algorithm is asymptotically the fastest?

(*i*) Algorithm 1. (*ii*) Algorithm 2. (*iii*) Algorithm 3.

(*iv*) All algorithms have the same performance because when $n = 1000$, which is a large input size, $f(1000) = g(1000) = h(1000) = 10^6$.

(*e*) Suppose that we have an algorithm of seven lines and the cost of execution each line is a constant, say c. By doing a line-by-line analysis, we found that the running time of the algorithm is $c[(n - n^8)/(1 - n)]$. How many times line i executes (for $i = 1, 2, \ldots, 7$)?

(*i*) n^i-times. (*ii*) $n^{(i-1)}$-times. (*iii*) $n^{(i+1)}$-times. (*iv*) Not any listed.

(*f*) In the fragment given in Algorithm 7.15, if the for-statement in line (2) is replaced with `for (i=1; i<=n; i++)`, and that in line (3) is replaced with with `for (j=1; j<=n; j++)`, then what is the new running time?

(*i*) The running time will not change.

(*ii*) It is $c_1 + c_2 n + c_2(n-1)n + c_3(n-1)^2$.

(*iii*) It is $c_1 + c_2(n+2) + c_2(n+1)(n+2) + c_3(n+1)^2$.

(*iv*) It is $c_1 + c_2(n+1) + c_2(n-1)(n+1) + c_3(n-1)^2$.

(*g*) Suppose that we have an algorithm of five lines, the cost of execution each line is the constant c, and the line i executes n^i times for each $i = 1, 2, \ldots, 5$. If a line-by-line runtime analysis is used, what is the running time of the algorithm?

(*i*) $\dfrac{c(n-n^5)}{1-n}$. (*ii*) $\dfrac{c(n-n^6)}{1-n}$. (*iii*) $\dfrac{c(1-n^5)}{1-n}$. (*iv*) $\dfrac{c(1-n^6)}{1-n}$.

(*h*) Let $f(n)$ be the running time function of the code given in Algorithm 7.30. Which one of the following summations represents $f(n)$?

(*i*) $f(n) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i}^{j} c.$

(*iii*) $f(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} c.$

(*ii*) $f(n) = \sum_{i=1}^{n-1} \sum_{j=i}^{n} \sum_{k=i}^{j} c.$

(*iv*) $f(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{k=i}^{j} c.$

Algorithm 7.30: The algorithm of Exercise 7.1 (h)

```
x = 0
for (i = 1; i < n; i++) do
    for (j = i + 1; j ≤ n; j++) do
        for (k = i; k ≤ n; k++) do
            x = x + (i − j − k)          // Execution cost is c
        end
    end
end
return x
```

Algorithm 7.31: The algorithm of Exercise 7.1 (i)

```
x = 0
for (i = 1; i ≤ n; i++) do
    for (j = log n; j ≥ 1; j−−) do
        x = x + (i − j)          // Execution cost is c(i, j)
    end
end
return x
```

(i) Given the program snippet in Algorithm 7.31. Which one of the following summations represents the running time of this program?

(i) $\sum_{i=1}^{n} \sum_{j=1}^{\log n} c(i, j)$.

(ii) $\sum_{i=1}^{n} \sum_{j=1}^{\log n} c(i, \log n - j)$.

(iii) $\sum_{i=1}^{n} \sum_{j=1}^{\log n} c(i, \log n - j + 1)$.

(iv) None of the above.

(j) If the running time of an algorithm is a polynomial in n and is represented by the sum $\sum_{i=1}^{n} \sum_{j=1}^{i^2} (j + (j+1) + (j+2) + \cdots + (i^2 - 1) + i^2)$, which one of the following correctly describes the algorithm's running time?

(i) quadratic. (ii) cubic. (iii) quartic. (iv) quintic.

(k) Which of the following asymptotic statements is/are true?

(i) $n^2 + n = O(n^2)$.

(ii) $n^2 + n = O(n^3)$.

(iii) $n^2 + n = O(n^4)$.

(iv) None of the above.

(l) Let $f(n) = \Theta(g(n))$ and $k < 0$. Which of the following asymptotic statements is/are true?

(*i*) $|kf(n)| = \Theta(g(n))$.
(*ii*) $|kf(n)| = O(g(n))$.
(*iii*) $|kf(n)| = \Omega(g(n))$.
(*iv*) All of the above.

(*m*) Which one of the following statements about Theorem 7.1 is incorrect?

(*i*) The converse of item (*i*) is false.
(*ii*) The converse of item (*i*) is true.
(*iii*) The converse of item (*ii*) is false.
(*iv*) The contrapositive of (*ii*) is true.

(*n*) Which one of the following is true?

(*i*) a^n is $O(b^n)$ if $1 < b < a$.
(*ii*) n^a is $O(n^b)$ if $1 < b < a$.
(*iii*) n^a is $O(b^n)$ for any a and $b > 1$.
(*iv*) a^n is $O(n^b)$ for any b and $a > 1$.

(*o*) Let $f(n)$ be the worst-case running time function of the code given in Algorithm 7.32. Which one of the following summations represents $f(n)$?

(*i*) $\sum_{i=0}^{64}\sum_{j=0}^{64}\sum_{k=1}^{64} c$.
(*ii*) $\sum_{i=0}^{64}\sum_{j=0}^{8}\sum_{k=1}^{4} c$.
(*iii*) $\sum_{i=0}^{64}\sum_{j=0}^{8}\sum_{k=0}^{4} c$.
(*iv*) $\sum_{i=0}^{63}\sum_{j=0}^{7}\sum_{k=1}^{3} c$.

(*p*) If the running time of an algorithm is a polynomial of degree 5 in n and is represented by the sum

$$\sum_{m=1}^{n}\sum_{k=1}^{m^2}[(k + (k+1) + (k+2) + \cdots + (m^2 - 1) + m^2)(64c)].$$

Which of the following is the greatest lower bound[8] for the algorithm's running time?

(*i*) $\frac{7c}{8}n^5$.
(*ii*) $\frac{3c}{4}n^5$.
(*iii*) $\frac{c}{2}n^5$.
(*iv*) $\frac{c}{4}n^5$.

Algorithm 7.32: The algorithm of Exercise 7.1 (o)

```
cnt = 0
for (i = 0; i < 64; i + +) do
    for (j = 0; j^2 < 64; j + +) do
        for (k = 1; k^2 < 64/k; k + +) do
            if (2i + j ≥ 3k) then
                cnt ++                                  // Execution cost is c
            end
        end
    end
end
return x
```

[8]By the greatest lower bound, we mean the largest polynomial bounds the running time from below.

(*q*) Which one of the following is false?

(*i*) n^a is $O(n^b)$ if $a \le b$.
(*ii*) n^a is not $O(n^b)$ if $b < a$.
(*iii*) a^n is $O(b^n)$ if $1 < b < a$.
(*iv*) a^n is not $O(b^n)$ if $1 < b < a$.

(*r*) Let $f(n) = (n^2 - 1)^5$. Which of the following is true about $f(n)$?

(*i*) $f(n) = \Theta(n^2)$. (*ii*) $f(n) = \Theta(n^5)$. (*iii*) $f(n) = \Theta(n^7)$. (*iv*) $f(n) = \Theta(n^{10})$.

(*s*) The tight Big-Oh bound is best expressed by

(*i*) O. (*ii*) Ω. (*iii*) Θ. (*iv*) All the above.

(*t*) Which one of the following functions is a simple and tight bound on the function $f(n) = (5.01)^{n-1} + 5^{n+1}$?

(*i*) $(5.01)^{n-1}$. (*ii*) $(5.01)^n$. (*iii*) 5^n. (*iv*) 5^{n+1}.

(*u*) Which one of the following functions is a simple and tight bound on the function $f(n) = (9.99)^{n+1} + (10)^n + (10.01)^{n-1}$?

(*i*) $(9.99)^{n+1}$. (*ii*) $(10)^n$. (*iii*) $(10.01)^{n-1}$. (*iv*) $(10.01)^n$.

(*v*) Which one of the following is/are true?

(*i*) If a logarithmic-time algorithm exists to solve a problem, then we can say that the problem is solved efficiently.
(*ii*) If a polynomial-time algorithm exists to solve a problem, then we can say that the problem is solved efficiently.
(*iii*) If a polylogarithmic-time algorithm exists to solve a problem, then we can say that the problem is solved efficiently.
(*iv*) All the above.

7.2 Write an algorithmic code of the following algorithm which solves the same problem in Example 7.6.

(i) If `L` is of length 1, return the first item of `L`.

(ii) Set `v1` to the first item of `L`.

(iii) Set `v2` to the output of performing `find-max()` on the rest of `L`.

(iv) If `v1` is larger than `v2`, return `v1`. Otherwise, return `v2`.

7.3 For the algorithm in Exercise 7.2, answer the following questions by “Yes" or “No" with any necessary comments.

(*a*) Does it have defined inputs?

(*b*) Does it have defined outputs?

(*c*) Is it guaranteed to terminate?

(*d*) Does it produce the correct result?

7.4 Find the running time equipped with worst-case performance for Algorithm 7.12 by carrying out a line-by-line analysis. Assume that the cost of the statement in line `(i)` is c_i for `i=1,2,...,5`.

7.5 Do a line-by-line analysis to find the running time of the fragment given in Algorithm 7.33.

Algorithm 7.33: The algorithm of Exercise 7.5

sum $= 0$ // Cost $= c_1$
for $(i = \log n; i \geq 1; i--)$ **do** // Cost $= c_2$
  for $(j = n; j > 1; j/ = 2)$ **do** // Cost $= c_3$
    sum $+ =$ array$[i][j]$ // Cost $= c_4$
  end
end

7.6 Let p and q be two positive integers such that $p \leq q$. How many times do we go around each of the following loops, as a function of p, q, and possibly k? (Here k is a positive integer. Also, in item (f), we assume that q is divisible by k and its powers).

(*a*) for $(i = p; i \leq q; i++)$.

(*b*) for $(i = q; i \geq p; i--)$.

(*c*) for $(i = p; i \leq q; i+ = k)$.

(*d*) for $(i = q; i \geq p; i- = k)$.

(*e*) for $(i = p; i \leq q; i* = k)$.

(*f*) for $(i = q; i \geq p; i/ = k)$.

7.7 Given the code in Algorithm 7.34, write the summations that represent the running time of the code and solve the summations to find an approximate running time. You must show all your work without doing a line-by-line analysis.

7.8 Given Algorithm 7.35, write the summations that represent the running time of the algorithm and solve them to determine an upper bound. You must show all your work without doing a line-by-line analysis.

Algorithm 7.34: The algorithm of Exercise 7.7

$x = 0$
for $(i = n^2; i \leq n^2 + 5; i++)$ **do**
  for $(j = 4; j \leq n; j++)$ **do**
    $x = x + (i - j)$ // Execution cost is c
  end
end
return x

Algorithm 7.35: The algorithm of Exercise 7.8

```
x = 0
for (i = 1; i ≤ n; i++) do
    for (j = 1; j ≤ 3i^3; j++) do
        x = x + (i − j)                // Execution cost is c
    end
end
return x
```

7.9 Find upper and lower bounds for the running time of the algorithm represented by the following summation $f(n) = \sum_{i=n}^{4n^3} \sum_{j=i}^{8n^3} c$, where c is the execution cost of the statement that contributed most to the running time of the algorithm.

7.10 Given Algorithm 7.30 which has a running time function $f(n)$. Use bounding to solve the summations that represent $f(n)$ and find a tight asymptotic bound for $f(n)$. You must show all your work.

7.11 Use the Big-Oh definition to prove the following asymptotic statements.

(*a*) $5n^2 - 3n + 20 = O(n^2)$.

(*b*) $4n^2 - 12n + 10 = O(n^2)$.

(*c*) $5n^5 - 4n^4 - 2n^2 + n = O(n^5)$.

(*d*) $n^{3/2} + \sqrt{n}\sin n + n\log n = O(n^2)$.

7.12 Use the Big-Omega definition to prove the following asymptotic statements.

(*a*) $4n^2 + n + 1 = \Omega(n^2)$.

(*b*) $n\log n - 2n + 13 = \Omega(n\log n)$.

7.13 Use the Big-Theta definition to prove the following asymptotic statements.

(*a*) $n^5 + n^3 + 7n + 1 = \Theta(n^5)$.

(*b*) $\frac{1}{2}n^2 - 3n = \Theta(n^2)$.

7.14 Let $f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_k$ be asymptotically nonnegative functions. Assume that $f_i(n) = O(g_i(n))$ for each $i = 1, 2, \ldots, k$, prove that

(*a*) $\sum_{i=1}^{k} f_i(n) = O\left(\max_{1\le i\le k}\{g_i(n)\}\right)$.[9]

(*b*) $\prod_{i=1}^{k} f_i(n) = O\left(\prod_{i=1}^{k} g_i(n)\right)$.[10]

7.15 Prove Properties 7.1 and 7.6. Prove also Properties 7.2 and 7.7 for Big-Oh.

[9] It is also true that $\sum_{i=1}^{k} f_i(n) = O\left(\sum_{i=1}^{k} g_i(n)\right)$.

[10] Recall that $\sum_{i=1}^{k} f_i(n) = f_1(n) + f_2(n) + \cdots + f_k(n)$. Likewise, $\prod_{i=1}^{k} f_i(n) = f_1(n)f_2(n)\cdots f_k(n)$.

7.16 Use limits to prove the following asymptotic statements.

(*a*) $\sqrt{4n^2+1} = \Theta(n)$. (*b*) $n^n = \Omega(n!)$. (*c*) $\sqrt{n+4} - \sqrt{n} = O(1)$.

(*d*) $n \log\left(n^2\right) + (n-1)^2 \log\left(\frac{n}{2}\right) = \Theta\left(n^2 \log n\right)$. [11]

7.17 Let $f(n) = 2^{(n+10)}$. Does $f(n) = O(2^n)$? Answer by YES or NO. If yes, use the Big-Oh definition to show that $f(n)$ is $O(2^n)$. If no, use the proof by contradiction to show that $f(n)$ is not $O(2^n)$.

7.18 Choose the correct answer for each of the following multiple-choice questions/items.

(*a*) Consider the selection statement given in Algorithm 7.2. If S_1 and S_2 are $O(f_1(n))$ and $O(f_2(n))$, respectively, and the `condition` is "`2 ==` $\sqrt{2}$", which can tell which branch of the selection statement is taken, then the running time of the selection statement is

(*i*) $O(f_1(n))$. (*iii*) $O(f_2(n))$.
(*ii*) $O(1 + f_1(n))$. (*iv*) $O(\max\{f_1(n), f_2(n)\})$.

(*b*) Consider the while statement given in Algorithm 7.6. If the `statement(s)` is (are) $O(f(n))$, and the `condition` is "`1 !=` $\sqrt{1}$", which is known to be true or false right from the start, then the running time of the while statement is

(*i*) $O(1)$. (*iii*) $O(1 + f(n))$.
(*ii*) $O(f(n))$. (*iv*) None of the above.

(*c*) Given the program snippet in Algorithm 7.36. What is the time complexity of this program?

(*i*) $O((n^2/2)\log(n))$. (*iii*) $O(n^2 \log(n))$.
(*ii*) $O(n \log^2(n))$. (*iv*) $O((n^2/2)\log^2(n))$.

(*d*) Given the program snippet in Algorithm 7.37. What is the time complexity of this program?

(*i*) $O(n^2 + m\log(m))$. (*iii*) $O(n^2 + m^2)$.
(*ii*) $O(n^2 + \log(m))$. (*iv*) $O(n^2 + m/2)$.

Algorithm 7.36: The algorithm of Exercise 7.18 (c)

```
for (i = n; i ≥ 1; i/ = 2) do
    for (j = 1; j ≤ log(i); j + +) do
        something O(n)
    end
end
```

[11] Hint: For (*b*), you may use the equation: $n\log(n^2) + (n-1)^2\log(n/2) = 2n\log n + (n-1)^2(\log n - \log 2)$.

Algorithm 7.37: The algorithm of Exercise 7.18 (d)

```
something O(n + m)
while (n > 0) do
    something O(n)
    n − −
end
while (m > 0) do
    something O(m)
    m/ = 2
end
```

7.19 Consider the fragment in Algorithm 7.26. Note that $f(n)$ is a function call. Give a simple and tight Big-Oh upper bound on the running time of Algorithm 7.26, as a function of n, on each of the assumptions below. Justify your answers with complete evidences.

(a) the running time of $f(n)$ is $O(n)$, and the value of $f(n)$ is n.

(b) the running time of $f(n)$ is $O(n)$, and the value of $f(n)$ is $n!$.

7.20 Suppose that line (17) of Algorithm 7.27 is replaced with "for ($i = 1; i \leq \texttt{cat}(n, n); i$++)". What would the running time of "`main`" be then? Justify your answer with complete evidence.

7.21 Algorithm 7.38 is called a stochastic[12] interior-point algorithm and is used for solving a class of two-stage stochastic linear optimization problems of K scenarios and with n variables in the first stage and m variables in the second stage. The input includes a matrix A and a barrier function f, and the output is an ϵ-optimal solution to the given stochastic linear program. No matter how the algorithm works and no matter how the mathematical notations therein look like and how they work, the graph structure (flowchart) of the algorithm can be visualized. Draw the graph structure for Algorithm 7.38 by completing the subgraph diagram shown in Figure 7.18.

7.22 Consider Algorithm 7.38. Suppose that the running time of the block in lines (5)-(7) is $O(m)$, and the running time of the block in lines (9)-(12) is $O(n)$. Similarly, suppose that the running time of the block in lines (17)-(19) is $O(m)$, that of the block in lines (21)-(24) is $O(n)$, and that of the statement in line (29) is $O(m)$. Suppose also that all other statements (including those of checking the conditions of the while loops) are of constant times $O(1)$. Let also N_{in} be the number of times we go around the inner while loop of line (13). In view of the graph structure obtained in Exercise 7.21, determine the running time of the algorithm as a whole in each of these cases:

(a) $N_{\text{in}} = O(n + Km)$ and $\gamma \in (0, 1)$ is an arbitrarily chosen constant.

(b) $N_{\text{in}} = O(1)$ and $\gamma = 1 - \sigma/\sqrt{n + Km}$, where $\sigma > 0$.

In each case, give your answer as a function of $n, m, K, \mu^{(0)}, \epsilon$ using Big-oh, and write a paragraph (or more) that justifies your determination.

[12]Deterministic problems do not consider any uncertainties in input data, whereas stochastic problems model the uncertainties in data with appropriate probability distributions.

Algorithm 7.38: The algorithm of Exercise 7.21

initialize $\epsilon > 0, \gamma \in (0,1), \theta > 0, \beta > 0, \boldsymbol{x}^{(0)}, \mu^{(0)} > 0, \boldsymbol{\lambda}^{(0)}$
set $\boldsymbol{x} \triangleq \boldsymbol{x}^{(0)}, \mu \triangleq \mu^{(0)}, \boldsymbol{\lambda} = \boldsymbol{\lambda}^{(0)}$
while $(\mu \geq \epsilon)$ **do**
  for $(k = 1; k < K+1; k++)$ **do**
    find $\left(\boldsymbol{y}^{(k)}, \boldsymbol{z}^{(k)}, \boldsymbol{s}^{(k)}\right)$
    choose a scaling element $\boldsymbol{p}$
    compute $\left(\boldsymbol{y}_{\boldsymbol{p}}^{(k)}, \boldsymbol{s}_{\boldsymbol{p}}^{(k)}\right)$ by scaling $\left(\boldsymbol{y}^{(k)}, \boldsymbol{s}^{(k)}\right)$ with $\boldsymbol{p}$
  end
  compute $\boldsymbol{g} = \nabla f(\mu, \boldsymbol{x}) - A^{\mathsf{T}}\boldsymbol{\lambda}$
  compute $\Delta \boldsymbol{x} = -\left((\nabla^2 f)^{-1} - (\nabla^2 f)^{-1} A^{\mathsf{T}} \left(A (\nabla^2 f)^{-1} A^{\mathsf{T}}\right)^{-1} A (\nabla^2 f)^{-1}\right)(\boldsymbol{g})$
  compute $\Delta \boldsymbol{\lambda} = \left(A \left(\nabla^2 f(\mu, \boldsymbol{x})\right)^{-1} A^{\mathsf{T}}\right)^{-1} A \left(\nabla^2 f(\mu, \boldsymbol{x})\right)^{-1} (\boldsymbol{g})$
  compute $\delta(\mu, \boldsymbol{x}) = \sqrt{\frac{1}{\mu}(\Delta \boldsymbol{x})^{\mathsf{T}} \nabla^2 f(\mu, \boldsymbol{x})(\Delta \boldsymbol{x})}$
  while $(\delta > \beta)$ **do**
    set $\boldsymbol{x} \triangleq \boldsymbol{x} + \theta \Delta \boldsymbol{x}$
    set $\boldsymbol{\lambda} \triangleq \boldsymbol{\lambda} + \theta \Delta \boldsymbol{\lambda}$
    for $(k = 1; k < K+1; k++)$ **do**
      find $\left(\boldsymbol{y}^{(k)}, \boldsymbol{z}^{(k)}, \boldsymbol{s}^{(k)}\right)$
      choose a scaling element $\boldsymbol{p}$
      compute $\left(\boldsymbol{y}_{\boldsymbol{p}}^{(k)}, \boldsymbol{s}_{\boldsymbol{p}}^{(k)}\right)$ by scaling $\left(\boldsymbol{y}^{(k)}, \boldsymbol{s}^{(k)}\right)$ with $\boldsymbol{p}$
    end
    compute $\boldsymbol{g} = \nabla f(\mu, \boldsymbol{x}) - A^{\mathsf{T}}\boldsymbol{\lambda}$
    compute $\Delta \boldsymbol{x} = -\left((\nabla^2 f)^{-1} - (\nabla^2 f)^{-1} A^{\mathsf{T}} \left(A (\nabla^2 f)^{-1} A^{\mathsf{T}}\right)^{-1} A (\nabla^2 f)^{-1}\right)(\boldsymbol{g})$
    compute $\Delta \boldsymbol{\lambda} = \left(A \left(\nabla^2 f(\mu, \boldsymbol{x})\right)^{-1} A^{\mathsf{T}}\right)^{-1} A \left(\nabla^2 f(\mu, \boldsymbol{x})\right)^{-1} (\boldsymbol{g})$
    compute $\delta(\mu, \boldsymbol{x}) = \sqrt{\frac{1}{\mu}(\Delta \boldsymbol{x})^{\mathsf{T}} \nabla^2 f(\mu, \boldsymbol{x})(\Delta \boldsymbol{x})}$
  end
  set $\mu \triangleq \gamma \mu$
end
for $(k = 1; k < K+1; k++)$ **do**
  apply inverse scaling to $\left(\boldsymbol{y}_{\boldsymbol{p}}^{(k)}, \boldsymbol{s}_{\boldsymbol{p}}^{(k)}\right)$ to compute $\left(\boldsymbol{y}^{(k)}, \boldsymbol{s}^{(k)}\right)$
end

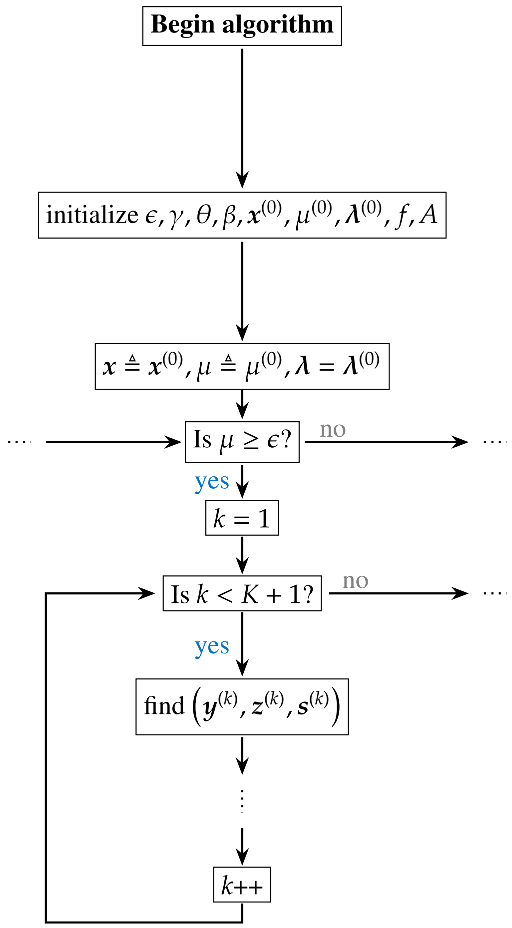


Figure 7.18: A subgraph structure of Algorithm 7.38.

Notes and sources

The history of algorithms and their analysis is intertwined with the development of mathematics and computing. As mentioned in the introductory paragraphs of this chapter, the term "algorithm" itself derives from the name of Al-Khwarizmi, who lived in the 9th century and made significant contributions to mathematics. Al-Khwarizmi's work laid the foundation for algebra and the systematic solving of linear and quadratic equations, essentially providing a structured approach or algorithm for these problems (refer to Khuwārizmī et al. [1963]). In the 20th century, the field of algorithm analysis saw substantial growth with the development of computer science. One of the key references in this field is Donald Knuth's multi-volume work Knuth [1997], which delves into the history and analysis of algorithms.

This chapter delved into the essential concepts of algorithmic analysis. By exploring asymptotic notation, readers gained a deep understanding of how to quantify and assess the running time of algorithms. This knowledge enabled us to evaluate and compare algorithm efficiency, ultimately aiding in the selection of efficient algorithms for finding more optimized solutions in mathematics and computer science.

As we conclude this chapter, it is worth noting that the cited references and others, such as Cormen et al. [2001], Rosen [2002], Cusack and Santos [2021], Mott et al. [1986], Liu [1968], Grimaldi [1999], Joshi [1989], Maurer and Ralston [2004], Hougardy and Vygen [2016], Higham [2002], Kagaris and Tragoudas [2008], Greene and Knuth [2009], Lewis and Papadimitriou [1998], Sipser [2013], also serve as valuable sources of information pertaining to the subject matter covered in this chapter. Exercise 7.22 is due to [Zhao, 2001, Section 5].

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CHAPTER 8

ARRAY AND NUMERIC ALGORITHMS

Chapter overview: This chapter describes and analyzes some fundamental algorithms that lie at the core of computer science and numerical analysis. More precisely, it covers some searching and sorting algorithms, namely linear search, binary search, insertion sort, selection sort, and merge sort, which are essential for data manipulation and organization. It also delves into specific numerical methods, namely Euclid's algorithm for greatest common divisor calculations and Newton's method for solving linear and nonlinear systems. The chapter concludes with a set of exercises to encourage readers to apply their knowledge and enrich their understanding.

Keywords: Searching algorithms, Sorting algorithms, Euclid's algorithm, Newton's method

Contents

B. Alzalg, *Combinatorial and Algorithmic Mathematics: From Foundation to Optimization*

Standard array and numeric algorithms, such as array multiplication algorithms, array searching algorithms, array sorting algorithms, and Newton's method algorithm for root finding are all discussed in this chapter. We also discuss the integer Euclidean algorithm (or Euclid's algorithm), which is one of the oldest and simplest number theoretic algorithms. Now we start with array multiplication algorithms.

8.1 Array multiplication algorithms

Vectors and matrices are arrays of real numbers. In this section, we will take a look at some simple matrix computations. Namely: Multiplying a matrix by a vector, and multiplying a matrix by a matrix.

Matrix-vector multiplication

Consider a matrix with n rows and m columns, that is, an $n \times m$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix}.$$

We assume that the entries of A are real. Given an m-tuple of real numbers:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix},$$

we can multiply A by $\boldsymbol{x}$ to get a product $\boldsymbol{b} = A\boldsymbol{x}$, where $\boldsymbol{b}$ is an n-tuple. The ith component of $\boldsymbol{b}$ is obtained by taking the dot product of ith row of A with $\boldsymbol{x}$. That is, its ith component is given by

$$b_i = \sum_{j=1}^{m} a_{ij} x_j. \tag{8.1}$$

Example 8.1 An example of matrix-vector multiplication with $n \times m = 3 \times 2$ is

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \times 7 + 2 \times 8 \\ 3 \times 7 + 4 \times 8 \\ 5 \times 7 + 6 \times 8 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 77 \end{bmatrix}.$$

■

A computer code to perform matrix-vector multiplication looks something like the one in Algorithm 8.1.

Algorithm 8.1: Matrix-vector multiplication (row-oriented)

$\boldsymbol{b} = 0$
for $(i = 1; i \leq n; i++)$ **do**
  for $(j = 1; j \leq m; j++)$ **do**
    $b_i = b_i + a_{ij}x_j$
  end
end

Note that in Example 8.1 we have with $n = 3$ and $m = 2$ is

$$\begin{bmatrix} 23 \\ 53 \\ 77 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$

In general, if $b = Ax$, then b is a linear combination of the columns of A. So, if $\boldsymbol{a}_j$ denotes the jth column of A, then we have

$$\boldsymbol{b} = \sum_{j=1}^{m} \boldsymbol{a}_j x_j.$$

Expressing these operations as a code, we have the one in Algorithm 8.2.

If each vector operation is performed by a loop, the code becomes the one in Algorithm 8.3.

Please take note that Algorithms 8.1 and 8.3 are essentially the same, with the only difference being the interchange of loops. Algorithm 8.1 accesses A by rows, while Algorithm 8.3 accesses A by columns.

Now, our objective is to determine the time required for the completion of Algorithm 8.3. It is important to note that each iteration of the inner loop involves two floating-point operations (flops). To simplify our analysis, we can represent each loop using a summation symbol Σ. Given that the inner loop executes for values of i ranging from 1 to n, and there are two flops performed in each iteration, the total number of flops executed during a single run of the inner loop can be expressed as $\sum_{i=1}^{n} 2$. Considering the outer loop, which runs from $j = 1$ to $j = m$, we can calculate the overall number of flops as $\sum_{j=1}^{m} \sum_{i=1}^{n} 2 = \sum_{j=1}^{m} 2n = 2nm$. This means that a total of $2nm$ flops are performed during the execution of Algorithm 8.3.

Algorithm 8.2: Writing $\boldsymbol{b} = \sum_{j=1}^{m} \boldsymbol{a}_j x_j$ in a code

$\boldsymbol{b} = 0$
for $(j = 1; j \leq m; j++)$ **do**
  $\boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}_j x_j$
end

Algorithm 8.3: Matrix-vector multiplication (column-oriented)

$\boldsymbol{b} = 0$
for $(j = 1; j \leq m; j++)$ **do**
  for $(i = 1; i \leq n; i++)$ **do**
    $b_i = b_i + a_{ij}x_j$
  end
end

Algorithm 8.4: Matrix-matrix multiplication

```
B=0
for (i = 1; i ≤ n; i + +) do
    for (j = 1; j ≤ l; j + +) do
        for (k = 1; k ≤ m; k + +) do
            b_ij = b_ij + a_ik x_kj
        end
    end
end
```

Matrix-matrix multiplication

If A is an $n \times m$ matrix, and X is an $m \times l$, we can form the product $B = AX$, which is $n \times l$. The (i, j) entry of B is the dot product of the ith row of A with the jth column of X. That is, the (i, j) entry of B is

$$b_{ij} = \sum_{k=1}^{m} a_{ik} x_{kj}. \tag{8.2}$$

A computer code to perform matrix-matrix multiplication looks something like the one in Algorithm 8.4.

Note that if $l = 1$, the matrix-matrix multiplication represented in (8.2) reduces to the vector-matrix multiplication represented in (8.1), and Algorithm 8.4 reduces to Algorithm 8.1.

Now, we want to know how long it will take to complete the task of Algorithm 8.4. Note that every iteration of the innermost loop includes two flops. Replacing each loop with a summation sign Σ, the total number of flops is

$$\sum_{i=1}^{n} \sum_{j=1}^{l} \sum_{k=1}^{m} 2 = 2nml.$$

In the important case when all of the matrices are square of dimension $n \times n$. We end this section by summarizing the above computational complexities in terms of n, where n is the size of the square matrix; see Corollary 8.1.

Corollary 8.1 *The time complexities of the above matrix algorithms are as follow:*

(*i*) *The matrix-vector multiplication algorithms are quadratic, running in $O(n^2)$ time.*

(*ii*) *The matrix-matrix multiplication algorithm is cubic, running in $O(n^3)$ time.*

8.2 Array searching algorithms

Searching algorithms are a class of algorithms that finds an element within an array or list. In this section, we introduce and analyze two well-known searching algorithms, namely linear search and binary search.

Linear search

Linear Search, also-called Sequential Search, is a linear-time searching algorithm that sequentially checks each element of the list to be searched until a target is found. The Linear Search is stated in Algorithm 8.5, in which we search an array $A[0:n-1]$ to find a value x that is expected to be present in the array. If the value x is not present in the array, the algorithm terminates unsuccessfully. To arrive at the Big-Oh of the running time for Algorithm 8.5, we depict a flowchart of this algorithm with running time complexity in Figure 8.1. It is evident that the while loop encompassing lines (3) to (5) within Algorithm 8.5 can be executed up to a maximum of n times, but not beyond that. Applying the summation rule, we can deduce that the time complexity of Algorithm 8.5 is $O(n)$, as this represents the maximum time among the following components: the time required for the assignment in line (2), the time for the while loop in lines (3)-(5), and the time for the selection statement in lines (6)-(11).

The running time found here (which is $O(n)$) is the worst-case performance of Algorithm 8.5. What are its best- and average-case performances? This question is left as an exercise for the reader (see Exercise 8.1 (c)).

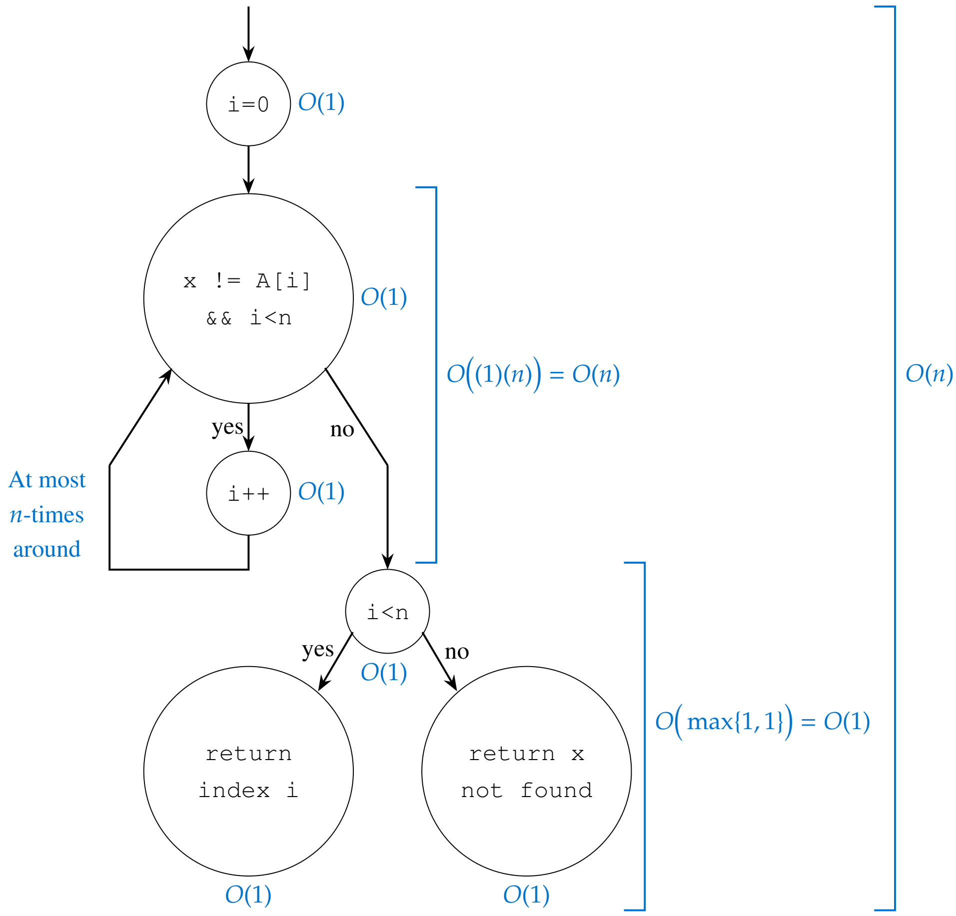

Figure 8.1: Linear Search graph structure with running time complexity.

Algorithm 8.5: Linear search algorithm

Input: An array $A[0 : n-1]$ and a sought value x
Output: Index i such that $x = A[i]$, otherwise x is not found

```
linear-search(int A[ ], int n, int x)
i = 0
while (x != A[i] && i < n) do
  | i++;
end
if (i < n) then
  | printf("x is present at index" i)
end
else
  | printf("x is not present in the array")
end
```

Algorithm 8.6: Binary search algorithm

Input: An array A[low:high] of length n and a sought value x
Output: Index i such that $x = A[i]$, otherwise x is not found

```
binary-search(int A[ ], int low, int high, int x)
if (low > high) then
  | printf("x is not present in array")            // constant time c1
end
else
  | mid = ⌊(low+high)/2⌋                           // constant time c2
  | if (x = A[mid]) then
  |   | printf("x is present at index" mid)       // constant time c3
  | end
  | if (x < A[mid]) then
  |   | binary-search(A,low, mid-1, x)          // same problem of size n/2
  | end
  | if (x > A[mid]) then
  |   | binary-search(A, mid+1, high, x)        // same problem of size n/2
  | end
end
```

Binary search

This section introduces another searching algorithm, called Binary Search. It is worth mentioning that Binary Search (and Merge Sort that we studied in Algorithm 8.9) are well-known examples of a class of algorithms called divide and conquer algorithms. Within this class of algorithms, the approach involves breaking down the problem into "simplified" iterations of itself. These simplified problems are then conquered using the same procedure, typically in a recursive manner, and the outcomes of these "simplified" versions are amalgamated to construct the ultimate solution.

Binary Search, also known as half-interval search, or logarithmic search, is a searching algorithm that finds the position of a target within a sorted array $A[low : high]$. The binary search algorithm is stated in Algorithm 8.6.

For more illustration of the binary search algorithm, in Table 8.1 we show in detail how to apply Algorithm 8.6 to an 8-element array. We first target the value $x = 8$, which is present in the array at index 6. Then, we also target $x = 6$ which is not found in the array.

Input: $A = [1\ 2\ 3\ 4\ 5\ 8\ 9\ 10]$,
$x = 8$.

Iteration #1:

low mid high

↓ ↓

1 2 3 4 5 6 7 8

1	2	3	**4**	5	8	9	10

Iteration #2:
As $x = 8 > A[\text{mid}] = 4$, we get

low mid high

↓ ↓

5 6 7 8

1	2	3	4	5	**8**	9	10

Iteration #3:
As $x = 8 = A[\text{mid}]$, we stop!

Output: x is present in A at index 6.

Input: $A = [1\ 2\ 3\ 4\ 5\ 8\ 9\ 10]$,
$x = 6$.

Iteration #1:

low mid high

↓ ↓

1 2 3 4 5 6 7 8

1	2	3	**4**	5	8	9	10

Iteration #2:
As $x = 6 > A[\text{mid}] = 4$, we get

low mid high

↓ ↓

5 6 7 8

1	2	3	4	5	**8**	9	10

Iteration #3:
As $x = 6 < A[\text{mid}] = 8$, we get

low mid high

↘ ↓ ↙

5

1	2	3	4	**5**	8	9	10

Iteration #4:
As $x = 6 > A[\text{mid}] = 5$, we get

high low

↓ ↓

5 6

1	2	3	4	5	8	9	10

Iteration #5:
As $low > high$, we stop!

Output: x is not present in the array A.

Table 8.1: A concrete implementation of the binary search algorithm.

Let $T(n)$ be the running time of Algorithm 8.6 on an array of size n. From Algorithm 8.6, it follows that the recurrence formula for this recursive program is

$$\begin{aligned} T(1) &= c, \\ T(n) &= c + T(n/2), \end{aligned} \tag{8.3}$$

where c might be taken equal to the constant time $c_1 + c_2 + c_3$. From Example 5.4, we have

$$T(n) = ck + T(n/2^k) = c \log n + T(1) = c \log n + c.$$

Thus, the recurrence formula (8.3) of Algorithm 8.6 describes the logarithmic function $T(n) = c \log n + c$. As a result, the binary search algorithm is logarithmic, running in $O(\log n)$ time. This is the worst-case performance of Algorithm 8.6. What are its best- and average-case performances? This question is left as an exercise for the reader (see Exercise 8.1 (e)).

We end this section with the following corollary which summarizes the computational complexity (worst behavior) in terms of the size of the list (n).

Corollary 8.2 *The (worst-case) time complexities of the above searching algorithms are as follow:*

(i) *The linear search algorithm is linear, running in $O(n)$ time.*

(ii) *The binary search algorithm is logarithmic, running in $O(\log n)$ time.*

8.3 Array sorting algorithms

Sorting algorithms are algorithms that put elements of an array or list in a certain order. In this section, we introduce and analyze three well-known sorting algorithms, namely insertion sort, selection sort, and merge sort.

Insertion sort

Insertion Sort is a sorting technique that resembles the way many individuals organize a hand of playing cards:

- Initiate with a single card in your hand.

- Select the subsequent card (to be sorted), create space for it by shifting the already sorted items, and place it in its appropriate position.

- Continue the above step for all the cards in the hand.

Figure 8.2 illustrates the insertion sort algorithm by applying the above steps to sort the array [57 32 41 60 11 33]. Note that, at each iteration, the array is divided into two sub-arrays. A formal description of the insertion sort method is given in Algorithm 8.7.

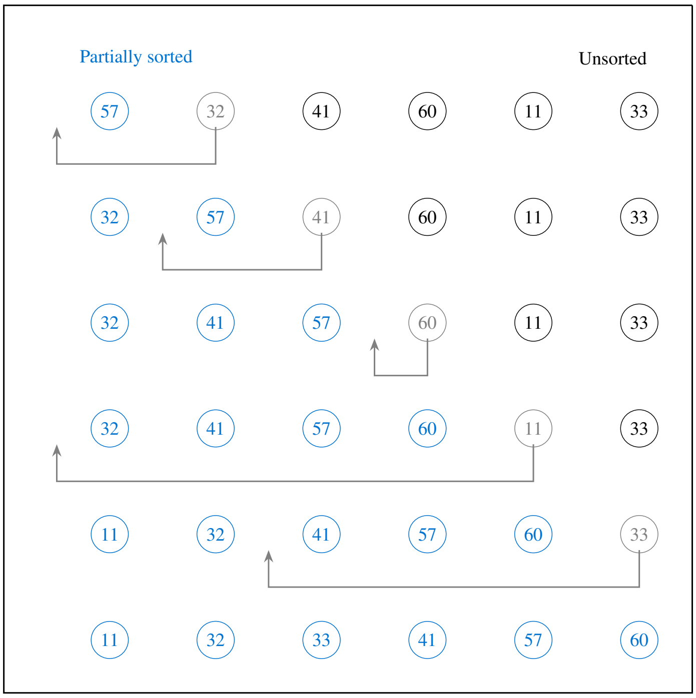

Figure 8.2: A concrete implementation of the insertion sort algorithm.

Algorithm 8.7: Insertion sort algorithm

Input: An integer array $A[0:n-1]$

Output: Array A sorted in ascending order

1: `insertion-sort`(int $A[\ ]$, int n)

2: **for** $(i = 1; i < n; i++)$ **do** // Cost $= c_1$, # times $= n$

3: next $= A[i]$ // Cost $= c_2$, # times $= n-1$

4: $j = i - 1$ // Cost $= c_3$, # times $= n-1$

5: **while** $(j > 0$ && $A[j] > next)$ **do** // Cost $= c_4$, # times $= \sum_{i=1}^{n-1} t_i$

6: $A[j+1] = A[j]$ // Cost $= c_5$, # times $= \sum_{i=1}^{n-1}(t_i - 1)$

7: $j = j - 1$ // Cost $= c_6$, # times $= \sum_{i=1}^{n-1}(t_i - 1)$

8: **end**

9: $A[j+1] =$ next // Cost $= c_7$, # times $= n-1$

10: **end**

Note that "next" in line (3) is the item to be sorted, line (6) shifts sorted items to make room for "next", and line (9) inserts "next" to the correct location.

In the comments in gray in Algorithm 8.7, t_i denotes the number of times the while statement is executed at iteration i for $i = 1, 2, \ldots, n-1$. Given this, the total cost is

$$\begin{aligned} f(n) &= c_1 n + c_2(n-1) + c_3(n-1) + c_4 \sum_{i=1}^{n-1} t_i \\ &\quad + c_5 \sum_{i=1}^{n-1}(t_i - 1) + c_6 \sum_{i=1}^{n-1}(t_i - 1) + c_7(n-1). \end{aligned} \tag{8.4}$$

We now present best- and worst-case analyses for the insertion sort method:

- Best-case analysis: The array is already sorted. So, $A[j] \leq$ next upon the first time the while-loop test is run, hence $t_i = 1$ for each $i = 1, 2, \ldots, n-1$. It follows that

$$f(n) = \underbrace{(c_1 + c_2 + c_3 + c_4 + c_7)}_{\bar{c}\ \text{(say)}}\, n - \underbrace{(c_2 + c_3 + c_4 + c_7)}_{\hat{c}\ \text{(say)}} = \bar{c}n + \hat{c}.$$

  Thus, when best-case analysis is considered, the rate of growth for the insertion sort method is n.

- Worst-case analysis: The array is sorted in reverse order. So, we always have `A[j] > next` in while-loop test. Then we have to compare "next" with all elements to the left of the i^{th}-position. That is, we compare with $i-1$ element. So, $t_i = i$ for $i = 1, 2, \ldots, n-1$.

  Using (8.4) and applying the arithmetic series formula (see Corollary 2.1), it follows that

$$\begin{aligned} f(n) &= c_1 n + c_2(n-1) + c_3(n-1) + c_7(n-1) \\ &\quad + c_4 \sum_{i=1}^{n-1} t_i + c_5 \sum_{i=1}^{n-1} (t_i - 1) + c_6 \sum_{i=1}^{n-1} (t_i - 1) \\ &= c_1 n + c_2(n-1) + c_3(n-1) + c_7(n-1) \\ &\quad + c_4 \left(\frac{(n-1)n}{2}\right) + c_5 \left(\frac{(n-2)(n-1)}{2}\right) + c_6 \left(\frac{(n-2)(n-1)}{2}\right) \\ &= \bar{c}n^2 + \hat{c}n + \tilde{c}, \end{aligned}$$

  for some constants $\bar{c}, \hat{c}$ and $\tilde{c}$. Thus, when worst-case analysis is considered, the rate of growth of the insertion sort algorithm is n^2.

Selection sort

Selection Sort is a sorting method that partitions the input list into two segments: a sorted sublist, which gradually forms on the right-hand side, and an unsorted sublist containing the remaining items on the left-hand side. At the outset, the sorted sublist is devoid of elements, while the unsorted sublist encompasses the entire input list. The algorithm advances by identifying the smallest element within the unsorted sublist, swapping it with the leftmost unsorted element, and shifting the boundaries of the sublist one element to the right.

For more illustration of this method, we apply the selection sort method to sort the 5-element list $(21, 35, 22, 32, 74)$. Table 8.2 shows all steps in this method for sorting the given list. The Selection Sort is formally stated in Algorithm 8.8.

To arrive at the Big-Oh of the running time for Algorithm 8.8, we represent the structure of this algorithm and the running time complexity by the tree structure shown in Figure 8.3.

Unsorted sublist	Least element in the unsorted list	Sorted sublist
(21, 35, 22, 32, 74)	11	( )
(35, 22, 32, 74)	12	(21)
(35, 32, 74)	32	(21, 22)
(35, 74)	35	(21, 22, 32)
(74)	74	(21, 22, 32, 35)
( )		(21, 22, 32, 35, 74)

Table 8.2: A concrete implementation of the selection sort algorithm.

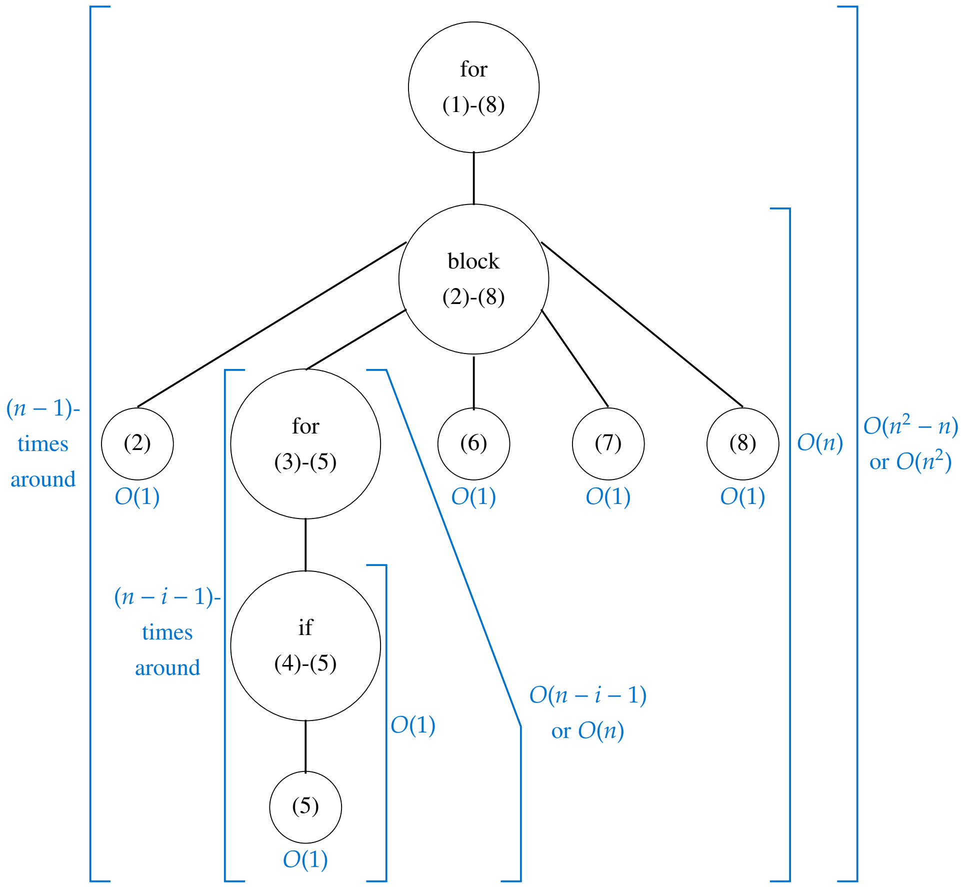

Figure 8.3: Selection Sort tree structure with running time complexity.

In closing, the running time of Selection Sort is $O(n^2)$. This is the worst-case performance of Algorithm 8.8. What are its best- and average-case performances? This question is left as an exercise for the reader (see Exercise 8.1 (c)).

Algorithm 8.8: Selection sort algorithm

Input: An integer array $A[0 : n-1]$
Output: Array A sorted in ascending order

```
selection-sort(int A[ ], int n)
for (i = 0; i < n - 1; i++) do
    small = i
    for (j = i + 1; j < n; j++) do
        if (A[j] < A[small]) then
            small = j
        end
    end
    swap = A[small]
    A[small] = A[i]
    A[i] = swap
end
```

Algorithm 8.9: Merge sort algorithm

Input: An integer array A[first:last] of length n
Output: Array A sorted in ascending order

```
merge-sort(int A[ ], int first, int last)
if (first < last) then
    mid = ⌊(first+last)/2⌋                  // Constant time c
    merge-sort(A, first, mid)               // Same problem of size n/2
    merge-sort(A, mid+1, last)              // Same problem of size n/2
    merge(A, first, mid, last)              // A call to "merge" takes time n
end
```

Merge sort

Merge Sort is a sorting algorithm in which the input is an unsorted array $A[\text{first} : \text{last}]$, and the output is the array A sorted in ascending order. The idea behind the merge sort method involves breaking down the unsorted list or array, consisting of n elements, into n sorted sublists, with each sublist initially containing a single element (where a single-element list is regarded as a sorted array). Subsequently, these sublists are systematically merged to generate fresh sorted sublists, and this process continues until only one sublist remains. That final sublist represents the sorted array. The merge sort algorithm is stated in Algorithm 8.9.

Note that Algorithm 8.9 uses the "merge" function, which is an inbuilt function in C++STL[1] that merges two sorted lists into a single sorted list.

For more illustration of the merge sort algorithm, in Figure 8.4 we show in detail how to apply Algorithm 8.9 to a 7-element array.

Let $T(n)$ be the running time of Algorithm 8.9 on an array of size n. From Algorithm 8.9, by dropping the constant time c in favor of the large time n, it follows that the recurrence formula for this recursive program is

$$T(n) = n + 2T(n/2). \tag{8.5}$$

[1] C++STL stands for the C++ Standard Template Library.

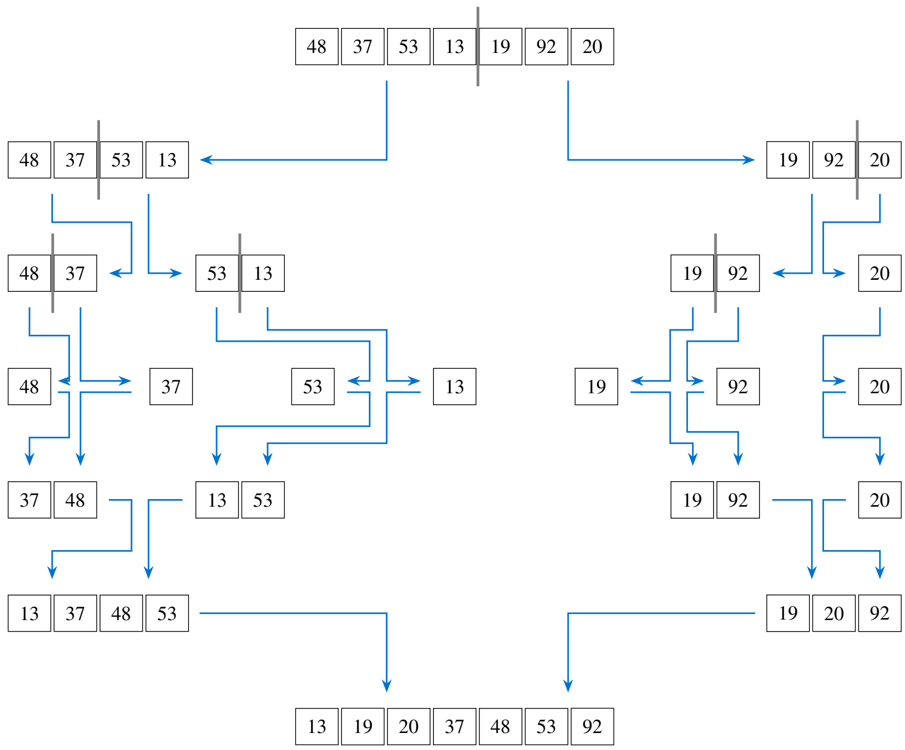

Figure 8.4: A concrete implementation of the merge sort algorithm.

From Example 5.5, we have

$$T(n) = nk + T(n/2^k) = n\log n + nT(1).$$

Thus, the recurrence formula (8.5) of Algorithm 8.9 describes the linearithmic function $T(n) = n(\log n + \text{constant time})$. As a result, the merge sort algorithm is linearithmic, running in $O(n \log n)$ time. This is the worst-case performance of Algorithm 8.9. What are its best- and average-case performances? This question is left as an exercise for the reader (see Exercise 8.1 (*e*)).

We end this section with the following corollary which summarizes the computational complexity (worst behavior) in terms of the size of the list (n).

Corollary 8.3 *The (worst-case) time complexities of the above sorting algorithms are as follow:*

(*i*) *The insertion sort algorithm is quadratic, running in* $O(n^2)$ *time.*

(*ii*) *The selection sort algorithm is quadratic, running in* $O(n^2)$ *time.*

(*iii*) *The merge sort algorithm is linearithmic, running in* $O(n \log n)$ *time.*

Besides the above sorting algorithms, there are many other sorting algorithms that are not presented in this book. This includes heap sort, quicksort and counting sort (see Cormen et al. [2001]). In practical implementations a few algorithms predominate. Insertion sort is widely used for small data sets, while for large data sets an asymptotically efficient sort is used, primarily merge sort, heap sort, or quicksort.

8.4 Euclid's algorithm

The greatest common divisor of two integers is the largest number that can evenly divide both integers without leaving a remainder. An effective approach for calculating the greatest common divisor of two integers is the Euclidean algorithm, also known as Euclid's algorithm. This algorithm was originally formulated by Euclid around 300 BC and can be found in Book VII of his Elements.

There are two versions of Euclid's algorithm: The division-based version which recursively calls a function while the larger number is not divisible by the smaller, and the subtraction-based version which recursively subtracts the smaller number from the larger.

The greatest common divisor of integers $a, b \in \mathbb{N}$ is written as $\gcd(a, b)$. We write $a \bmod n$ to denote the unique integer b such that $0 \leq b < n$ and $a \equiv b \pmod{n}$ (i.e., the remainder of a when divided by n). For example, $39 \equiv 3 \pmod{12}$ because $38 - 3 = 36$, which is a multiple of 12. It is known that $\gcd(a, b) = b$ when b divides a. In addition, if a is not divisible by b, the reader can show that $\gcd(a, b) = \gcd(b, a \bmod b)$ (see Exercise 8.5). This leads us to the correctness of Algorithm 8.10.

For example, the gcd(1377, 594) is calculated from the equivalent gcd(594, 1377 mod 594) = gcd(594, 189). The latter greatest common divisor is calculated from the gcd(189, 462 mod 189) = gcd(189, 27), which in turn is calculated from the gcd(21, 189 mod 27) = gcd(27, 0) = 27. Thus, gcd(1377, 594) = 27.

Note that, in Algorithm 8.10, if negative inputs are allowed, then the mod function may return negative values. In this case, the instruction "return a" must be replaced with "return $\max(a, -a)$".

Contrary to the recursive version stated in Algorithm 8.10, which works with two arbitrary integers as input, the subtraction-based version works with two positive integers and stops when they are both identical. This version of Euclid's Algorithm is stated in Algorithm 8.11.

For illustration, the reader can use Algorithm 8.11 to check that gcd(1377, 594) = 27. Now, let us analyze the worst-case time complexity of Algorithm 8.11 with respect to the sum, denoted as $n = a + b$, of a and b. Excluding our base case, this sum decreases with each recursive step. Considering that the smallest possible reduction in each step is 1 (for instance, $\gcd(x, 1)$), and since every positive integer is assured to have a common divisor of 1, we can conclude that the time complexity is $O(n)$. Consequently, the number of iterations in Algorithm 8.11 can grow linearly in relation to the sum of a and b.

We now estimate the worst-case time complexity of Algorithm 8.10 based on the minimum, $n = \min(a, b)$, of a and b. Note that the number of iterations Algorithm 8.10 will take is maximized when the two inputs are consecutive Fibonacci numbers. As introduced in Section 3.1, the Fibonacci numbers $f_0, f_1, f_2, \ldots$, are defined by the recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \text{ for } n = 2, 3, 4, \ldots, \quad \text{where } f_0 = 0 \text{ and } f_1 = 1.$$

Using this recurrence formula, we have $f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13$, etc. More specifically, if $a > b > 0$ and $\gcd(a, b)$ requires $n \geq 1$ steps, then we can verify that the smallest possible values of a and b are f_{n+2} and f_{n+1}, respectively (see Exercise 8.6). Therefore, $b \geq f_{n+1}$.

Algorithm 8.10: Euclid's Algorithm (the division-based version)

```
gcd(a, b)
if b = 0 then
    return a
end
else
    return gcd(b, a mod b)
end
```

Algorithm 8.11: Euclid's Algorithm (the subtraction-based version)

```
gcd(a, b)
while (a ≠ b) do
    if (a > b) then
        a ≜ a − b
    end
    else
        b ≜ b − a
    end
end
return a
```

Exercise 5.6 uses the generating function method to solve the Fibonacci recurrence $f_n = f_{n-1} + f_{n-2}$. One can find (see the solution to Exercise 5.6) that

$$f_n = \frac{1}{\sqrt{5}}\left(\alpha^n - (1-\alpha)^n\right),$$

where, for $n \geq 0$, $\alpha \triangleq (1+\sqrt{5})/2$ is the 'golden ratio'. A more simplified version of the above formula is the following:

$$f_n \approx \frac{\alpha^n}{\sqrt{5}}.$$

It follows that

$$b \geq f_{n+1} \approx \frac{1}{\sqrt{5}}\alpha^{(n+1)} = \frac{3+\sqrt{5}}{2\sqrt{5}}\alpha^{(n-1)} \geq \alpha^{(n-1)}.$$

Solving for n, we get

$$n \leq \log_\alpha b + 1 = O(\log_\alpha n).$$

Thus, the number of iterations in Algorithm 8.10 can be logarithmic based on the minimum of a and b.

8.5 Newton's method algorithm

Newton's method (also called Newton-Raphson method) is one of the root-finding algorithms. Given a function f, the goal of a root-finding algorithm is to find a root of f, or solution of the equation $f(x) = 0$. The desired value r that satisfies $f(r) = 0$ is also called a zero of the function f.

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a twice continuously differentiable function. Let $r_0 \in [a, b]$ be an approximation to a root r of f such that $f'(r_0) \neq 0$ and $|r - r_0|$ is small. Consider the following Taylor expansion for $f(x)$ about r_0 evaluated at $x = r$.

$$f(r_0) + (r - r_0)f'(r_0) + \frac{(r - r_0)^2}{2} f''(r_0) + \frac{(r - r_0)^3}{6} f'''(r_0) + \cdots .$$

Newton's method can be derived by assuming that the terms involving $(r - r_0)^2, (r - r_0)^3, \ldots$ are tiny because $|r - r_0|$ is small. So

$$0 \approx f(r_0) + (r - r_0)f'(r_0).$$

After solving for r, we immediately have

$$r \approx r_0 - \frac{f(r_0)}{f'(r_0)} \equiv r_1.$$

Figure 8.5 shows the geometry of Newton's method. The tangent line to the curve $y = f(x)$, namely $y = f(r_0) + (x - r_0)f'(r_0)$, intersects the x-axis at r_1 and has the slope of $f'(r_0)$. Initializing at the approximation r_0, the x-intercept of the tangent line to the graph of f at $(r_0, f(r_0))$, say r_1, becomes a better approximation to r. Re-initializing at the approximation r_1, the x-intercept of the tangent line to the graph of f at $(r_1, f(r_1))$, becomes even better approximation than r_1 to r, and so on.

Accordingly, Newton's method initializes with an initial point r_0 and generates a sequence of points $\{r_k\}_{k=0}^{\infty}$ using

$$r_{k+1} = r_k - \frac{f(r_k)}{f'(r_k)}, \text{ for } k = 0, 1, 2, \ldots. \tag{8.6}$$

The following example is extracted from Burden and Faires [2010].

Example 8.2 Approximate a root of the function $f(x) = \cos x - x$ using Newton's Method and starting with $r_0 = \pi/4 \approx 0.7853981635$.

Solution Equation (8.6) for our setting becomes

$$r_{k+1} = r_k - \frac{\cos r_k - r_k}{-\sin r_k - 1},$$

for $k = 0, 1, 2, \ldots$. This gives the approximations in Table 8.3. ■

k	r_{k+1}
0	0.7395361337
1	0.7390851781
2	0.7390851332
3	0.7390851332

Table 8.3: Numerical results of Example 8.2.

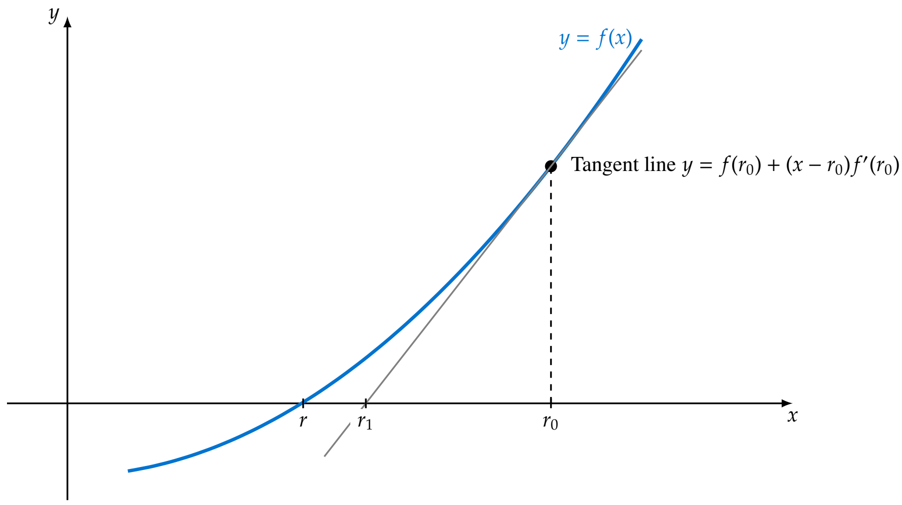

Figure 8.5: The geometry of Newton's method.

Algorithm 8.12: Newton's method algorithm

Input: Initial guess r_0, maximum number of iterations N, tolerance TOL
Output: Approximate solution r to $f(x) = 0$ or message of failure

```
set k = 0
while (k ≤ N) do
    compute r_{k+1} using (8.6)
    if |r_{k+1} − r_k| < TOL then
        printf("r equals" r_{k+1})          // The procedure is successful
        break
    end
    set k + +
    set r_k = r_{k+1}
end
printf("Fails after N iterations")          // The procedure is unsuccessful
```

Newton's method is formally stated in Algorithm 8.12.

We give, without proof, the following convergence and complexity result in this context. For a proof, see Cheney and Kincaid [2007].

Theorem 8.1 *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a twice continuously differentiable function. If $r \in (a,b)$ is such that $f(r) = 0$ and $f'(r) \neq 0$, then there exists $\delta > 0$ such that Newton's method generates a sequence $\{r_k\}_{k=1}^{\infty}$ converging quadratically to r for any initial approximation satisfying $|r - r_0| < \delta$.*

Newton's method for nonlinear systems

Newton's method can be also used to find roots of nonlinear equations. In general, given a system of equations when the number of unknowns equals the number of equations. We are interested to find solutions to a system of the form:

$$\begin{aligned} f_1(x_1; x_2; \dots; x_n) &= 0, \\ f_2(x_1; x_2; \dots; x_n) &= 0, \\ &\vdots \\ f_n(x_1; x_2; \dots; x_n) &= 0, \end{aligned} \tag{8.7}$$

where $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ for $i = 1, 2, \dots, n$. In vector form, System (8.7) can be written as

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}, \tag{8.8}$$

where $\boldsymbol{f} \triangleq (f_1; f_2; \dots, f_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\boldsymbol{x} \triangleq (x_1; x_2; \dots; x_n) \in \mathbb{R}^n$.

Newton's method can be extended to the nonlinear system (8.8) using

$$\boldsymbol{r}^{k+1} = \boldsymbol{r}^k - [\mathcal{J}_x \boldsymbol{f}]^{-1}|_{x=r^k} \boldsymbol{f}(\boldsymbol{r}^k)$$

where $\mathcal{J}_x \boldsymbol{f}$ is the $n \times n$ Jacobian matrix defined as

$$\mathcal{J}_x \boldsymbol{f} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{x}) & \frac{\partial f_1}{\partial x_2}(\boldsymbol{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{x}) \\ \frac{\partial f_2}{\partial x_1}(\boldsymbol{x}) & \frac{\partial f_2}{\partial x_2}(\boldsymbol{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\boldsymbol{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\boldsymbol{x}) & \frac{\partial f_n}{\partial x_2}(\boldsymbol{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} (\nabla_x f_1(\boldsymbol{x}))^\mathsf{T} \\ (\nabla_x f_2(\boldsymbol{x}))^\mathsf{T} \\ \vdots \\ (\nabla_x f_n(\boldsymbol{x}))^\mathsf{T} \end{bmatrix},$$

and, for a function $g : \mathbb{R}^n \to \mathbb{R}$, $\nabla g(\boldsymbol{x})$ is the gradient vector (the vector of the first derivatives of g) defined as

$$\nabla_x g(\boldsymbol{x}) \triangleq \begin{bmatrix} \frac{\partial g}{\partial x_1}(\boldsymbol{x}) \\ \frac{\partial g}{\partial x_2}(\boldsymbol{x}) \\ \vdots \\ \frac{\partial g}{\partial x_n}(\boldsymbol{x}) \end{bmatrix}. \tag{8.9}$$

Note that the method presumes that the Jacobian $\mathcal{J}_x \boldsymbol{f}$ is nonsingular at each $\boldsymbol{r}^k$, so its inverse exists. The two-step procedure in the following workflow, followed by an example, will teach us to solve System (8.8).

> **Workflow 8.1** *We solve System (8.8) by following two steps:*
>
> (*i*) *Solve* $\mathcal{J}_x \boldsymbol{f}\ \Delta \boldsymbol{r} = -\boldsymbol{f}(\boldsymbol{x})$ *for* $\Delta \boldsymbol{r}$. (*ii*) *Let* $\boldsymbol{r}^{k+1} = \boldsymbol{r}^k + \Delta \boldsymbol{r}$.

The vector $\Delta \boldsymbol{r}$ defined in Workflow 8.1 is called a Newton direction.

Example 8.3 Let us perform one iteration of Newton's method to approximate the solution of the following set of equations with a starting guess $r_0 = (1/2, 1/2)^\mathsf{T}$.

$$\begin{aligned} x_1^3 + x_2 &= 1, \\ -x_1 + x_2^3 &= -1. \end{aligned}$$

Letting

$$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_3) \end{bmatrix} = \begin{bmatrix} x_1^3 + x_2 - 1 \\ -x_1 + x_2^3 + 1 \end{bmatrix},$$

we have

$$\mathcal{J}_x f = \begin{bmatrix} 2x_1^2 & 1 \\ -1 & 3x_2^2 \end{bmatrix}.$$

Then

$$f(r_0) = \begin{bmatrix} -3/8 \\ 5/8 \end{bmatrix}, \mathcal{J}_x f(r_0) = \begin{bmatrix} 3/4 & 1 \\ -1 & 3/4 \end{bmatrix}, \text{ and hence } [\mathcal{J}_x f(r_0)]^{-1} = \frac{16}{25}\begin{bmatrix} 3/4 & -1 \\ 1 & 3/4 \end{bmatrix}.$$

Performing one iteration of Newton's method, we get the following Newton direction

$$\Delta r = -[\mathcal{J}_x f(r_0)]^{-1} f(r_0) = -\frac{16}{25}\begin{bmatrix} 3/4 & -1 \\ 1 & 3/4 \end{bmatrix}\begin{bmatrix} -3/8 \\ 5/8 \end{bmatrix} = \begin{bmatrix} 29/50 \\ 22/50 \end{bmatrix}.$$

As a result, we have

$$r^1 = r^0 + \Delta r = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 29/50 \\ -3/50 \end{bmatrix} = \begin{bmatrix} 27/25 \\ 11/25 \end{bmatrix},$$

which is the approximation after the first iterate. Note that, with a sharp eye, the reader can see that there is one and only one solution to $f(x) = \mathbf{0}$, namely $(x_1, x_2) = (1, 0)$. ■

Newton's method for optimization

In this part, Newton's method is exploited to optimize twice differentiable convex functions. Consider the function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, which is a real-valued function of n independent variables. We say that g is differentiable on a domain $D \subseteq \mathbb{R}^n$ if $\nabla_x g$ exists for all $x \in D$, where $\nabla_x g$ is the gradient vector defined in (8.9). Similarly, g is twice differentiable on D if $\nabla^2_{xx} g$ exists for all $x \in D$, where $\nabla^2_{xx} g$ is the Hessian matrix (the matrix of the second derivatives of g) defined as

$$\nabla^2_{xx} g \triangleq \begin{bmatrix} \frac{\partial^2 g}{\partial x_1^2}(x) & \frac{\partial^2 g}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 g}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 g}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 g}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 g}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial x_n \partial x_1}(x) & \frac{\partial^2 g}{\partial x_n \partial x_2}(x) & \cdots & \frac{\partial^2 g}{\partial x_n^2}(x) \end{bmatrix}. \tag{8.10}$$

Note that the Hessian of a function is the Jacobian of its gradient. That is

$$\nabla^2_{xx} g = \mathcal{J}_x \nabla_x g.$$

In addition, g is called twice continuously differentiable on D if $\nabla^2_{xx} g$ is continuous on D. As a matter of fact, if a function is a twice continuously differentiable function, then its Hessian matrix is symmetric (positive semidefinite). Moreover, if a function is a twice continuously differentiable convex function, then its Hessian matrix is positive semidefinite (see Definition 3.3).

Algorithm 8.13: Newton's algorithm for unconstrained optimization

Input: Initial guess $\boldsymbol{x}^0$, maximum number of iterations N, tolerance TOL
Output: Approximate optimal solution to Problem (8.11)

set $k = 0$
while $(k \leq N)$ **do**
  compute $\Delta\boldsymbol{x} = -[\nabla^2_{\boldsymbol{x}\boldsymbol{x}} g(\boldsymbol{x}^k)]^{-1} \nabla_{\boldsymbol{x}} g(\boldsymbol{x}^k)$
  choose step-size θ_k
  set $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \theta_k \Delta\boldsymbol{x}$
  if $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| <$ TOL **then**
    `printf`(*"Approximate optimal solution is"* $\boldsymbol{x}^{k+1}$)
    break
  end
  set $k++$
  set $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k$
end
`printf`(*"The method fails after N iterations"*)

Assume that we are interested in minimizing a twice continuously differentiable real valued function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$. That is, we are interested in a problem of the form:

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} g(\boldsymbol{x}). \tag{8.11}$$

In this case, we want to its gradient to zero. That is, we need to find the roots of the system of nonlinear equations

$$\nabla_{\boldsymbol{x}} g(\boldsymbol{x}) = \boldsymbol{0}. \tag{8.12}$$

Note that, in (8.12), the number of unknowns equals the number of equations. Applying Newton's method here involves the Jacobian matrix of $\nabla_{\boldsymbol{x}} g$, which is its Hessian matrix defined in (8.10).

Algorithm 8.13 is used to minimize the function g. In Algorithm 8.13, the vector $\Delta\boldsymbol{x}$ is called a Newton direction or Newton step, $\boldsymbol{x}^k$ is called the Newton iterate, and the parameter θ_k is called the step-size. Also, in Algorithm 8.13, $\|\cdot\|$ denotes the Euclidean norm.

Example 8.4, which is taken from Freund [2004], performs one iteration of Algorithm 8.13 to get an approximation of the optimal solution of a minimization problem.

Example 8.4 With a starting guess $\boldsymbol{x}^0 = (0.85; 0.05)$, let us perform one iteration of Algorithm 8.13 to approximate the optimal solution of the problem:

$$\min_{(x_1;x_2) \in D} -\ln(1 - x_1 - x_2) - \ln x_1 - \ln x_2$$

where $D = \{(x_1; x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0, x_1 + x_2 < 1\}$. Letting

$$g(\boldsymbol{x}) = -\ln(1 - x_1 - x_2) - \ln x_1 - \ln x_2,$$

we have

$$\nabla_x g(\boldsymbol{x}) = \begin{bmatrix} \dfrac{1}{1-x_1-x_2} - \dfrac{1}{x_1} \\ \dfrac{1}{1-x_1-x_2} - \dfrac{1}{x_2} \end{bmatrix},$$

and

$$\nabla^2_{xx} g(\boldsymbol{x}) = \begin{bmatrix} \dfrac{1}{(1-x_1-x_2)^2} + \dfrac{1}{x_1^2} & \dfrac{1}{(1-x_1-x_2)^2} \\ \dfrac{1}{(1-x_1-x_2)^2} & \dfrac{1}{(1-x_1-x_2)^2} + \dfrac{1}{x_2^2} \end{bmatrix}.$$

Then

$$\nabla_x g(\boldsymbol{x}^0) \approx \begin{bmatrix} 8.824 \\ -10 \end{bmatrix}, \text{ and } \nabla^2_{xx} g(\boldsymbol{x}^0) \approx \begin{bmatrix} 100.2 & 100 \\ 100 & 500 \end{bmatrix}.$$

Performing one iteration of Newton's method, we get the following Newton direction

$$\begin{aligned} \Delta\boldsymbol{x} &= -[\nabla^2_{xx} g(\boldsymbol{x}^k)]^{-1} \nabla_x g(\boldsymbol{x}^k) \\ &\approx -\frac{1}{40100} \begin{bmatrix} 500 & -100 \\ -100 & 100.2 \end{bmatrix} \begin{bmatrix} 8.824 \\ -10 \end{bmatrix} \\ &= \begin{bmatrix} -0.012468 & 0.002493 \\ 0.002493 & -0.002498 \end{bmatrix} \begin{bmatrix} 8.824 \\ -10 \end{bmatrix} = \begin{bmatrix} -0.134947 \\ 0.046978 \end{bmatrix}. \end{aligned}$$

As a result, we have

$$\boldsymbol{x}^1 = \boldsymbol{x}^0 + \Delta\boldsymbol{x} \approx \begin{bmatrix} 0.85 \\ 0.05 \end{bmatrix} + \begin{bmatrix} -0.134947 \\ 0.046978 \end{bmatrix} = \begin{bmatrix} 0.715053 \\ 0.096978 \end{bmatrix},$$

which is the approximation after the first iterate. The step size was chosen to be $\theta_1 = 1$.

As an exercise, the reader is encouraged to perform more iterations to find $\boldsymbol{x}^k$ for $k = 2, 3, 4, 5$. The approximate optimal solution after the fifth iteration is $\boldsymbol{x}^5 = (0.333338; 0.333259)$. It is not hard to check that the optimal solution $(x_1, x_2) = (1/3, 1/3)$. ■

The Newton direction (or generally, any search direction) is called a decent direction if $g(\boldsymbol{x} + \theta\Delta\boldsymbol{x}) < g(\boldsymbol{x})$ for all sufficiently small values of θ. In general, without some further assumptions, there is no guarantee of decent directions in Algorithm 8.13. We have the following theorem.

Theorem 8.2 *If $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a twice continuously differentiable strictly convex real valued function, then the Newton direction $\Delta\boldsymbol{x}$ is descent.*

Proof If g is strictly convex, then its Hessian matrix, $\nabla^2_{xx} g(\boldsymbol{x})$, is positive definite (the proof for this fact is left as an exercise for the reader). It follows that

$$(\nabla_x g(\boldsymbol{x}))^{\mathsf{T}} \Delta\boldsymbol{x} = -(\nabla_x g(\boldsymbol{x}))^{\mathsf{T}} [\nabla^2_{xx} g(\boldsymbol{x})]^{-1} \nabla_x g(\boldsymbol{x}) < 0,$$

where the inequality follows from the fact that $[\nabla^2_{xx} g(\boldsymbol{x})]^{-1}$ is positive definite (see Exercise 3.7).

Now, g can be approximated by its first-order Taylor expansion at $\boldsymbol{x}$:

$$g(\boldsymbol{x} + \theta\Delta\boldsymbol{x}) \approx g(\boldsymbol{x}) + \theta(\nabla_{\boldsymbol{x}} g(\boldsymbol{x}))^{\mathsf{T}}\Delta\boldsymbol{x} < g(\boldsymbol{x}).$$

Thus, $\Delta\boldsymbol{x}$ is a descent direction. ■

It can be shown that Algorithm 8.13 has a quadratic convergence under some certain circumstances (see for example Freund [2004]).

Exercises

8.1 Choose the correct answer for each of the following multiple-choice questions/items.

(*a*) Let n be a positive integer. The total number of flops when we multiply an $n^2 \times n$ matrix with an n-tuple of real numbers is

(*i*) $2n$. (*ii*) $2n^2$. (*iii*) $2n^3$. (*iv*) $2n^4$.

(*b*) Let n be a positive integer. If A and B are $n \times n^2$ and $n^2 \times n^3$ matrices, respectively, then the total number of flops after multiplying A with B is

(*i*) $2n^3$. (*ii*) $2n^4$. (*iii*) $2n^5$. (*iv*) $2n^6$.

(*c*) The best-case time complexities of Linear Search and Selection Sort are respectively

(*i*) $O(1)$ and $O(1)$.
(*ii*) $O(1)$ and $O(n)$.
(*iii*) $O(1)$ and $O(n^2)$.
(*iv*) $O(n)$ and $O(n^2)$.

(*d*) If the function `main` in Algorithm 8.14 calls not only the function `karger` but also the function `selection-sort` that we analyzed in Algorithm 8.8. What is the worst-case time complexity of the new resulting program?

(*i*) $O(mn + n^2)$.
(*ii*) $O(m^2 + n^2)$.
(*iii*) $O(mn + m^2)$.
(*iv*) $O(mn)$.

(*e*) The best-case time complexities of Binary Search and Merge Sort are respectively

(*i*) $O(1)$ and $O(1)$.
(*ii*) $O(1)$ and $O(n)$.
(*iii*) $O(1)$ and $O(n \log n)$.
(*iv*) $O(\log n)$ and $O(n \log n)$.

(*f*) Which of the following sorting algorithms, if any, is an example of an algorithm for which the best- and worst-case time complexities differ?

(*i*) Merge Sort.

(*ii*) Insertion Sort.

(*iii*) Selection Sort.

(*iv*) None of the above because, in all sorting algorithms, the best- and worst-case time complexities always coincide.

(*g*) Which of the following, if any, is an example of a searching algorithm for which the best- and worst-case time complexities coincide?

(*i*) Linear Search.

(*ii*) Selection Sort.

(*iii*) No such algorithm exists because the best- and worst-case time complexities always differ.

(*iv*) None of the above.

(*h*) Using Algorithm 8.10, we find that gcd(55, 99) =

(*i*) 9. (*ii*) 10. (*iii*) 11. (*iv*) 12.

(*i*) Using Algorithm 8.11, we find that gcd(840, 3220) =

(*i*) 120. (*ii*) 140. (*iii*) 160. (*iv*) 180.

8.2 When we perform the multiplication of two matrices, both of size $n \times n$, it typically involves $2n^3$ flops. However, in 1969, Strassen introduced a novel approach, documented in Strassen [1969], which accomplishes this task using an algorithm that scales with $O(n^s)$ flops, where the exponent s is approximately equal to 2.81 (computed using $\log_2 7$). Given that 2.81 is less than 3, Strassen's method proves to be more efficient than Algorithm 8.4 when the value of n is sufficiently large. Empirical tests have revealed that Strassen's method can even outperform the conventional approach when n is relatively small, such as around 100. Nevertheless, because the difference between 2.81 and 3 is relatively small, we need to increase the value of n substantially before Strassen's method demonstrates a significantly superior performance. Moreover, it's worth noting that there exist even faster matrix multiplication techniques. For instance, Coppersmith and Winograd introduced a method, as discussed in Higham [2002], that can multiply two $n \times n$ matrices in approximately $O(n^{2.376})$ flops. However, despite this impressive reduction in complexity, the computing community has observed that this algorithm does not outperform Strassen's method. How can we explain this observation?

8.3 Given the program in Algorithm 8.14. What is the worst-case time complexity of this program?

8.4 What is the worst-case time complexity of the program shown in Algorithm 8.15? Justify your answer.

8.5 Prove that $\gcd(a, b) = \gcd(a \bmod b, b)$ for any $a, b \in \mathbb{N}$.

8.6 Consider Algorithm 8.11. Assume that $a > b > 0$ and that $\gcd(a, b)$ requires $n \geq 1$ steps. Show that the smallest possible values of a and b are f_{n+2} and f_{n+1}, respectively, where

$\{f_n\}_{n=0}^{\infty}$ is the Fibonacci sequence defined as

$$f_n = f_{n-1} + f_{n-2}, \text{ for } n = 2, 3, 4, \ldots, \quad \text{with } f_0 = 0 \text{ and } f_1 = 1.$$

8.7 Write a code to find a root of the function $f(x) = \frac{1}{2}x^2 + x + 1 - e^x$ on $[0, 4]$. Apply Newton's method starting with $x_0 = 1$.

Algorithm 8.14: The algorithm of Exercise 8.3

```
# include <stdio. h >
int karger(int x, int m)
int linear-search(int A[ ], int n, int x)
// This line is left intentionally blank
main( ) begin
    int A[0 : n − 1], m
    a = karger(0, m)
    printf("Value a: \ n")
    return a
end
// This line is left intentionally blank
karger(int x, int m) begin
    for (i = 1; i ≤ m; i + +) do
        x += linear-search(A[ ], n, m)
    end
    return x
end
```

Algorithm 8.15: The algorithm of Exercise 8.4

```
# include <stdio. h >
int dinic(int x, int m)
int binary-search(int A[ ], int low, int high, int x)
int merge-sort(int A[ ], int first, int last)
// This line is left intentionally blank
main( ) begin
    int A[0 : n − 1], m
    A[n − 1] = dinic(0, m)
    B = merge-sort(A[ ], 0, n − 1)
    printf("Sorted array: \ n")
    return B
end
// This line is left intentionally blank
dinic(int x, int m) begin
    for (i = 1; i ≤ m; i∗ = 2) do
        x += binary-search(A[ ], 0, n − 1, m)
    end
    return x
end
```

Notes and sources

The origins of the numeric and array algorithms date back to ancient times. Euclid's algorithm, developed by the ancient Greek mathematician Euclid in the 3rd century BCE, is one of the earliest known numeric algorithms (see, for example, Heath et al. [1952]). Newton's method, named after Sir Isaac Newton in the 17th century, is also a historical numeric algorithm that laid the groundwork for the development of numerical analysis (refer to Newton [1687]). When it comes to array searching and sorting algorithms, the history is also rich. One of the earliest sorting algorithms, known as the "bubble sort", was described in 1956 by John Mauchly. Additionally, John von Neumann's work on merge sorting algorithms in the 1940s significantly contributed to the development of array sorting techniques (refer to Von Neumann and Burks [1966]).

This chapter described and analyzed fundamental algorithms that form the core of computer science and numerical analysis. Specifically, it covered various searching and sorting algorithms, including linear search, binary search, insertion sort, selection sort, and merge sort, which play a crucial role in data manipulation and organization. The chapter also delved into specific numerical methods, such as Euclid's algorithm for greatest common divisor calculations and Newton's method for solving linear and nonlinear systems.

As we conclude this chapter, it is worth noting that the cited references and others, such as Cusack and Santos [2021], Mott et al. [1986], Grimaldi [1999], Maurer and Ralston [2004], Hougardy and Vygen [2016], Kagaris and Tragoudas [2008], Greene and Knuth [2009], Sipser [2013], Watkins [1991], also serve as valuable sources of information pertaining to the subject matter covered in this chapter. Exercise 8.2 is due to Watkins [1991].

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CHAPTER 9

ELEMENTARY COMBINATORIAL ALGORITHMS

Chapter overview: Within this chapter, we introduce fundamental algorithms designed for the purpose of graph exploration and search. Our primary emphasis is placed on two key searching algorithms: The breadth-first search and depth-first search. We also describe the applications of these algorithms in computing spanning trees, computing shortest paths, testing bipartiteness, detecting cycles, and finding connected components. Additionally, we present the topological sort for ordering directed acyclic graphs. To adequately prepare for these algorithms, we first learn the techniques for graph representation. The chapter concludes with some exercises to encourage readers to enrich their understanding.

Keywords: Breadth-first search, Depth-first search, Topological sort

Contents

The graph-searching algorithms are the main focus of this chapter. Before studying graph-searching algorithms, it is crucial to first acquire an understanding of graph representation. This foundational knowledge is essential as it forms the basis for employing these algorithms effectively. Exploring the techniques for graph representation becomes the cornerstone of our initial section, serving as a fundamental stepping stone leading to the subsequent investigation and exploration of search algorithms.

9.1 Graph representations

There are two standard ways to represent a graph, namely the adjacency list and the adjacency matrix. They both can be used for directed and undirected graphs.

The adjacency list representation

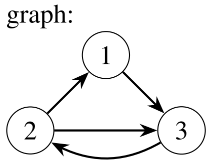
An adjacency list is a collection of unordered lists used to represent a (directed or undirected) graph. We write an array of $|V|$ lists. Each list describes the set of neighbors of a vertex in the graph. The list that describes the set of neighbors of a vertex u in a graph $G = (V, E)$ is denoted by $\text{Adj}[u]$ and is defined to contain all vertices v that are adjacent u. That is,

$$\text{Adj}[u] \triangleq \{v \in V : (u, v) \in E\}.$$

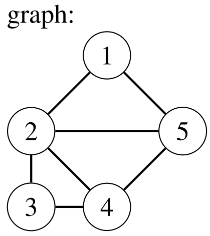
The following example illustrates this.

Example 9.1 Use adjacency lists to represent each of the following graphs.

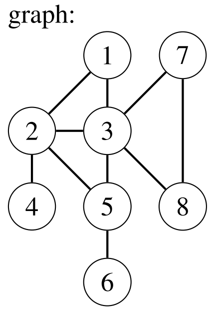
(a) The directed graph:



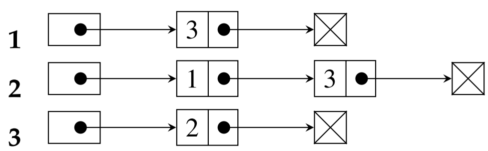
(b) The undirected graph:



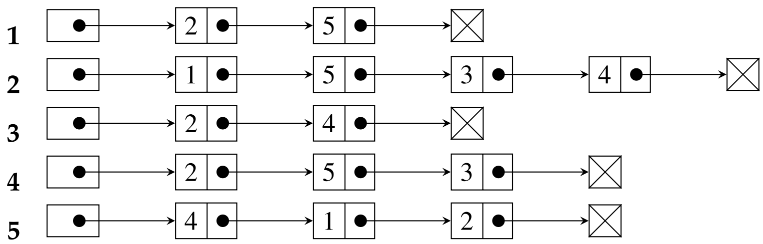
(c) The undirected graph:



Solution (a) Note that $\text{Adj}[1] = \{3\}, \text{Adj}[2] = \{1, 3\}$, and $\text{Adj}[3] = \{2\}$. So, the adjacency lists that represent the graph are:



(*b*) Note that Adj[1] = $\{2, 5\}$, Adj[2] = $\{1, 5, 3, 4\}$, Adj[3] = $\{2, 4\}$, Adj[4] = $\{2, 5, 3\}$, and Adj[5] = $\{4, 1, 2\}$. So, the adjacency list representation for the graph is:



(*c*) This part is left as an exercise for the reader (see Exercise 9.3 (*a*)).

Note that, for an undirected graph, in the adjacency list there are two representations of each edge in the graph.

The adjacency matrix representation

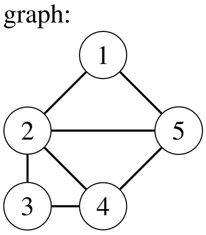
An adjacency matrix is a square binary matrix[1] used to represent a (directed or undirected) graph. We assume that the vertices of the given graph $G = (V, E)$ are numbered $1, 2, \ldots, |V|$. The representation consists of a $(0, 1)$-matrix $A_{|V| \times |V|}$, where the entry of the ith row and jth column in the matrix is labeled a_{ij} and is given by

$$a_{ij} \triangleq \begin{cases} 1, & \text{if } (i, j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

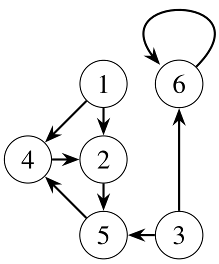
The following example illustrates this.

Example 9.2 Use adjacency matrices to represent each of the following graphs.

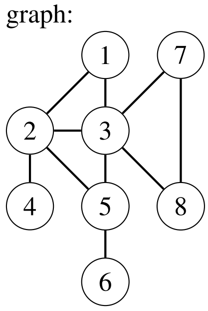
(*a*) The undirected graph:



(*b*) The directed graph:



(*c*) The undirected graph:



Solution The adjacency matrices that represent the graphs are as follow:

[1] A binary matrix is a matrix with entries from the set $\{0, 1\}$.

(a)

	1	**2**	**3**	**4**	**5**
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

(b)

	1	**2**	**3**	**4**	**5**	**6**
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

(c) This part is left as an exercise for the reader (see Exercise 9.3 (b)).

It is worth noting that in the case of a simple graph, the adjacency matrix exhibits a specific characteristic: All the diagonal elements of the matrix are zero. This is because, in a simple graph, no vertex is connected to itself directly, so the diagonal entries of the adjacency matrix must be zero. Note also that each edge in the graph is represented twice. This is due to the fact that in an undirected graph, the connection between two vertices is mutual. Consequently, the adjacency matrix of an undirected graph turns out to be symmetric, as it contains mirrored entries above and below the main diagonal. This symmetry further underscores the fundamental difference between directed and undirected graphs when analyzing and manipulating graph data represented by adjacency matrices.

We end this section with Table 9.1, which shows some differences between the adjacency list representation and the adjacency matrix representation. From Table 9.1, we can see that the choice between these representations depends on the graph's characteristics and the specific operations or algorithms being applied.

	Adjacency list	Adjacency matrix
Space Complexity:	$O(V+E)$ space	$O(V^2)$ space
Time to list all vertices adjacent to $u \in V$:	$O(\deg(u))$ time	$O(V)$ time
Time to determine if an edge $(u, v) \in E$:	$O(\deg(u))$ time	$O(1)$ time
Is it efficient for sparse matrices?	Yes	No
Is it efficient for dense matrices?	No	Yes
Is it a quick way to list the vertices adjacent to a given vertex?	Yes	No
Is it a quick way to determine if there is an edge between two vertices?	No	Yes

Table 9.1: Comparison between the adjacency list and adjacency matrix representations of a given graph $G = (V, E)$.

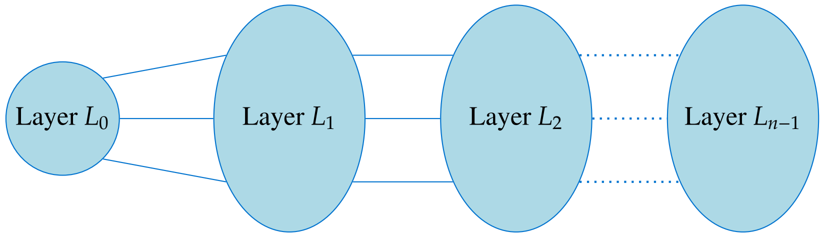


Figure 9.1: The breadth-first search scheme.

9.2 Breadth-first search algorithm

Breadth-first search is a method for exploring a graph. In this section, we describe and analyze this method. In the next section, we will describe some of its applications.

In a breadth-first search, we explore outward from a source vertex, say s, of a graph $G = (V, E)$ in all possible directions, adding vertices one layer (or level) at a time. Breadth-first search is used for both directed and undirected graphs. Figure 9.1 shows the breadth-first search scheme, where L_0 represents the zeroth move (which only includes the a source vertex S), L_1 represents the first move, L_2 represents the second move, etc.

In Figure 9.1, $L_0 \triangleq \{s\}$, $L_1 \triangleq \text{Adj}[s]$ (i.e., the neighbors of L_0), L_2 contains all vertices that do not belong L_0 or L_1 and that have an edge to a vertex in L_1, and L_{i+1} contains all vertices that do not belong to an earlier layer, which is needed to avoid duplication, and that have an edge to a vertex in L_i. We have the following remark.

Remark 9.1 *For each i, the layer L_i consists of all vertices at distance exactly i from the source vertex s.*

The following example explains the steps outlined in the breadth-first search method.

Example 9.3 For each of the following graphs, use the breadth-first search method to determine the smallest number of layers (or hopes) among the vertices starting from vertex s.

(a) The undirected graph in the left-hand side of Figure 9.2 (take s to be the vertex 1).

(b) The directed graph in the right-hand side of in Figure 9.2.

Solution (a) In this part, the breadth-first search method is explained move-by-move as shown in Figure 9.3. We find that the smallest number of layers among the vertices starting from the vertex s (vertex 1) equals 4.

(b) As shown in Figure 9.4, the smallest number of layers among the vertices starting from the vertex s equals 6.

■

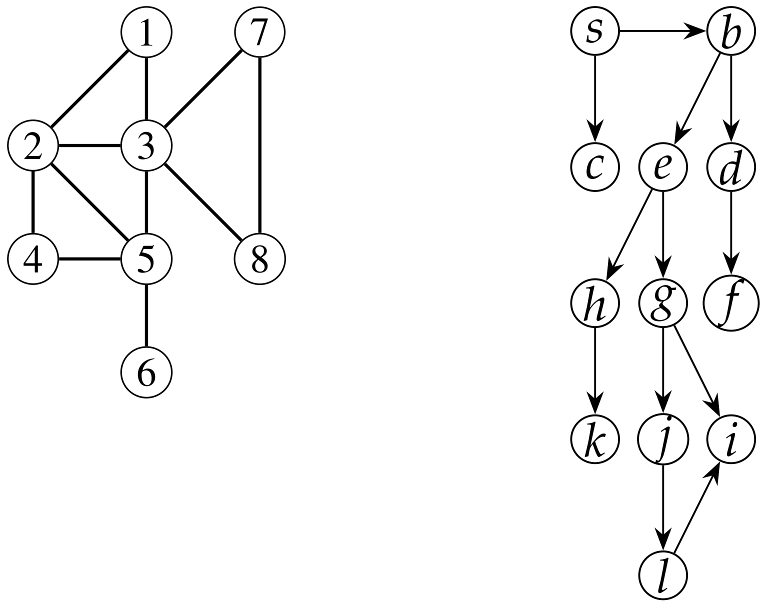


Figure 9.2: The graphs of Example 9.3.

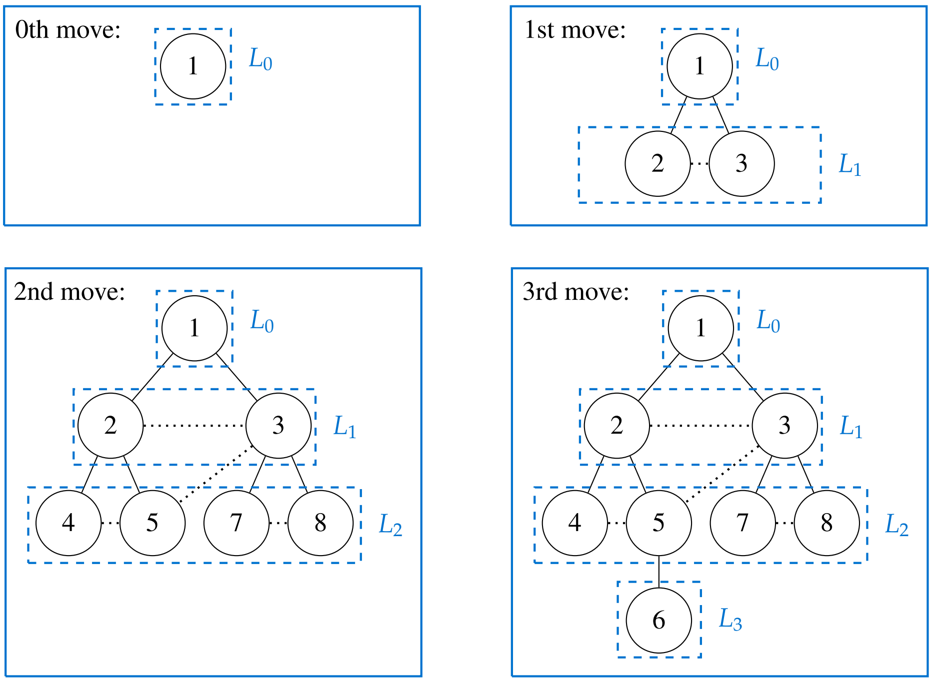


Figure 9.3: Illustrating the progress of breadth-first search on the graph of Example 9.3 (*a*).

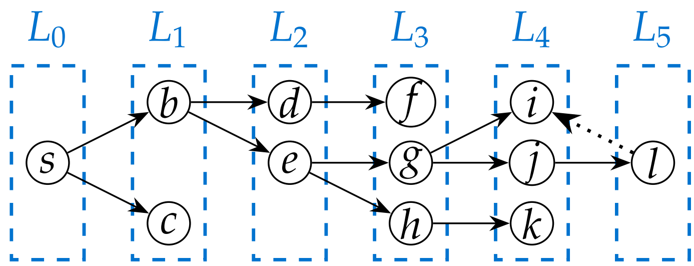


Figure 9.4: Breadth-first search on the digraph of Example 9.3 (b).

Algorithm 9.1: Breadth-first search algorithm

```
BFS(s, Adj)
begin
    level = {s : 0}
    parent = {s : None}
    i = 1
    frontier = [s]
    while frontier do
        next = [ ]
        for u in frontier do
            for v in Adj[u] do
                if v not in level then
                    level[v] = i
                    parent[v] = u
                    next.append(v)
                end
            end
        end
        frontier = next
        i+ = 1
    end
    return parent
end
```

A version of a breadth-first search algorithm is presented in Algorithm 9.1.

Theorem 9.1 gives the running time of Algorithm 9.1 on an input graph $G = (V, E)$ given in its adjacency lists.

Theorem 9.1 *Algorithm 9.1 runs in $O(V + E)$ time.*

Proof Note that, on an input graph $G = (V, E)$ given in its adjacency lists, we have at most $|V|$ layers (lists) and each vertex occurs on at most one layer and is captured at most once. So, the while-loop and the first for-loop in Algorithm 9.1 are executed at most $|V|$ times. Then we have $O(V)$. The second for-loop in Algorithm 9.1 runs $|E_{\text{adj}}|$ times, where E_{adj} is the set of edges that are incident to current vertex. The reason is that every vertex captured at most once and we examine (u, v) only when u is captured. Note that every edge examined at most once if G is directed, and at most twice if G is undirected. Therefore, from Theorem 4.1 and 4.14,

$|V| \times O(E_{\text{adj}})$ is $O(E)$. The other lines in Algorithm 9.1 take constant time to execute. Thus, the total running time is $|V| \times (O(1) + O(E_{\text{adj}}))$ which turns out to $O(V + E)$. This completes the proof. ■

From Theorem 9.1, we conclude that the breadth-first search algorithm runs in time linear in size of the adjacency list representation of G.

9.3 Applications of breadth-first search

The breadth-first search has some applications, and there are some $O(V + E)$-time algorithms based on the breadth-first search algorithm for the following problems: Computing a spanning tree (forest) of a graph, computing a path with the minimum number of edges between start vertex and assigned vertex or reporting that no such path exists, testing bipartiteness of an input graph, testing whether a graph is connected, and finding the connected component that contains a given vertex. Below we describe these applications.

Computing spanning trees (forests)

For a given graph $G = (V, E)$, the breadth-first search algorithm inherently creates a spanning tree that has its root at the source vertex. If the given graph G is disconnected, then the breadth-first search produces a spanning forest of G. The tree (forest) produced by performing a breadth-first search is called breadth-first tree (forest), and its edges are called breadth-first edges.

For the given graph shown to the left of Figure 9.5, we compute a spanning tree by drawing the produced breadth-first tree. As shown to the right of Figure 9.5, the breadth-first edges are colored blue and the vertices are enumerated in alphabetical order starting at a (the src vertex), where each vertex is also super-indexed by (i) to mean that it occurs on the layer L_i for some i. As breadth-first search does not define the order of the neighbors, we assume that in this graph the children of each vertex are ordered from top to bottom.

Figure 9.5 leads us to deduce the following remark.

Remark 9.2 *If (u, v) is a breadth-first edge, then the levels of u and v differ by at most 1.*

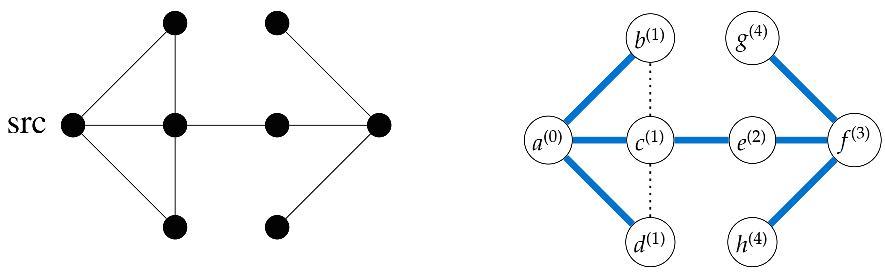


Figure 9.5: A graph and its breadth-first tree.

Computing shortest paths

Finding shortest paths is one of the applications of the breadth-first search algorithm. The shortest path problem is a well-known problem in graph theory and it has numerous applications in the real world. It involves finding a path from a source vertex to a destination vertex which has the minimum number of edges (i.e., with the least length) among all such paths, or report that no such path exists. The following remark gives evidence when such existence is affirmed.

Remark 9.3 *There is a path from a (source) vertex s to a vertex t if and only if t appears in some layer while performing the breadth-first search algorithm.*

The procedure in following workflow, followed by an example taken from Section 22.2 in Cormen et al. [2001] with some modifications, will show us how to find the shortest path(s) in an input graph G given by its adjacency representation.

Workflow 9.1 *We find the shortest path(s) from a source vertex to a destination vertex t in an input graph G by following two steps:*

(i) *Run a breadth-first search starting at the vertex s.*

(ii) *Construct a path whose edges are all from breadth-first tree edges by moving backward from vertex t to its predecessors along the tree edges until we reach vertex s.*

From Workflow 9.1, we conclude that

$$(s, \ldots, \mathrm{parent}[\mathrm{parent}[t]], \mathrm{parent}[t], t)$$

is a shortest path from vertex s to vertex t. We also conclude that the length of this path equals level[t]. That is, if vertex t occurs on a layer L_k (i.e., level k), then the length of the above shortest path is k.

Example 9.4 As shown Figure 9.6, by performing the breadth-first search algorithm, the layers obtained among the vertices of the graph in Figure 9.6 (i), starting from vertex s, are shown in Figure 9.6 (ii). The smallest number of such layers is 4. The graphs in Figure 9.6 (iii)-(v) visually show step-by-step how to find the shortest path from vertex s to vertex v. It is clear that parent[v] = c, parent[parent[v]] = parent[c] = x, and parent[x] = s. So, (s, x, c, v) is the shortest path from s to v. Because vertex v occurs in the layer L_3, the length of the above shortest path is 3. The graph in Figure 9.6 (vi) shows that (s, x, c, f) is a shortest path from vertex s to vertex f.

Note that the shortest path may not be unique (in some cases may not even exist (see Remark 9.3)). This can be seen in Figure 9.6 (vi) and ($viii$). In fact, Figure 9.6 (vii) shows another breadth-first tree for the graph shown in Figure 9.6 (i), and accordingly Figure 9.6 ($viii$) shows another shortest path from vertex s to vertex f, namely the path (s, x, d, f), which is different from that shown in Figure 9.6 (vi). ■

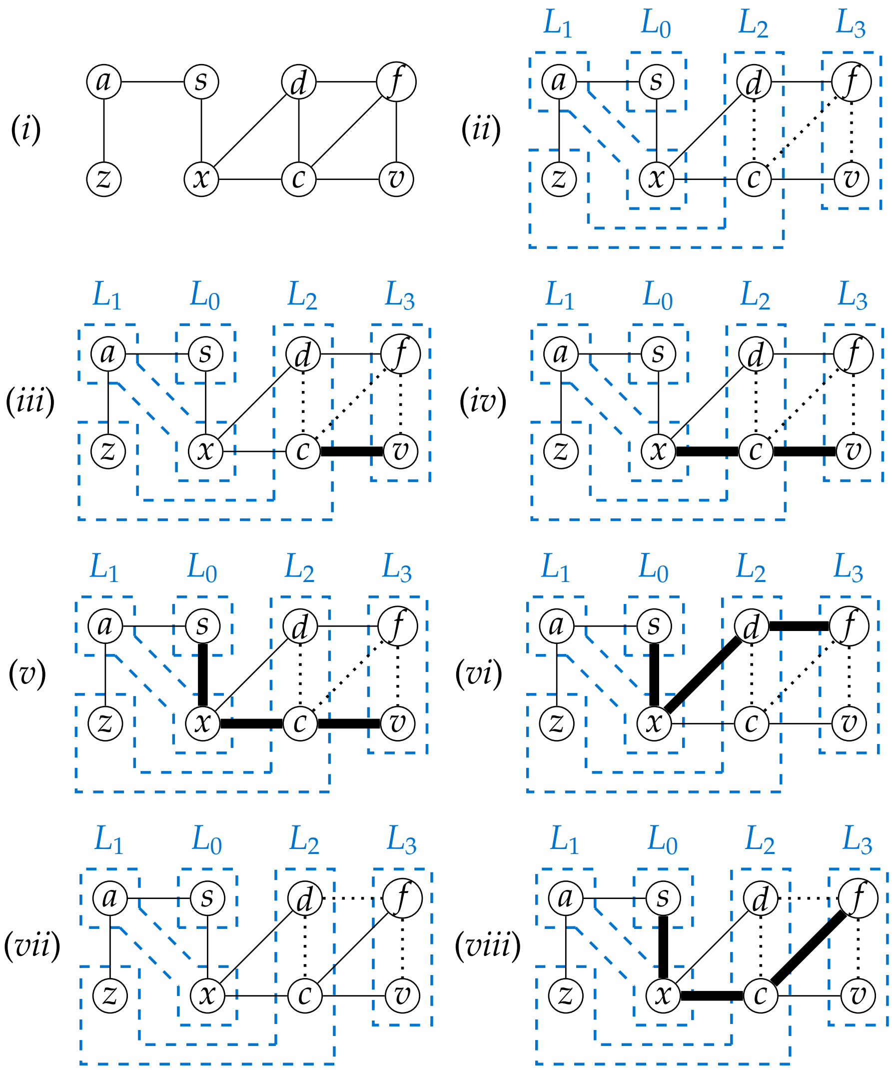


Figure 9.6: Finding shortest paths by running breadth-first searches.

Testing bipartiteness

Breadth-first search can be used to test bipartiteness. The algorithm commences its exploration from any vertex and assigns alternating labels to the visited vertices during the search process. That is, give label 0 to the starting vertex, 1 to all its neighbors, 0 to those neighbors' neighbors, and so on. Should a vertex, at any point during the process, possess neighboring vertices with identical labels (indicating they have been visited), it signifies that the graph is not bipartite. Conversely, if the search concludes without encountering such a scenario, it implies that the graph is bipartite.

Workflow 9.2 uses graph coloring (see Theorem 4.11) and breadth-first search to test bipartiteness. We visually show the steps of this workflow in Figure 9.7.

Workflow 9.2 *We test the bipartiteness of an input graph G by following five steps:*

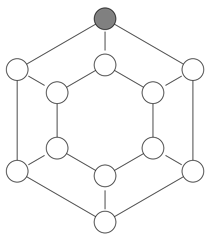
(*i*) *Assign a color (say red) to a random source vertex.*

(*ii*) *Assign all the neighbors of the above vertex another color (say blue).*

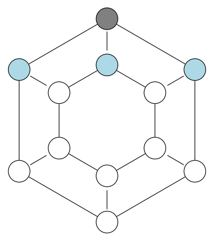
(*iii*) *Take one neighbour at a time to assign all the neighbour's neighbors the color red.*

(*iv*) *Continue in this manner till all the vertices have been assigned a color.*

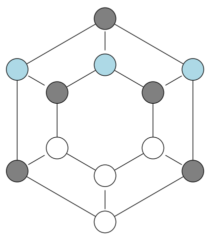
(*v*) *If at any stage, we find a neighbour which has been assigned the same color as that of the current vertex, stop the process. The graph cannot be colored using two colors. Thus the graph is not bipartite.*



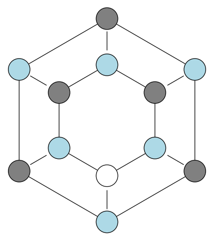
(*i*) We test the bipartiteness of this graph. Color source vertex, say gray.



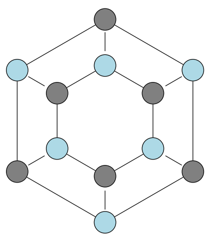
(*ii*) Assign neighbors of the source vertex another color, say light blue.



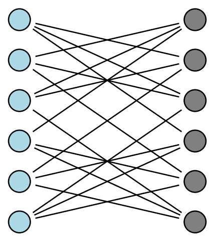
(*iii*) Assign all neighbors of the vertices colored in (*ii*) the color gray.



(*iv*) Assign all neighbors of the vertices colored in (*iii*) the color light blue.



(*v*) Repeat until all vertices are colored, or a conflicting assignment occurs.



(*vi*) Since no conflicting evidence was found, the graph is bipartite.

Figure 9.7: Illustrating the progress of breadth-first search on testing bipartiteness of a graph.

Finding connected components is another application of breadth-first search that is not covered in this section. In the next two sections, we present depth-first search as an alternative to breadth-first search. Since finding connected components is an application of both breadth- and depth-first searches and it can be described in both methods by the same scheme, finding connected components will be discussed together with the applications of depth-first search.

9.4 Depth-first search algorithm

In the previous two sections, we have presented the breadth-first search method for exploring a graph. In this section, we present depth-first search as another method for exploring a graph. In the next section, we will describe some of its applications.

Recall that the breadth-first search involves a systematic exploration starting from a source vertex, extending in all feasible directions, and including vertices one layer at a time. In contrast, depth-first search distinguishes itself by visiting vertices in sequence until it encounters a "dead end," after which it retraces its steps (backtracking); see Figure 9.8. Like breadth-first search, depth-first search is used for both directed and undirected graphs.

In a depth-first search, we follow a path starting from a source vertex, say s, of a graph $G = (V, E)$ until we get stuck, then we backtrack until we reach unexplored neighbor, being careful not to repeat a vertex. See Figure 9.9.

Example 9.5 explains the steps outlined in the depth-first search method. A version of a depth-first search algorithm is presented in Algorithm 9.2.

Algorithm 9.2: Depth-first search algorithm

```
begin
    parent = {s : None}
    DFS-visit(V, Adj, s)
    begin
        for v in Adj[s] do
            if v not in parent then
                parent[v] = s
                DFS-visit(V, Adj, v)
            end
        end
    end
    DFS(V, Adj)
    begin
        parent = { }
        for s in V do
            if s not in parent then
                parent[s] = None
                DFS-visit(V, Adj, s)
            end
        end
    end
end
```

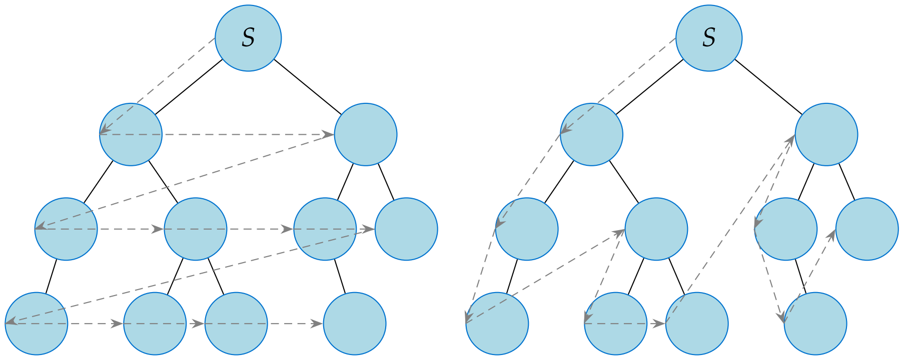


Figure 9.8: The operation of breadth-first search (left) versus that of depth-first search (right) on an undirected graph.

In the context of the depth-first search, we make assumptions about the arrangement of a vertex's children based on their positions relative to the parent vertex. For vertices placed horizontally to the left or right of the parent, we consider their order from top to bottom as illustrated in the digraph shown to the right. In the case of vertices positioned vertically above or below the parent, we establish their order from left to right as illustrated in the graph shown to the right of Figure 9.8. This means that while exploring the neighborhood, we assume the children's arrangement moves from top to bottom and left to right.

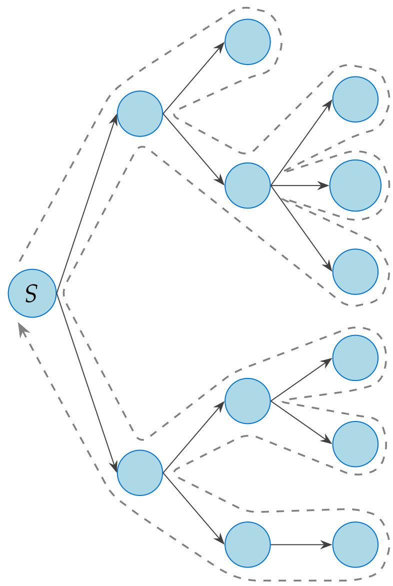
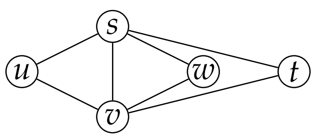


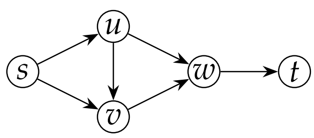
Figure 9.9: The depth-first search scheme.

Example 9.5 Show the progress of the depth-first search method on each of the following graphs starting from vertex s.

(a) The undirected graph:



(b) The directed graph:



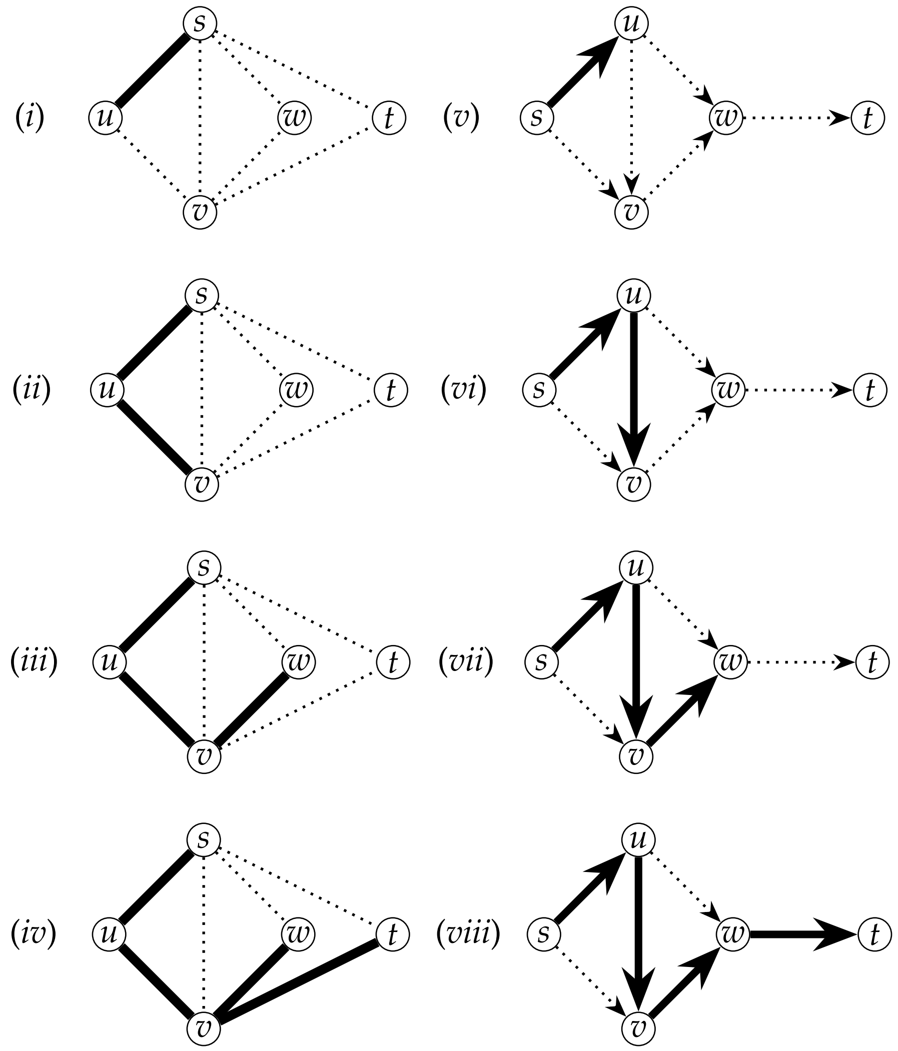


Figure 9.10: Visualization of the progress of the depth-first search method on the graphs of Example 9.5.

Solution (*a*) The progress of the depth-first search method on the graph given in item (*a*) is visualized in Figure 9.10 (*i*)-(*iv*).

(*b*) The progress of the depth-first search method on the graph given in item (*b*) is visualized in Figure 9.10 (*v*)-(*viii*).

■

Theorem 9.2 gives the running time of Algorithm 9.2 on an input graph $G = (V, E)$ given in its adjacency lists. This theorem concludes that the depth-first search algorithm runs in time linear in size of the adjacency list representation of G.

Theorem 9.2 *Algorithm 9.2 runs in $O(V + E)$ time.*

Proof Note that the function `DFS-Visit` in Algorithm 9.2 calls itself once for each vertex in V since each vertex is added to the resulting tree at most once. The for-loop within `DFS-Visit` function is executed a total of $|E|$ times in the case of a directed graph, or $2|E|$ times for an undirected graph, as each edge is explored exactly once. The for-loop in the function `DFS` in Algorithm 9.2 adds $O(V)$ time. Therefore, the total running time is $O(V + E)$. This completes the proof. ■

9.5 Applications of depth-first search

The depth-first search has some applications, and there are some $O(V+E)$-time algorithms based on the depth-first search algorithm for the following problems: Computing a spanning tree (forest) of a graph, detecting a cycle in a graph or reporting that no such cycle exists, testing whether a graph is connected, and finding the connected component that contains a given vertex. Below we describe some of these applications.

Computing spanning trees (forests)

For a given graph $G = (V, E)$, the depth-first search algorithm directly produces a spanning tree rooted at the source vertex. If the given graph G is disconnected, then the depth-first search produces a spanning forest of G. The tree (forest) produced by performing a depth-first search is called a depth-first tree (forest).

For a directed graph G, we define the following four edge types in terms of its depth-first tree (forest):

- A tree edge: An edge of a depth-first tree (forest) of G.
- A back edge: A non-tree edge connecting a vertex to one of its ancestors (but not parents) in a depth-first tree (forest) of G. Note that a self-loop is a back edge.
- A forward edge: A non-tree edge connecting a vertex to one of its descendants (but not children) in a depth-first tree (forest) of G.
- A cross edge: A non-tree edge connecting some pair of vertices that have no ancestor-descendent relationship in a depth-first tree (forest) of G.

For an undirected graph G, there are no forward edges because they become back edges when we move in the opposite direction, and there are no cross edges because every edge of an undirected graph G must connect an ancestor vertex to a descendant vertex. This leads us to the following remark.

Remark 9.4 *For an undirected graph G, there are tree edges and back edges only in terms of its depth-first tree (forest). No forward or cross edges.*

For the given digraph shown to the left of Figure 9.11, we compute a spanning forest by drawing the produced depth-first forest. The computed spanning forest consists of two trees with two different roots.

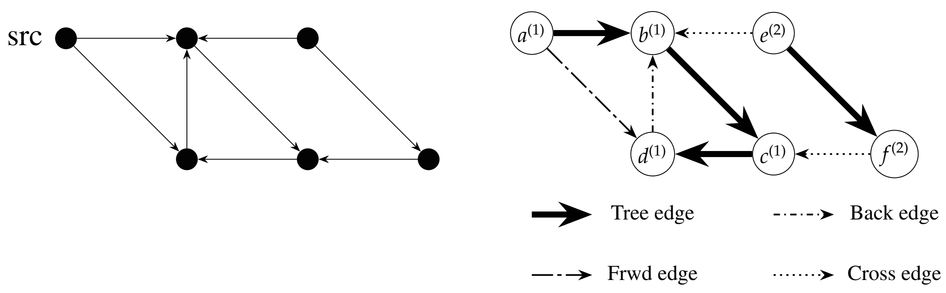


Figure 9.11: A graph and its depth-first forest.

As shown to the right of Figure 9.11, the tree edges are colored blue and the vertices are enumerated in alphabetical order starting at a (the src vertex), where each vertex is also super-indexed either by (1) to mean that it is located on the first tree that has root a, or by (2) to mean that it is located on the second tree that has root e. Note that (d, b) is a back edge, (a, d) is a forward edge, and (g, e) and (d, c) are cross edges.

Detecting cycles

Detecting cycles is one of the applications of the depth-first search algorithm. The following remark gives evidence when the existence a cycle in an input graph is affirmed.

Theorem 9.3 *A graph G has a cycle if and only if there exists a back edge in G after performing any depth-first search.*

Proof To begin, let us establish that if a graph $G = (V, E)$ contains a cycle, then it also includes a back edge. By performing a depth-first search on G, we can arrange the vertices in a descending order based on their appearance in the depth-first tree, starting from a source vertex. It is important to note that tree edges, forward edges, and cross edges all progress forward in this ordering. Conversely, back edges point backward towards an earlier vertex within this sequence. Given that, according to our assumption, these vertices form a cycle, it logically follows that there must be at least one back edge. Otherwise, there would be no means to traverse from a later vertex in the ordering back to an earlier vertex, which directly contradicts the initial assumption of the existence of a cycle.

Conversely, let us assume that $(u, v) \in E$ represents a back edge. Following the definition of back edges, there must exist a path P from vertex v to vertex u within the depth-first tree. Now, let's define a new set C as $C = P + (u, v)$; this set C forms a cycle within graph G. The proof is complete. ■

Recall that a directed acyclic graph (DAG) is a digraph that has no directed cycles. Theorem 9.3 leads immediately us to the following corollary.

Corollary 9.1 *A digraph is a DAG if and only if it has no back edges after performing any depth-first search.*

Workflow 9.3 tells us how to detect and report a cycle in an input graph G.

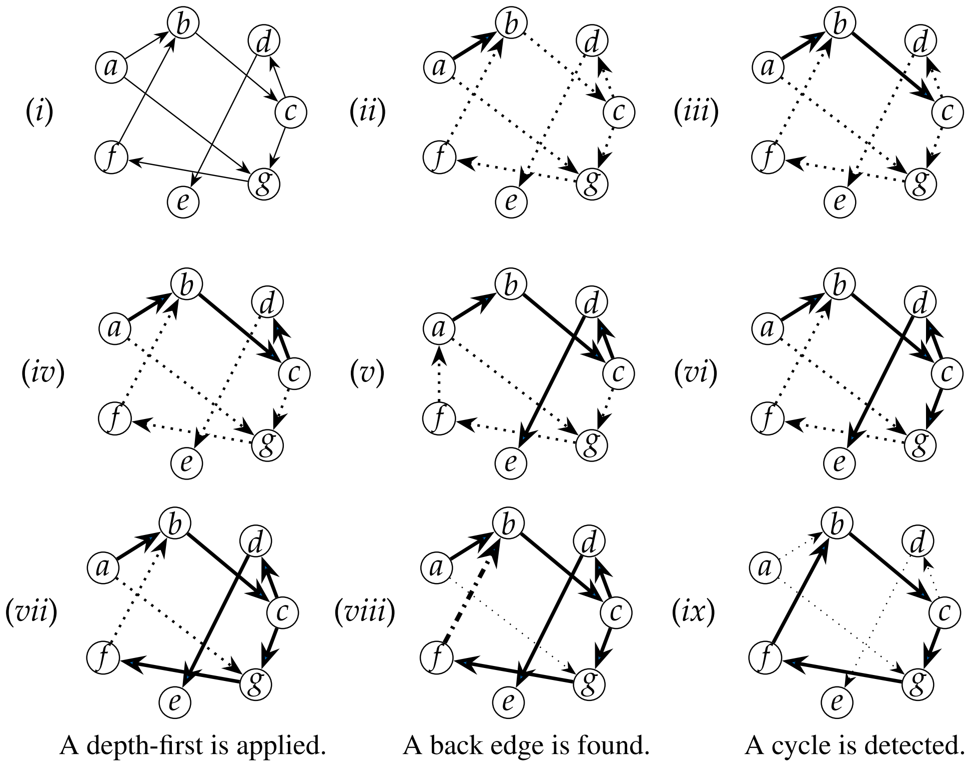


Figure 9.12: Detecting a cycle by running a depth-first search.

Workflow 9.3 *We detect and report a cycle in an input graph G by four steps:*

(i) *Run a depth-first search.*

(ii) *If at any stage, we find a back edge* (u, v)*, stop the depth-first search process. A cycle is detected in the graph.*

(iii) *Find the (unique) path P from v to u in the produced depth-first tree.*

(iv) *Add the edge* (u, v) *to the path P to localize the detected cycle.*

We visually show the steps of Workflow 9.3 in Figure 9.12. After running a depth-first search on the graph shown in Figure 9.12 (i), the back edge (f, b) is found, as illustrated in Figure 9.12 ($viii$), and the cycle (b, c, g, f) is detected, as localized in Figure 9.12 (ix).

Finding connected components

Depth- and breadth-first searches can be used to test if a graph is connected as follows: Starting from a random source vertex, if on termination of algorithm, all vertices are visited, then the graph is connected, otherwise it is disconnected. The following remark proves the correctness of this assertion.

Remark 9.5 *Depth- and breadth-first searches visit all vertices that are reachable from source vertex s.*

Proof We prove this remark by induction. Firstly, both depth-first and breadth-first searches unquestionably cover all vertices at level 0, which comprises only the vertex s itself. This serves as our base case. Next, we consider the vertices located at the minimum distance i from vertex s and label them as "level i" vertices. If either the depth-first or breadth-first search successfully reaches all the vertices at level i, then it naturally follows that they will also reach all the vertices at level $i + 1$, because every vertex at a distance of $i + 1$ from s must be connected to some vertex at a distance of i from s. This constitutes the inductive step and completes the proof. ∎

Depth- and breadth-first searches can also be used to find the connected components in an input graph (i.e., finding all vertices within each connected component). To find a connected component within a graph, we initiate the process by selecting an arbitrary source vertex and commence either a depth-first or breadth-first search from that vertex. All the vertices reachable from this chosen vertex constitute a single connected component. To uncover all the connected components within the graph, we systematically examine every vertex, discovering their connected components one by one through graph traversal. It is important to note, however, that there is no need to initiate a search from a vertex, say, v, if we have previously identified it as a part of a previously found connected component. Therefore, by maintaining a record of encountered vertices, we can streamline the process, ensuring that we only need to conduct one search for each connected component.

The connected component, say R, which encompasses an initial vertex s, can be determined by executing the code outlined in Algorithm 9.3. During a search starting from a specific vertex s, it is evident that we will never extend our reach to any vertices situated beyond the boundaries of the connected component when utilizing depth-first or breadth-first search methods. Consequently, by Remark 9.5, Algorithm 9.3 will find each connected component correctly. That is, upon termination, R will consist of vertices that are reachable from vertex s.

Algorithm 9.3: Finding connected components

```
R = {s}
while (there is an edge (u, v) where u ∈ R and v ∉ R) do
    add v to R
end
return R
```

There are other applications of depth-first search such as testing bipartiteness and topological sorting. We decided to not include testing bipartiteness here because it was previously described as an application of breadth-first search. Topological sorting will presented in the next section as an application of depth-first search.

9.6 Topological sort

A topological ordering or sorting of an n-vertex directed graph $G = (V, E)$ is an ordering of its vertices as $v_1, v_2, \ldots, v_n$ so that for every edge $(v_i, v_j) \in E$ we have $i < j$.

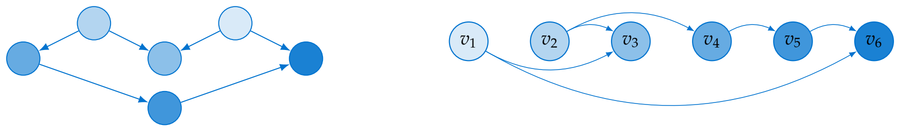


Figure 9.13: A directed graph (left) and its topological ordering (right).

Less informally, a topological ordering arranges the vertices of a directed graph along a horizontal line so that all edges point from left to right; see Figure 9.13. Formally, we have the following definition.

Definition 9.1 *A topological ordering of a directed graph* $G = (V, E)$ *is a total order "$\prec$" such that* $u \prec v$ *for every edge* $(u, v) \in E$.

Topological sorting is used to schedule tasks under precedence constraints. Assume that we have some tasks to execute, but certain tasks must be completed before others can begin. This dependency between tasks can be visualized as a directed graph, and topological sorting comes to the rescue in scheduling tasks while adhering to these precedence constraints. Our objective is to ascertain whether there exists an order in which this set of tasks can be successfully executed, taking into account the specified constraints. This order, if it exists, essentially constitutes a topological sort of a digraph G. In this graph, the vertices represent individual tasks, while the edges signify the precedence constraints governing the sequence in which these tasks must be undertaken. Edge (v_i, v_{i+1}), for instance, means that task v_i must be completed before task v_{i+1} can be started; see Figure 9.14. Topological sort is often useful in selecting course prerequisites of a target program, for which the course v_i must be taken before course v_{i+1}. It is generally useful in scheduling jobs in their proper sequence, for which the output of job v_i is needed to determine the input of job v_{i+1}.

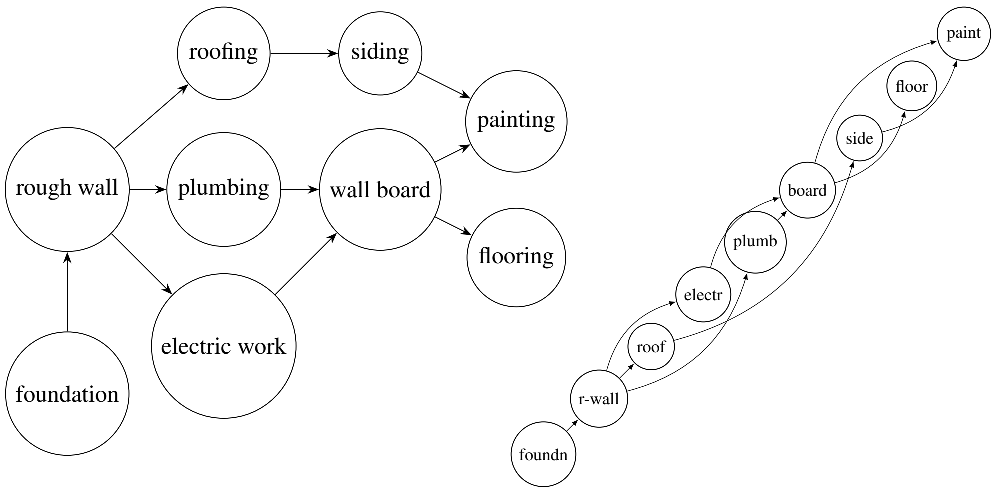


Figure 9.14: A house construction order, indicating which to execute first, is visualized as a digraph (left), while topological sorting reveals precedence constraints (right).

Note that the graphs in Figures 9.13 and 9.14 are DAGs (recall that a DAG is a directed acyclic graph). We have the following theorem.

Theorem 9.4 *A digraph has a topological order if and only if it is a DAG.*

Proof Let us begin by demonstrating that if a digraph G possesses a topological order, then G is a DAG. Contrarily, suppose that G does have a directed cycle. In such a scenario, it becomes evident that obtaining a topological ordering for G is impossible. This is because the rightmost vertex within the cycle would inevitably have an edge pointing towards the left, directly contradicting the existence of a topological order for G.

Conversely, consider a DAG represented as $G = (V, E)$. Our goal now is to establish that G indeed possesses a topological order. We begin by considering an arbitrary ordering designated as "$\prec$" for the vertices in G. In instances where $v \prec u$ holds true for any edge $(u, v) \in E$, it signifies that within G, there exists a directed path originating from vertex v and concluding at vertex u. Consequently, it also indicates the presence of a directed cycle encompassing the edge (u, v). Alternatively, if G lacks cycles (i.e., is acyclic), then $u \prec v$ is applicable for any edge $(u, v) \in E$. In simpler terms, any DAG inherently possesses a topological order. ■

The three-step procedure in the following workflow, followed by an example, will learn us how to compute a topological ordering of a DAG.

Workflow 9.4 *We compute a topological ordering of a DAG $G = (V, E)$ by following three steps:*

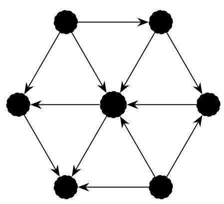
(i) *Find a vertex $v \in V$ with no incoming edges and order it first.*

(ii) *Delete v from G.*

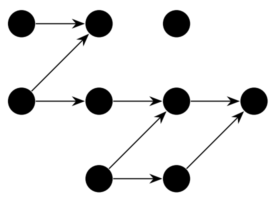
(iii) *Recursively compute a topological ordering of $G - \{v\}$ and append this order after v.*

Example 9.6 Compute a topological sort for each of the following DAGs.

(a) The connected digraph:



(b) The disconnected digraph:



Solution The progress of Workflow 9.4 steps to compute a topological sort for the DAG given in part (a) is visualized in Figure 9.15. A topological sort for the DAG given in part (b) is computed and shown in Figure 9.16. ■

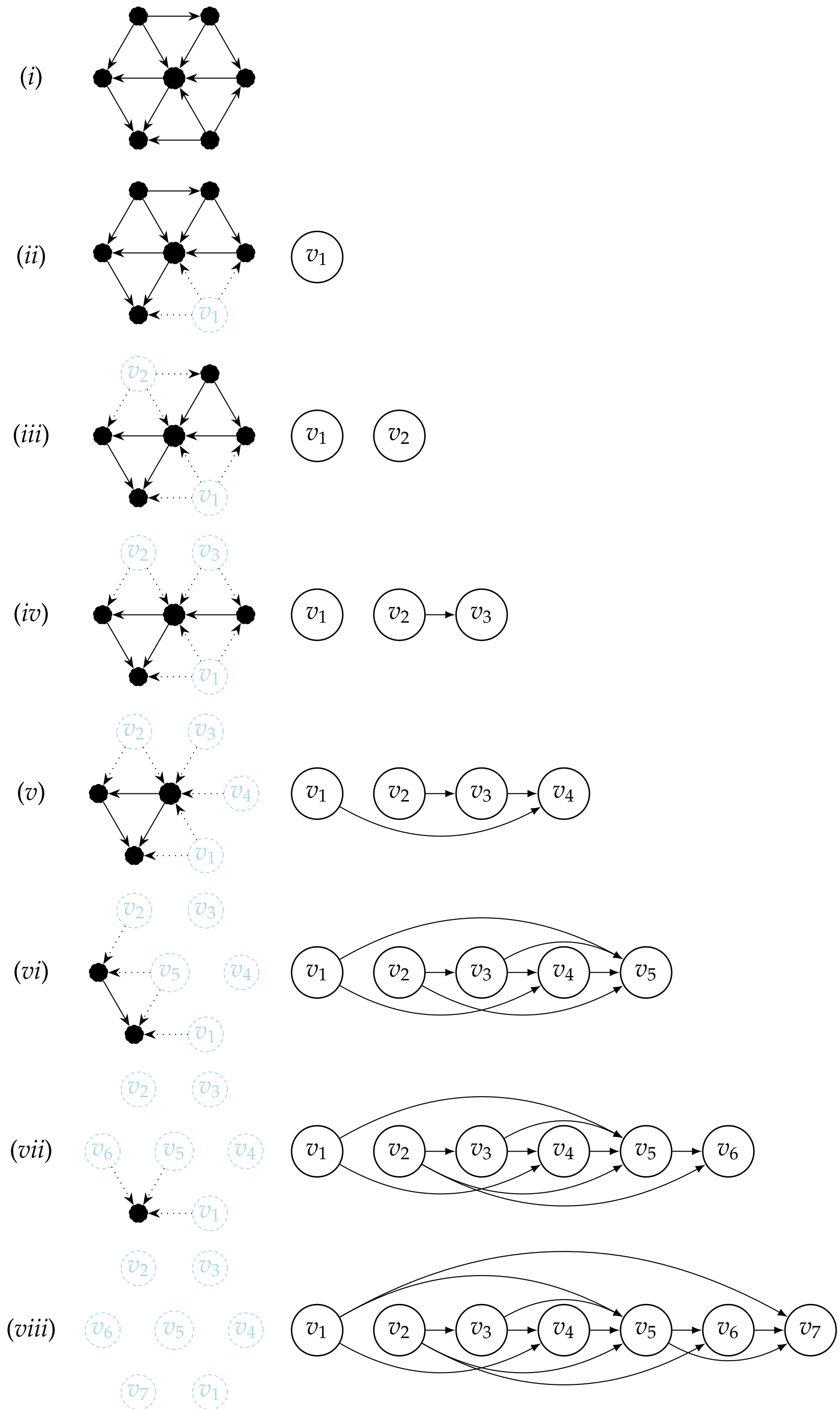


Figure 9.15: Visualization of the progress of Workflow 9.4 steps to compute a topological ordering for the DAG in Example 9.6 (*a*).

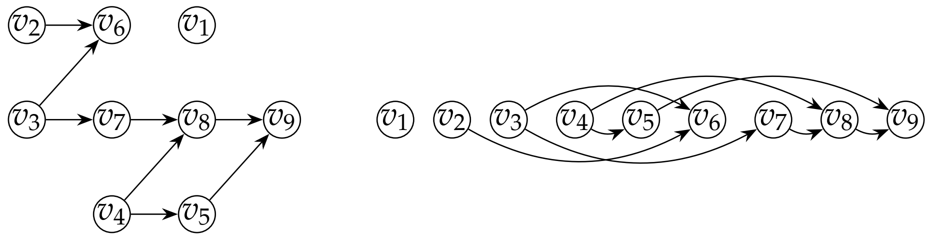


Figure 9.16: The DAG in Example 9.6 (*b*) and its topological ordering (shown to the left).

It is not hard to find that the topological sort has finishing time of the reverse of depth-first search. We have shown in Theorem 9.2 that the depth-first search runs in $O(V + E)$ time. Hence we can conclude the following corollary.

Corollary 9.2 *The topological sort runs in* $O(V + E)$ *time.*

In conclusion, topological sorting stands as a fundamental concept in graph theory, providing a systematic arrangement of vertices in a DAG. Its practical applications span various fields, aiding in the optimization of scheduling, task sequencing, and dependency resolution.

Exercises

9.1 Choose the correct answer for each of the following multiple-choice questions/items.

(*a*) Which one of the following statements is false?

(*i*) The adjacency list is preferred when the graph is a cycle while the adjacency matrix is preferred when the graph is complete.

(*ii*) The adjacency list is faster than the adjacency matrix for listing the vertices adjacent to a given vertex.

(*iii*) The adjacency matrix is faster than the adjacency list for determining if there is an edge between two vertices.

(*iv*) The adjacency matrix is preferred when the graph is sparse while the adjacency list is preferred when the graph is dense.

(*b*) If we use a breadth-first search for testing bipartiteness of an input graph, we stop the process when

(*i*) all vertices are visited.

(*ii*) all vertices are colored red or blue, or a conflicting color assignment occurs.

(*iii*) a back edge is found.

(*iv*) it exceeds 10 minutes.

(c) If we run a depth-first search on a directed graph, and then remove all of the back edges found, the resulting graph is

(i) a tree. (ii) cyclic. (iii) acyclic. (iv) bipartite.

(d) Assume in some general connected graph G we run the breadth-first search and the depth-first search on G starting from the same vertex s and we find that the resulting trees are the same, then this necessarily means that G is

(i) a cycle.
(ii) a tree.
(iii) a complete graph.
(iv) a complete bipartite graph.

(e) Suppose that G is a graph with n vertices and at most $n \log n$ edges, represented in its adjacency list representation. Then the run time of the depth-first search on G is:

(i) $O(n)$. (ii) $O(\log n)$. (iii) $O(n \log n)$. (iv) $O(n/\log n)$.

(f) A person wants to visit some places. He starts from a vertex and then wants to visit every place connected to this vertex and so on. What algorithm he should use?

(i) Breadth-first search.
(ii) Depth-first search.
(iii) Topological sorting.
(iv) Any of the above.

(g) Consider the undirected graph $G = (V, E)$, where $V = \{m, n, o, p, q, r\}$ and $E = \{(m, n), (n, q), (q, m), (n, o), (o, p), (p, q), (m, r)\}$. If we run the breadth-first search on G starting at any vertex, which one of the following is a possible order for visiting the vertices?

(i) m, n, o, p, q, r. (ii) n, q, m, p, o, r. (iii) r, m, q, p, o, n. (iv) q, m, n, p, r, o.

(h) Suppose that G is a graph with n vertices and at most $n^2 \log n$ edges, represented in its adjacency list representation. Then the run time of the breadth-first search on G is:

(i) $O(n)$. (ii) $O(\log n)$. (iii) $O(n \log n)$. (iv) $O(n^2 \log n)$.

(i) If we use a depth-first search for detecting cycles in an input graph, we stop the process when

(i) all vertices are visited.
(ii) all vertices are colored red or blue, or a conflicting color assignment occurs.
(iii) a back edge is found.
(iv) it exceeds 10 minutes.

(j) Three students were given a DAG, $G = (V, E)$, where $V = \{2, 3, 5, 7, 8, 9, 10, 11\}$ and $E = \{(5, 11), (7, 8), (7, 11), (3, 8), (3, 10), (8, 9), (11, 2)(11, 9), (11, 10)\}$. Each one of these students run the topological sorting on G twice and obtained a result of the following. Which student made a mistake?

(i) The first student who got the sorts: 5, 7, 3, 11, 8, 2, 9, 10 and 3, 5, 7, 8, 11, 2, 9, 10.

(ii) The second student who got the sorts: 3, 5, 7, 8, 11, 2, 9, 10 and 5, 7, 3, 8, 11, 10, 9, 2.

(iii) The third student who got the sorts: 7, 5, 11, 3, 10, 8, 9, 2 and 5, 3, 11, 2, 7, 8, 9, 10.

(iv) No student made a mistake.

9.2 Read each of the following statements, and decide whether it is true or false.

(a) For undirected graphs, in both the adjacency list and the adjacency matrix, there are two representations of each edge in the graph.

(b) Breadth- and depth-first searches can both be used for testing the connectedness of a graph.

(c) A graph G is acyclic if and only if there does not exist a back edge in G after performing any depth-first search.

(d) We can find a topological sorting for any digraph, even if it contains a dicycle.

(e) If a topological sort exists for the vertices in a graph, then a depth-first search on the graph will produce no back edges.

(f) For directed graphs, in both the adjacency list and the adjacency matrix, there are two representations of each edge in the graph.

(g) Breadth- and depth-first searches can both be used for finding connected components.

(h) Given two vertices of a graph G, the shortest path in G between them is not always unique.

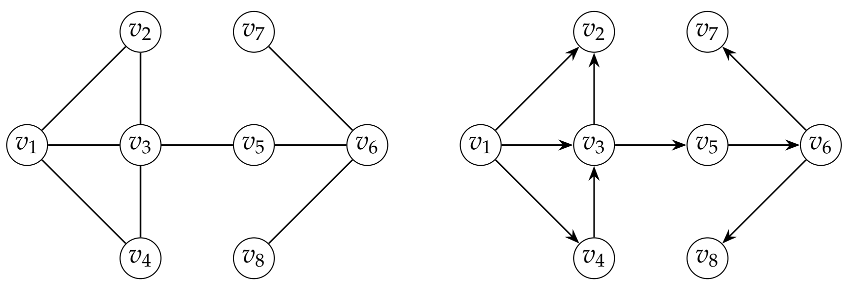
(i) For an undirected graph G, there are tree edges and forward edges only in terms of its depth-first tree (forest). No back or cross edges.

(j) A digraph G has a topological order if and only if there exists a back edge in G after performing any depth-first search.

9.3 Represent the graph in Example 9.1 (c) using:

(a) Adjacency list representation. (b) Adjacency matrix representation.

9.4 Consider the following two graphs; one of them is undirected and the other is directed. Let v_1 be the source vertex in each.



(*a*) We run a breadth-first search on the undirected graph and obtained the breadth-first tree in Figure 9.5. Use this to find the shortest path from vertex v_1 to vertex v_8. Is this shortest path unique?

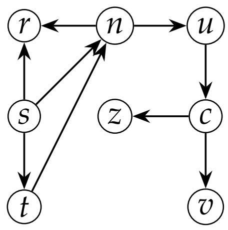
(*b*) Use breadth-first search to test the undirected graph for bipartiteness.

(*c*) Run a depth-first search on the directed graph to compute a depth-first tree. Classify the edges as tree edges, back edges, forward edges, and cross edges.

(*d*) Does the directed graph has a directed cycle? Why or why not? Link your answer to the depth-first search completed in item (*c*).

(*e*) Compute a topological ordering for the directed graph.

9.5 Consider the following the directed graph, and let s be its source vertex.



(*a*) Run a breadth-first search on the graph to obtain the breadth-first tree.

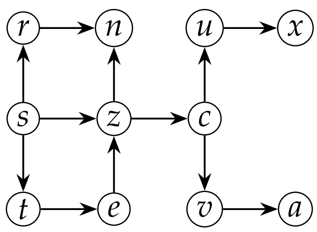
(*b*) Use breadth-first search to test the undirected version of the graph for bipartiteness.

(*c*) Run a depth-first search on the graph to compute a depth-first tree.

(*d*) Does the graph have a directed cycle? Why or why not? Link your answer to the depth-first search completed in item (*c*).

(*e*) Compute a topological ordering for the graph.

9.6 Answer items (*a*) through (*e*) in Exercise 9.5 for the following graph, which has s as its source vertex.



9.7 Prove or disapprove: A depth-first search of a directed graph always produces the same number of tree edges (i.e., independent of the order in which the vertices are provided and independent of the order of the adjacency lists).

Notes and sources

The history of combinatorial algorithms, particularly breadth-first search and depth-first search, can be traced back to the early 19th century. These graph algorithms were first described by Charles Pierre Trémaux in 1857 for solving mazes and puzzles. The earliest reference to depth-first search is referred to Gaston Tarry in 1883, while the first explicit description of the breadth-first search was published in 1957 by Edward Moore (refer to Moore [1959]). Over the years, combinatorial algorithms have become essential tools in computer science and graph theory, used for traversing and analyzing the structure of graphs, trees, and networks.

This chapter introduced these fundamental algorithms designed for graph exploration and search. To prepare adequately for these algorithms, we first learned the techniques for graph representation. The primary emphasis of the chapter was on the two key searching algorithms: Breadth-first search and depth-first search. We also described the applications of these algorithms in computing spanning trees, finding shortest paths, testing bipartiteness, detecting cycles, and identifying connected components. Additionally, we presented the topological sort for ordering directed acyclic graphs.

As we conclude this chapter, it is worth noting that the cited references and others, such as Erickson [2019], Maurer and Ralston [2004], Hougardy and Vygen [2016], Even and Even [2011], Sedgewick [2003], Thulasiraman and Swamy [2011], Lawler [2001], Kádek and Pánovics [2014], Ajwani et al. [2006], Klöckner et al. [2004], Goldfarb et al. [1991], Manber [1990], Siklóssy et al. [1973], Allender et al. [2022], Nishihara and Minamide [2004], Berghammer and Hoffmann [2001], Tarjan [1972], Hagerup [1990], Aggarwal et al. [1990], also serve as valuable sources of information pertaining to the subject matter covered in this chapter. The code that created Figure 9.8 is due to StackExchange [2016]. The code that created Figure 9.9 is due to StackExchange [2013]. We used and modified a code due to StackExchange [2019] to create the adjacency matrix representations in Section 9.1. We used and modified a code due to StackExchange [2020] to create the adjacency list representations in Section 9.1. We used some parts of a code due to StackExchange [2017] to draw topological sort diagrams in Section 9.6.

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Part IV

OPTIMIZATION

CHAPTER 10

LINEAR PROGRAMMING

Chapter overview: In linear programming (LP) problems, we optimize a linear function subject to linear equality and inequality constraints. Within this chapter, we initiate our study of LP, commencing with the graphical method. We delve into the intricacies of LP geometry. Subsequently, we shift our focus to the study of the simplex method, the most prevalent algorithm for solving LP problems. Following that, we transition to an exploration of LP duality. As we approach the conclusion of this chapter, we address the LP problems outside the realm of the simplex method by investigating an interior-point method. The chapter concludes with a set of exercises to encourage readers to enrich their understanding.

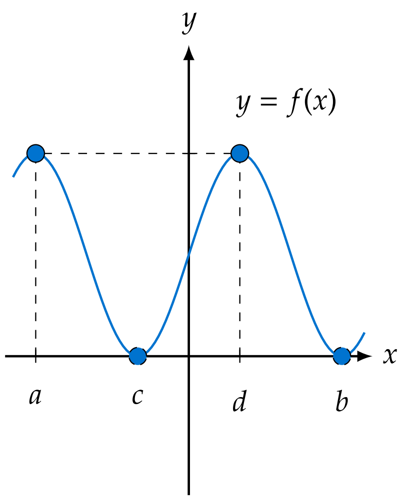
Keywords: Linear programming, LP duality, LP geometry, Simplex method

Contents

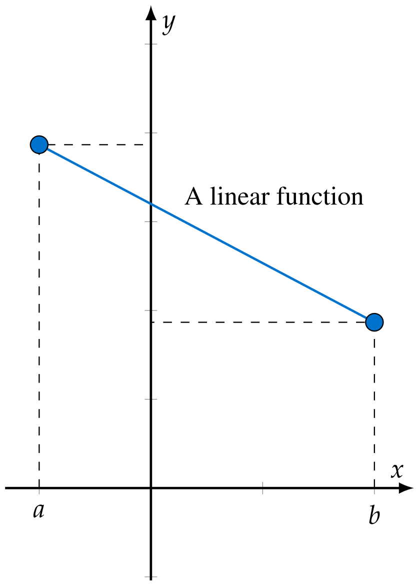
B. Alzalg, *Combinatorial and Algorithmic Mathematics: From Foundation to Optimization*

Ever since its inception in the 1940s, initially in the context of military planning, linear programming (also known as linear optimization) has found extensive application across various industries and disciplines.

Let $f : [a, b] \to \mathbb{R}$ be a nonlinear continuous function. In Calculus, if we want to minimize/maximize the function $f(x)$, we must take the derivative, and then find the critical points. We also check the endpoints, if there are any. We can justify our maxima or minima either by the first derivative test, or the second derivative test. In the graph shown to the right, a and b are endpoints of the function $f(x)$, and c and d are their critical numbers ($f'(x) = 0$ when $x = c, d$). The function f has maximum values at $x = a, d$, and has minimum values at $x = c, b$.



Now, instead of optimizing a nonlinear function on $[a, b]$, consider a linear function on $[a, b]$. In this case, we need to check only the endpoints as in the figure shown to the right. Linear optimization studies generalizations of this easy (linear) case to higher dimensions. More specifically, instead of optimizing a linear function of only one variable, say cx, on the closed interval $[a, b]$, we optimize a linear function of a finite number of variables, say $\boldsymbol{c}^\mathsf{T} x = c_1x_1 + c_2x_2 + \cdots + c_nx_n$, on polytopes, which are generalizations of polygons from $\mathbb{R}^2$ to $\mathbb{R}^n$, where the set $\mathbb{R}^n$ consists of all n-tuples of real numbers, $\mathbb{R}$. This study is “easy" to understand because of linearity, but it is “difficult" to carry out because of high dimensionality.



In this chapter, we introduce linear programming, the graphical method, and study the linear programming duality and geometry. The references Bertsimas and Tsitsiklis [1997], Nemhauser and Wolsey [1988], for example, is a good source for information relative to this topic. Then we study the most common linear programming algorithm, the simplex method. We also study an interior-point method as one of the non-simplex methods. The references Bertsimas and Tsitsiklis [1997], Nemhauser and Wolsey [1988], for example, is a good source for information relative to this topic.

10.1 Linear programming formulation and examples

In this section, we will see that applications of linear programming touch a vast range of real-world areas. First, we present the general form of a linear programming problem.

General form linear programs

A linear programming (LP) problem is the problem of minimizing a linear cost function subject to linear equality and inequality constraints. We have the following example.

Example 10.1 The following is a linear programming problem.

$$\begin{array}{lcccccccccc}
\text{minimize} & 4x_1 & - & x_2 & + & 3x_3 & & & & & \\
\text{subject to} & x_1 & + & x_2 & & & + & x_4 & \leq & 7, \\
& & & 2x_2 & - & x_3 & & & = & 6, \\
& & & & & x_3 & + & x_4 & \geq & 4, \\
& x_1 & & & & & & & \geq & 0, \\
& & & & & x_3 & & & \leq & 0.
\end{array}$$

Here, x_1, x_2, x_3 and x_4 are the decision variables whose values are to be chosen to minimize the linear cost function $4x_1 - x_2 + 3x_3$ subject to linear equality and inequality constraints. ■

Generally speaking, assume that we are given a cost vector $\boldsymbol{c} = (c_1; c_2; \ldots; c_n)^{\mathsf{T}}$ and we minimize a linear cost function $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} = \sum_{i=1}^{n} c_i x_i$ over all nth-dimensional vectors $\boldsymbol{x} = (x_1; x_2; \ldots; x_n)^{\mathsf{T}}$ subject to linear equality and inequality constraints. Then we are interested in a problem of the form:

$$\begin{array}{lll}
\min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & \\
\text{s.t.} & \boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{x} \geq b_i, & i = 1, 2, \ldots, m_1, \\
& \boldsymbol{a}_j^{\mathsf{T}}\boldsymbol{x} \leq b_j, & j = 1, 2, \ldots, m_2, \\
& \boldsymbol{a}_k^{\mathsf{T}}\boldsymbol{x} = b_k, & k = 1, 2, \ldots, m_3, \\
& x_p \geq 0, & p = 1, 2, \ldots, m_4, \\
& x_q \leq 0, & q = 1, 2, \ldots, m_5.
\end{array} \tag{10.1}$$

Problem (10.1) is said to be the general form LP. We have the following definition.

Definition 10.1 *Consider the minimization problem* (10.1). *Then:*

(a) *The variables $x_1, x_2, \ldots, x_n$ are called decision variables;*

(b) *A vector $\boldsymbol{x}$ satisfying all of the constraints is called a feasible solution;*

(c) *The set of all feasible solutions is called the feasible set or feasible region;*

(d) *If $x_i \geq 0$ or $x_i \leq 0$, then x_i is called a restricted variable, otherwise it is called a free or unrestricted variable (urs);*

(e) *The function $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ is called the objective function or cost function;*

(*f*) *A feasible solution* $\boldsymbol{x}^\star$ *that minimizes the objective function (that is,* $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}^\star \leq \boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ *for any feasible solution* $\boldsymbol{x}$*) is called an optimal solution;*

(*g*) *The value of* $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}^\star$*, corresponding to an optimal solution* $\boldsymbol{x}^\star$*, is called the optimal cost or optimal value;*

(*h*) *If the optimal cost is* $-\infty$*, we say that the minimization problem is unbounded.*

Example 10.2 Consider the following nonlinear minimization problem.

$$\begin{array}{lll} \min & 2x_1 + |x_2| & \\ \text{s.t.} & 5x_1 + 7x_2 & \leq \; 3, \\ & |x_1| + x_2 & \leq \; 4, \\ & x_1, x_2 \text{ urs.} & \end{array}$$

Using the fact that $|t| = \max\{t, -t\}$ for $t \in \mathbb{R}$, this problem can be expressed as

$$\begin{array}{lll} \min & 2x_1 + y & \\ \text{s.t.} & 5x_1 + 7x_2 & \leq \; 3, \\ & x_1 + x_2 & \leq \; 4, \\ & -x_1 + x_2 & \leq \; 4, \\ & y - x_2 & \geq \; 0, \\ & y + x_2 & \geq \; 0, \\ & x_1, x_2 \text{ urs,} & \end{array}$$

which is an LP problem. ■

Note that there is no need to study maximization problems separately, because maximizing $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ subject to some constraints is equivalent to minimizing $(-\boldsymbol{c})^\mathsf{T}\boldsymbol{x}$ subject to the same constraints.

Examples of linear programming problems

This part presents some examples of linear programming problems and allows the reader gain to some familiarity with the art of constructing mathematical optimization models.

The procedure given in the following workflow, followed by some examples, will teach us to formulate linear optimization models.

Workflow 10.1 *There are three steps involved in the formation of a linear programming problem:*

(*i*) *Identify the decision variables of interest to the decision maker and express them as* $x_1, x_2, x_3, \ldots$.

(*ii*) *Ascertain the objective function in terms of the decision variables. This would be a cost in case of minimization problem or a profit in case of maximization problem.*

(*iii*) *Ascertain the constraints representing the maximum availability or minimum commitment or equality.*

Example 10.3 (Maximizing profit in product manufacturing) A company is involved in the production of two items, denoted as P_1 and P_2. The manufacturing process for each unit of product P_1 necessitates 2 kilograms of raw material and 4 labor hours for processing, while each unit of product P_2 requires 5 kilograms of raw material and 3 labor hours of the same type. On a weekly basis, the company has access to 45 kilograms of raw material and 55 labor hours. For the financial aspect, the company gains a profit of JD 25 for every unit of product P_1 sold and JD 35 for every unit of product P_2 sold. Formulate this problem as an LP problem that maximizes the total profit.

Solution The given data can be summarized in the following table.

Product	Row material	Labour hours	Profit
P_1	2 kg	4 hrs	JD 25
P_2	5 kg	3 hrs	JD 35
Restrictions	45 kg	55 hrs	

• The first step is to identify the decision variables. Let x_i denote the number of units that should be produced from product P_i per week, $i = 1, 2$.

• The second step is to find out the objective function. The objective is to maximize the total profit. So, our objective function is $z = 25x_1 + 35x_2$.

• The third step is formulate the constraints. In this example, the constraints are:

- A restriction on the row material. This can be formulated as $2x_1 + 5x_2 \leq 45$.
- A restriction on the labor hours. This can be formulated as $4x_1 + 3x_2 \leq 55$.
- Non-negativity constraints. This can be formulated as $x_1 \geq 0$ and $x_2 \geq 0$.

As a result, this problem can be formulated as the following LP model.

$$\begin{array}{ll} \max & 25x_1 + 35x_2 \\ \text{s.t.} & 2x_1 + 5x_2 \leq 45, \\ & 4x_1 + 3x_2 \leq 55, \\ & x_1, x_2 \geq 0. \end{array}$$

■

Example 10.4 (Maximizing profit in corn chip production) A corn chip company operates with two distinct departments, each responsible for producing two types of corn chips: "extra larges" and "really smalls". The company earns a profit of 225 per kilobag of extra larges and 175 per kilobag of really smalls (where a kilobag contains 1000 bags). Each department adheres to specific production regulations per day. The company's primary objective is to maximize its profit while complying with these regulations.

(*a*) Identify the decision variables.

(b) Write the objective function z in terms of the decision variables (x and y).

(c) Write inequalities expressing the following constraints:

(i) The production of extra larges should not exceed 20 kilobags per day, and the production of really smalls should not exceed 30 kilobags per day.

(ii) No more than a total of 45 kilobags can be produced each day.

(iii) The number of extra larges produced daily must be at least 2/3 of the number of really smalls produced.

(iv) The company must utilize more than 250 hours of labor each day to satisfy union requirements. Making one kilobag of extra larges consumes 10 hours, and making one kilobag of really smalls consumes 15 hours.

Solution (a) The decision variables are:

x: The number of extra large corn chips produced per day;

y: The number of really small corn chips produced per day.

(b) $z = 225x + 175y$.

(c) (i) $x \leq 20,\ y \leq 30$. (iii) $x \geq \frac{2}{3}y$.

(ii) $x + y \leq 45$. (iv) $10x + 15y > 250$.

■

Example 10.5 (Minimizing nutritional costs) Two different food items, denoted as F_1 and F_2, contain vitamins A and B. Food F_1 provides 2 units of vitamin A and 5 units of vitamin B per unit, while food F_2 offers 4 units of vitamin A and 2 units of vitamin B per unit. The cost of one unit of food F_1 is JD 10, and for food F_2, it is JD 12.50. The objective is to meet or exceed the minimum daily nutritional requirements for vitamins A and B, which are 40 and 50 units respectively, at the lowest possible cost. Formulate this problem as an LP problem.

Solution The given data can be summarized in the following table.

Food/Vitamin	A	B	Cost
F_1	2 units	5 units	JD 10
F_2	4 units	2 units	JD 12.5
Restrictions	40 units	50 units	

Let x_i be the number of units that should be daily produced from food F_i for a person, $i = 1, 2$. This problem can be formulated as the following LP model.

$$\begin{aligned} \min \quad & 10x_1 + 12.5x_2 \\ \text{s.t.} \quad & 2x_1 + 4x_2 \geq 40, \\ & 5x_1 + 2x_2 \geq 50, \\ & x_1, x_2 \geq 0. \end{aligned}$$

■

Example 10.6 (Maximizing advertising audience) A marketing manager has an annual advertising budget of JD 25,000, which he intends to allocate to two advertising media, A and B. Media A, a monthly magazine, costs JD 1,000 per message, and media B costs JD 1,500 per message. The following conditions apply: For media A, not more than one insertion is desired in the issue. For media B, at least five messages should be placed. The expected effective audience for one message in media A is 40,000 people, while for media B, it is 50,000 people. Formulate this problem as an LP problem to maximize the total audience reached through advertising while staying within the budget constraints.

Solution The given data can be summarized in the following table.

Media	Media A	Media B	Restrictions
Audience	40,000 people	50,000 people	
One message cost	JD 1,000	JD 1,500	JD 25,000
Number of messages	at most 1	at least 5	

Let x_1 and x_2 be the number of messages that should appear in media A and B, respectively. This problem can be formulated as the following LP model.

$$\begin{array}{ll} \max & 40,000x_1 + 50,000x_2 \\ \text{s.t.} & 1,000x_1 + 1,500x_2 \leq 25,000, \\ & x_1 \leq 1, \\ & x_2 \geq 5, \\ & x_1, x_2 \geq 0. \end{array}$$

■

Example 10.7 (Minimizing cost in sheep nutrition) A farmer is actively involved in breeding sheep, and the sheep's diet primarily consists of various products grown on the farm. To ensure that the sheep receive the required nutrient constituents, the farmer must consider purchasing additional products, which we will refer to as Product A and Product B. The essential nutrient constituents (vitamins and protein) contained in each of these products are detailed in the table below:

Nutrient Constituents	Nutrient in product A	Nutrient in product B	Minimum requirement of nutrient constituents
X	36	6	108
Y	3	12	36
Z	20	10	100

Product A is priced at JD 20 per unit, while Product B is priced at JD 40 per unit. Formulate an LP problem that can minimize the total cost and satisfy the requirements.

Solution Let x_1 and x_2 be the number of units that must be purchased from products A and B, respectively. This problem can be formulated as the following LP model.

$$\begin{array}{ll} \min & 20x_1 + 40x_2 \\ \text{s.t.} & 36x_1 + 6x_2 \geq 108, \\ & 3x_1 + 12x_2 \geq 36, \\ & 20x_1 + 10x_2 \geq 100, \\ & x_1, x_2 \geq 0. \end{array}$$

Example 10.8 (Nurse scheduling at a university hospital) A university hospital is seeking your assistance in scheduling nurses for their intensive care unit. In this scenario, it is assumed that the same daily schedule repeats, and the nurse requirements remain constant. Each workday is divided into four shifts: 12AM-6AM, 6AM-12PM, 12PM-6PM, and 6PM-12AM. Every day, each nurse is assigned to work two of these shifts. Nurses working two consecutive shifts are compensated at a rate of \$20 per hour, while those working a "split schedule" (e.g., 12AM-6AM and 12PM-6PM) receive \$25 per hour. (It is important to note that the shifts 6PM-12AM and 12AM-6AM are considered consecutive). The following table indicates the daily nurse requirements for each shift:

Shift	Number required
12AM-6AM	5
6AM-12PM	12
12PM-6PM	7
6PM-12AM	10

Formulate an LP that can assist this hospital in determining the optimal nurse scheduling to meet daily requirements and minimize the total nurse compensation cost.

Solution The decision variables are:

x_1: The number of nurses that work from 12AM-12PM;
x_2: The number of nurses that work from 6AM-6PM;
x_3: The number of nurses that work from 12PM-12AM;
x_4: The number of nurses that work from 6PM-6AM;
x_5: The number of nurses that work from 12AM-6AM and 12PM-6PM;
x_6: The number of nurses that work from 6AM-12PM and 6PM-12AM.

Minimizing the total cost, we obtain the following LP problem.

$$\begin{array}{lrrrrrrl} \min & \multicolumn{7}{l}{12(20(x_1+x_2+x_3+x_4)+25(x_5+x_6))} \\ \text{s.t.} & x_1+ & & & x_4+ & x_5 & & \geq 5, \\ & x_1+ & x_2+ & & & & x_6 & \geq 12, \\ & & x_2+ & x_3+ & & x_5 & & \geq 7, \\ & & & x_3+ & x_4+ & & x_6 & \geq 10, \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & \geq 0. \end{array}$$

10.2 The graphical method

In this section, we discuss the graphical method for linear optimization problems of two variables. We will also visually demonstrate different LP cases which may result in different types of solutions. We start by presenting the following workflow of six steps to find the extremum (maximum or minimum) solution graphically.

Workflow 10.2 *The following steps involved in solving 2-dimensional LP problems graphically:*

(*i*) *Graph constraint equations on a rectangular coordinate plane.*

(*ii*) *Determine the valid side of each constraint equation.*

(*iii*) *Isolate and identify the feasible region.*

(*iv*) *Determine the direction of improvement.*

(*v*) *Locate the extreme corner.*

(*vi*) *Find the optimum solution and the corresponding optimal value.*

As a direct application of the above steps, we have the following example.

Example 10.9 Use the graphical method to solve the following LP problem.

$$\begin{aligned} \min\ & z = 2x + 5y \\ \text{s.t.}\quad & 3x + 2y \le 6, \\ & -x + 2y \le 4, \\ & x + y \ge 1, \\ & x, y \ge 0. \end{aligned}$$

Solution Following the steps in Workflow 10.2, we obtain the graphical solution visualized in Figure 10.1. Note that the given objective function $z = 2x + 5y$ is perpendicular to the vector $\boldsymbol{c} = (2, 5)$ for any given scalar z. For simplicity, we represent this using the vector $\boldsymbol{c}$ in Figure 10.1. Furthermore, decreasing z corresponds to moving the line $z = 2x + 5y$ in the direction of $-\boldsymbol{c}$. Therefore, to minimize z, we move the line $2x + 5y = z$ as much as possible in the direction of $-\boldsymbol{c}$, as long as we do not leave the feasible region. From Figure 10.1, we find that the unique optimal solution is $\boldsymbol{x} = (1, 0)$ and the optimal value is $z = 2\times1+5\times0 = 2$. ■

For a system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$, we have three possibilities: The system has a unique solution, it has infinitely many solutions, or it is inconsistent. For an LP, we have the corresponding three possibilities, but we have one more possibility in addition. An LP problem may have:

- A unique/finite optimal solution;
- An unbounded solution;
- An infeasible solution;
- Alternative (multiple or infinite number of) optimal solutions.

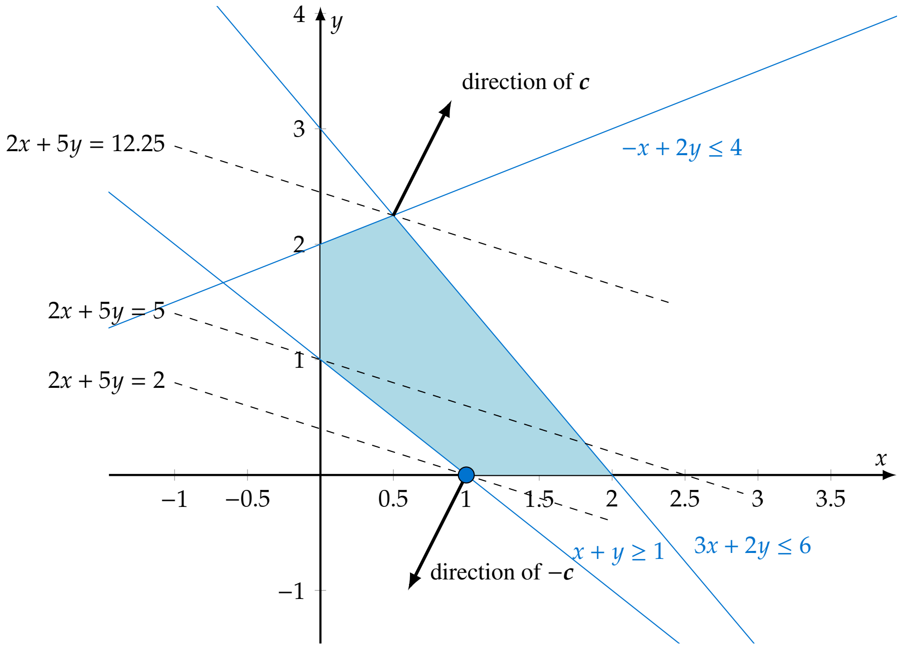


Figure 10.1: Graphical solution of the LP problem in Example 10.9.

In the context of graphical method, it is easy to visualize these four different cases. We have the following examples.

Example 10.10 Use the graphical method to solve the following LP problems.

(a) $$\begin{array}{lrl} \max & z = & 13x_1 + 23x_2 \\ \text{s.t.} & x_1 + 3x_2 & \le 96, \\ & x_1 + x_2 & \le 40, \\ & 7x_1 + 4x_2 & \le 238, \\ & x_1, \quad x_2 & \ge 0. \end{array}$$

(b) $$\begin{array}{lrl} \max & z = & x_1 + x_2 \\ \text{s.t.} & x_1 + 3x_2 & \le 96, \\ & x_1 + x_2 & \le 40, \\ & 7x_1 + 4x_2 & \le 238, \\ & x_1, \quad x_2 & \ge 0. \end{array}$$

Solution (a) The graphical representation of the given LP problem is shown in Figure 10.2, with the feasible region shaded in cyan. From the graph, we find that the maximum value for z is 800 at $\boldsymbol{x} = (12, 28)$. So, this LP problem has a unique optimal solution.

(b) The graphical representation of the given LP problem is shown in Figure 10.3, with the feasible region shaded in cyan. Note that the z-line hits the entire line segment between the points $(12, 28)$ and $(26, 14)$. From the graph, we find that the maximum value for z is 40, and that every point in the line segment between $(12, 28)$ and $(26, 14)$ is an optimal solution. So, this LP problem has alternative optimal solutions.

■

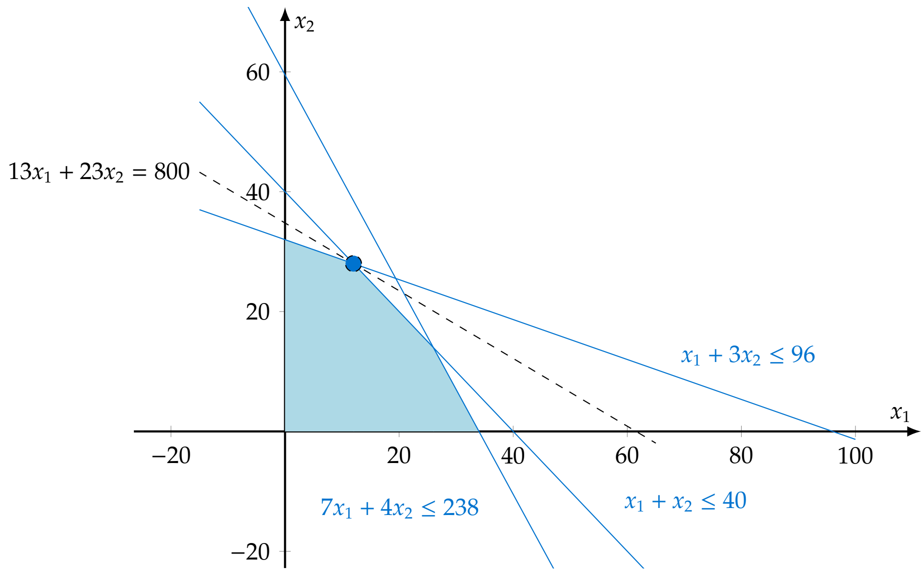


Figure 10.2: Graphical solution of the optimization problem in Example 10.10 (a).

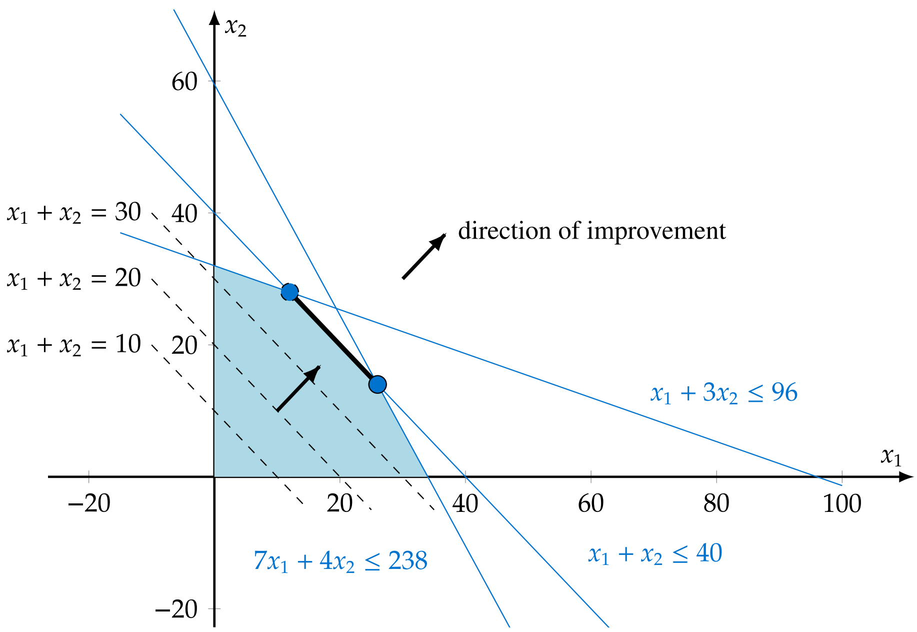


Figure 10.3: Graphical solution of the optimization problem in Example 10.10 (b).

Example 10.11 Use the graphical method to solve the following LP problems.

$$(a)\ \begin{array}{ll} \min & z = 3x_1 + x_2 \\ \text{s.t.} & 5x_1 + x_2 \geq 42, \\ & 2x_1 + x_2 \geq 30, \\ & x_1, \quad x_2 \geq 0. \end{array} \qquad (b)\ \begin{array}{ll} \max & z = 3x_1 + x_2 \\ \text{s.t.} & 5x_1 + x_2 \geq 42, \\ & 2x_1 + x_2 \geq 30, \\ & x_1, \quad x_2 \geq 0. \end{array}$$

Solution (a) In Figure 10.4, we have provided a graphical representation of the LP problem at hand. The feasible region is distinctly shaded in a cyan color for clarity. Upon inspecting the graph, we can readily deduce that the lowest attainable value for the objective function z occurs at 34. This minimal value of z is achieved when the decision variables are set to $\boldsymbol{x} = (4, 22)$.

(b) We provide a visual depiction of the given LP problem in Figure 10.5. Within this graph, the feasible region is distinctly highlighted in cyan. One can observe that the z-line, which represents the objective function's values, can be continuously extended toward the upper-right corner of the feasible region without any bound or limit. This observation implies that there is no finite or optimal value of z that can be achieved within the problem's constraints. Consequently, we can conclude that this LP problem is unbounded, emphasizing the open-ended nature of this particular problem.

■

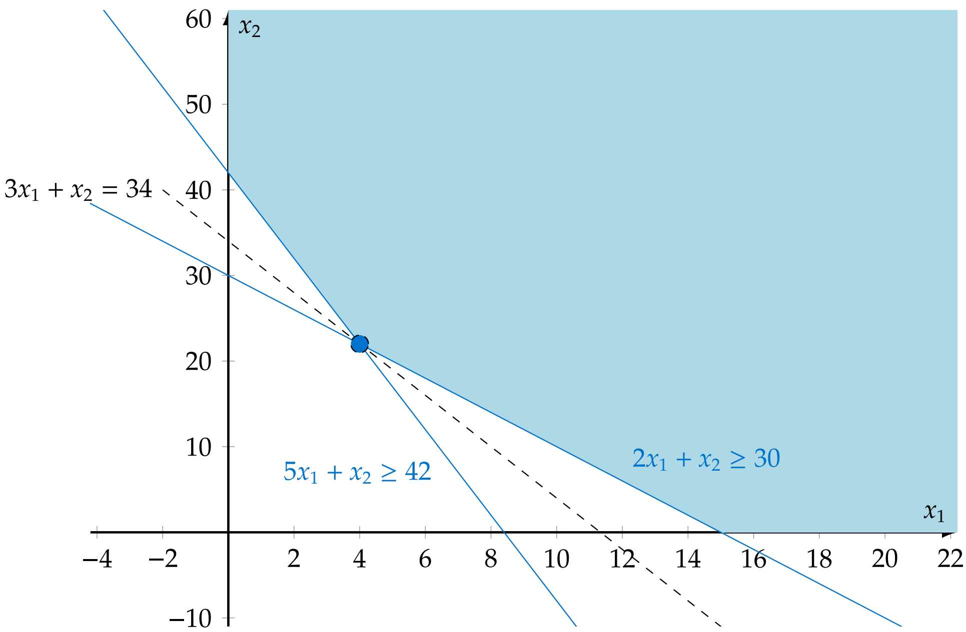


Figure 10.4: Graphical solution of the optimization problem in Example 10.11 (a).

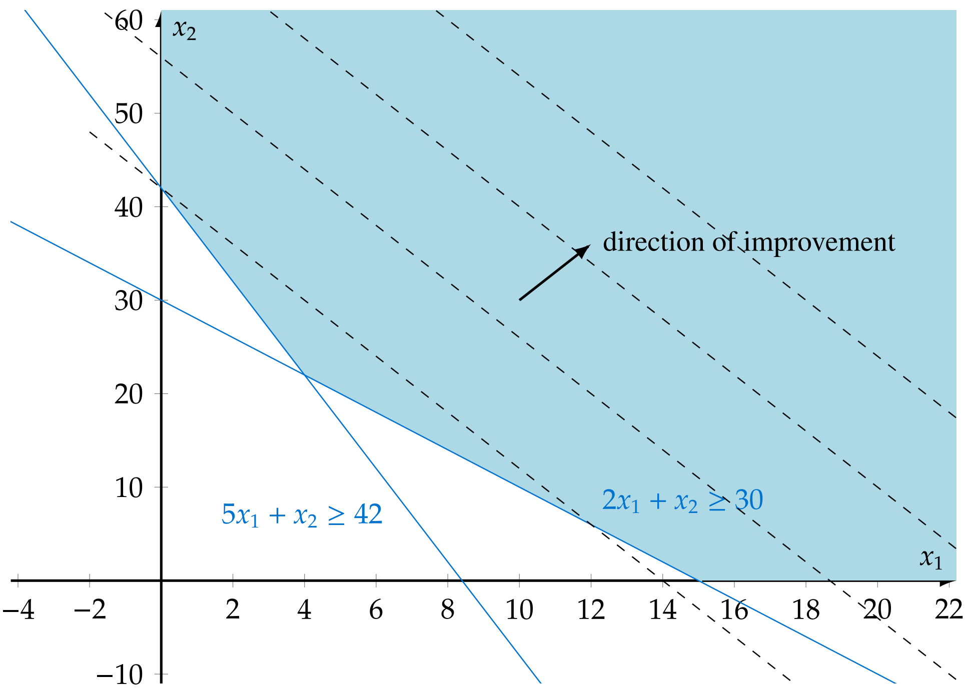


Figure 10.5: Graphical solution of the optimization problem in Example 10.11 (b).

Example 10.12 Use the graphical method to solve the following LP problems.

(*a*) $$\begin{aligned} \max\ z = 13x_1 + 23x_2 & \\ \text{s.t.} \quad x_1 + 3x_2 &\leq 96, \\ x_1 + x_2 &\geq 30, \\ 7x_1 + 4x_2 &\leq 238, \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

(*b*) $$\begin{aligned} \max\ z = 13x_1 + 23x_2 & \\ \text{s.t.} \quad x_1 + 3x_2 &\geq 96, \\ x_1 + x_2 &\leq 30, \\ 7x_1 + 4x_2 &\geq 238, \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

Solution (*a*) The graphical representation of the given LP problem is shown in Figure 10.6, with the feasible region shaded in cyan. From the graph, we find that the maximum value for z is 839.52 at $\boldsymbol{x} = (19.41, 25.53)$. So, this LP problem has a unique optimal solution.

(*b*) The graphical representation of the given LP problem is shown in Figure 10.7. Note that there are no feasible solutions, i.e., there are no points satisfying all constraints. Therefore, the feasible region is empty, and the LP problem is infeasible.

■

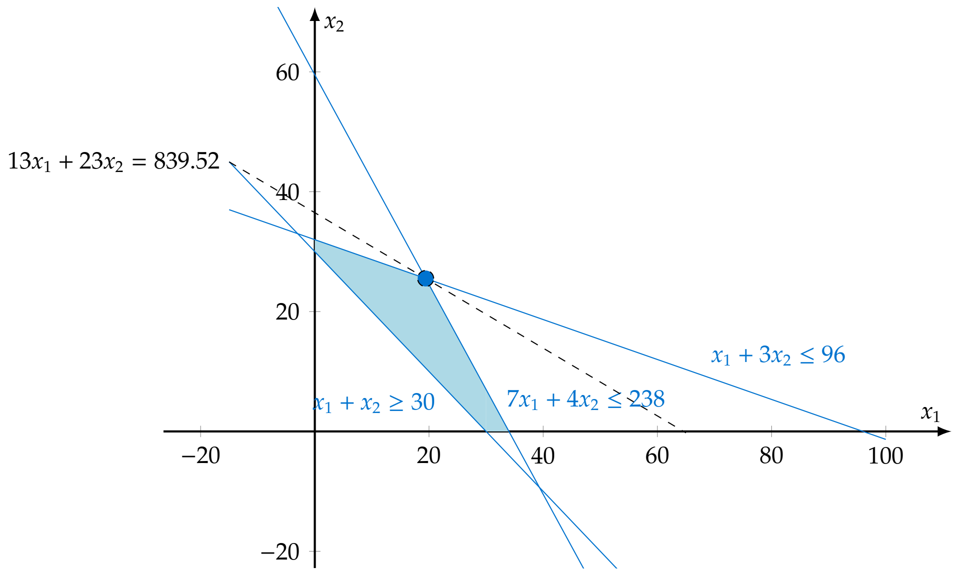


Figure 10.6: Graphical solution of the optimization problem in Example 10.12 (a).

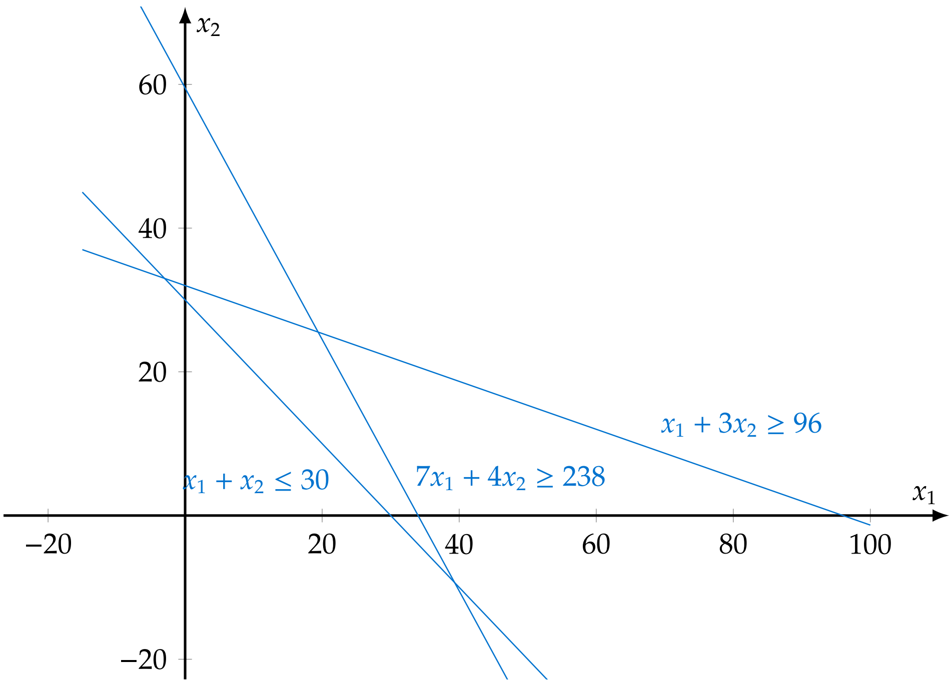


Figure 10.7: Graphical solution of the optimization problem in Example 10.12 (b).

LP problem	LP case	Feasible region type
The LP in Example 10.10 (a)	Unique optimal solution	Bounded
The LP in Example 10.10 (b)	Alternative optimal solutions	Bounded
The LP in Example 10.11 (a)	Unique optimal solution	Unbounded
The LP in Example 10.11 (b)	Unbounded solution	Unbounded
The LP in Example 10.12 (a)	Unique optimal solution	Bounded
The LP in Example 10.12 (b)	Infeasible solution	Empty

Table 10.1: The answer of Example 10.13.

Example 10.13 For the LP problems given in Examples 10.10–10.12, indicate which case the LP belongs to (that is, if the LP has a unique optimal solution, has many optimal solutions, is unbounded, or is infeasible), and which type the feasible region is found (that is, if the feasible region is bounded, unbounded, or empty).

Solution The answer is given in Table 10.1. ■

Example 10.14 Consider the following LP problem.

$$\begin{array}{lrl} \max & y & \\ \text{s.t.} & -x + y & \leq 1, \\ & 3x + 2y & \leq 12, \\ & 2x + 3y & \geq 12, \\ & x, \quad y & \geq 0. \end{array}$$

(a) Sketch the feasible region of this LP and solve it using the graphical method.

(b) Generally speaking, if (some of) the variables are restricted to be integer-valued, then the underlying optimization problem is called an integer (a mixed-integer) program. In this example, assume that x and y are restricted to be integer-valued. Sketch its feasible region and solve it graphically.

Solution (a) The graphical representation of the LP is shown in Figure 10.8, with the feasible region shaded in cyan. We find that the optimal solution is 3 at $\boldsymbol{x} = (2, 3)$.

(b) Introducing the condition $x, y \in \mathbb{Z}$ changes the feasible region, which is now indicated by the blue bullet shown in Figure 10.8. The optimal solution remains the same. ■

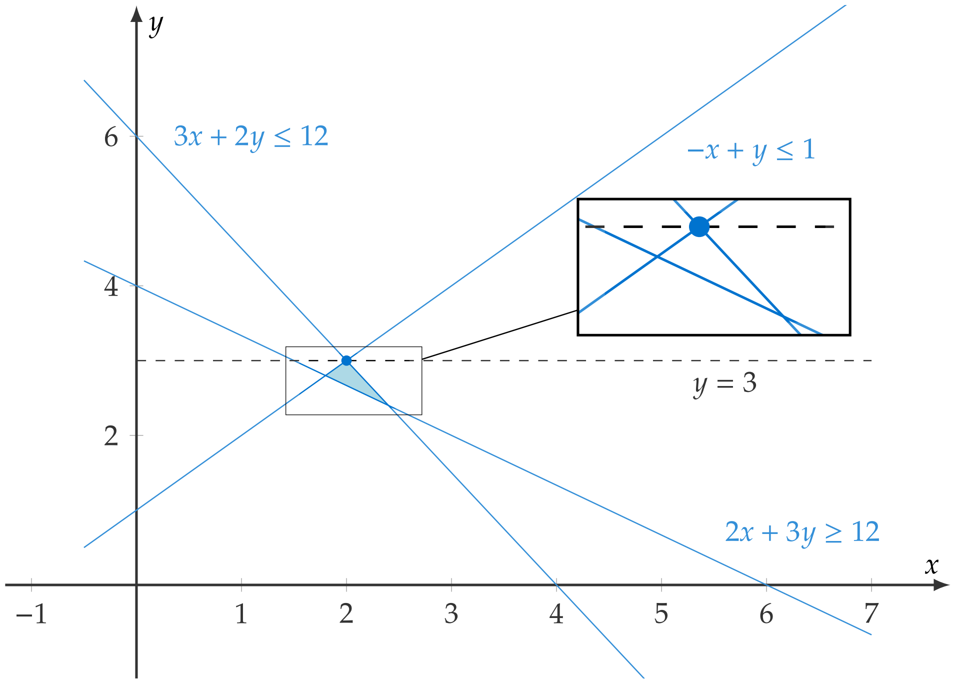


Figure 10.8: Graphical solution of the optimization problem in Example 10.14.

10.3 Standard form linear programs

Recall that the general form LP is:

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{a}_i^\mathsf{T}\boldsymbol{x} \geq b_i, \quad && i = 1, 2, \ldots, m_1, \\
& \boldsymbol{a}_j^\mathsf{T}\boldsymbol{x} \leq b_j, \quad && j = 1, 2, \ldots, m_2, \\
& \boldsymbol{a}_k^\mathsf{T}\boldsymbol{x} = b_k, \quad && k = 1, 2, \ldots, m_3, \\
& x_p \geq 0, \quad && p = 1, 2, \ldots, m_4, \\
& x_q \leq 0, \quad && q = 1, 2, \ldots, m_5.
\end{aligned}
\tag{10.2}
$$

where $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$.

Recall also that there is no need to study maximization problems separately, because maximizing $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ subject to some constraints is equivalent to minimizing $-\boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ subject to the same constraints. In addition, because

- $\boldsymbol{a}_i^\mathsf{T}\boldsymbol{x} = b_i$ is equivalent to $\boldsymbol{a}_i^\mathsf{T}\boldsymbol{x} \leq b_i$ and $\boldsymbol{a}_i^\mathsf{T}\boldsymbol{x} \geq b_i$;
- $\boldsymbol{a}_i^\mathsf{T}\boldsymbol{x} \leq b_i$ can be written as $(-\boldsymbol{a}_i)^\mathsf{T}\boldsymbol{x} \geq -b_i$;
- $x_i \geq 0$ and $x_i \leq 0$ are special cases of $\boldsymbol{u}^\mathsf{T}\boldsymbol{x} \geq 0$ and $(-\boldsymbol{u})^\mathsf{T}\boldsymbol{x} \geq 0$, respectively, where $\boldsymbol{u}$ is a unit vector in $\mathbb{R}^n$,

Problem (10.2) can be expressed exclusively in terms of inequality constraints of the form $\boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{x} \geq b_i$. As a result, Problem (10.2) can be formulated in vector form as

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{x} \geq b_i,\ i = 1, 2, \ldots, m, \end{array} \tag{10.3}$$

or, more compactly, in matrix form as

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \geq \boldsymbol{b}, \end{array} \tag{10.4}$$

where $A \in \mathbb{R}^{m\times n}$ is the matrix whose rows are the row vectors $\boldsymbol{a}_1^{\mathsf{T}}, \boldsymbol{a}_2^{\mathsf{T}}, \ldots, \boldsymbol{a}_m^{\mathsf{T}}$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)^{\mathsf{T}}$. We have the following example.

Example 10.15 The LP problem in Example 10.1 can be written as

$$\begin{array}{lrcrcrcrcr} \min & 2x_1 & - & x_2 & + & 4x_3 & & & & \\ \text{s.t.} & -x_1 & - & x_2 & & & - & x_4 & \geq & -2, \\ & & & 3x_2 & - & x_3 & & & \geq & 5, \\ & & - & 3x_2 & + & x_3 & & & \geq & -5, \\ & & & & & x_3 & + & x_4 & \geq & 3, \\ & x_1 & & & & & & & \geq & 0, \\ & & & & - & x_3 & & & \geq & 0. \end{array}$$

This can be also written in the matrix form (10.4) with

$$\boldsymbol{c} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix}, A = \begin{bmatrix} -1 & -1 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} -2 \\ 5 \\ -5 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

An LP problem of the form

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s.t.} & A\,\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \tag{10.5}$$

is said to be the standard form LP problem.

We can convert an LP problem to the standard form by eliminating of free variables and eliminating of inequality constraints as detailed in the following workflow.

Workflow 10.3 *We can convert a linear programming problem to the standard form by following three steps:*

(*i*) *Elimination of free variables: We replace each unrestricted variable x_i with $x_i^+ - x_i^-$, where $x_i^+, x_i^- \geq 0$.*

(*ii*) *Elimination of "$\leq$" constraints: We replace $\sum_{j=1}^n a_{ij}x_j \leq b_i$ with $\sum_{j=1}^n a_{ij}x_j + s_i = b_i$, where $s_i \geq 0$ is called a slack variable.*

(*iii*) *Elimination of "$\geq$" constraints: We replace $\sum_{j=1}^n a_{ij}x_j \geq b_i$ with $\sum_{j=1}^n a_{ij}x_j - e_i = b_i$, where $e_i \geq 0$ is called an excess variable.*

Example 10.16 The LP problem:

$$\begin{array}{llcl}
\min & 3x_1 + 7x_2 & & \\
\text{s.t.} & x_1 + x_2 & \geq & 3, \\
& 5x_1 + 3x_2 & = & 19, \\
& x_1 & \geq & 0,
\end{array}
\quad
\begin{array}{l}
\text{is equivalent to} \\
\text{standard form} \\
\text{LP problem:} \\
(\text{letting } x_3 = s_3)
\end{array}
\quad
\begin{array}{llcl}
\min & 3x_1 + 7x_2^+ - 7x_2^- & & \\
\text{s.t.} & x_1 + x_2^+ - x_2^- - x_3 & = & 3, \\
& 5x_1 + 3x_2^+ - 3x_2^- & = & 19, \\
& x_1, x_2^+, x_2^-, x_3 & \geq & 0.
\end{array}$$

For instance, given the feasible solution $(x_1, x_2) = (2, 3)$ to the original problem, we obtain the feasible solution $(x_1, x_2^+, x_2^-, x_3) = (2, 3, 0, 2)$ to the standard form problem. In Exercise 10.12, we seek the point (x_1, x_2) for the original problem given the feasible solution $(x_1, x_2^+, x_2^-, x_3) = (4, 0, 1/3, 2/3)$ to the standard form problem. ■

10.4 Geometry of linear programming

The graphical method for linear optimization problems indicates that an optimal solution to an LP lies at a "corner" of a polyhedron. A vertex, an extreme point, and a basic feasible solution all describe corners of a polyhedron, with the first two being geometric definitions.

Extreme points, vertices, and basic feasible solutions

In this section, we define a vertex, an extreme point, and a basic feasible solution of a given nonempty polyhedron.

Definition 10.2 *Let P be a nonempty polyhedron. A vector $\boldsymbol{x} \in P$ is called a vertex of P if there is some $\boldsymbol{c}$ such that $\boldsymbol{c}^\top \boldsymbol{x} < \boldsymbol{c}^\top \boldsymbol{y}$ for all $\boldsymbol{y} \in P$ different from $\boldsymbol{x}$.*

From Definition 10.2, we observe that $\boldsymbol{x}$ is a vertex of a polyhedron P if it is the optimal solution of some linear program with P as the feasible region. In Figure 10.9, we show two polyhedra. In each polyhedron, the hyperplane $\{\boldsymbol{y} : \boldsymbol{c}^\top \boldsymbol{y} = \boldsymbol{c}^\top \boldsymbol{v}\}$ on the right-hand side touches P as a single point and the point $\boldsymbol{v}$ is a vertex. In contrast, the point $\boldsymbol{w}$ is not a vertex since there is no hyperplane intersecting solely at $\boldsymbol{w}$ within P.

Figure 10.9: Vertices versus ($\boldsymbol{v}$'s) nonvertices ($\boldsymbol{w}$'s).

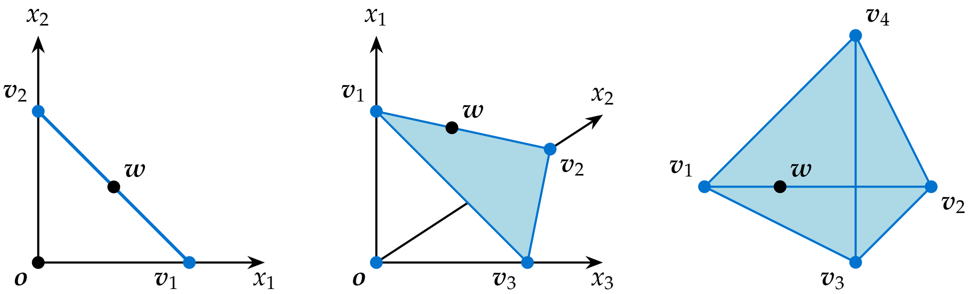


Figure 10.10: Extreme points ($\boldsymbol{v}_i$'s) versus nonextreme points ($\boldsymbol{w}$'s).

Definition 10.3 *Let P be a nonempty polyhedron. A vector $\boldsymbol{x} \in P$ is called an extreme point of P if there are no $\boldsymbol{y}, \boldsymbol{z} \in P$ and a scalar $\lambda \in (0, 1)$ such that $\boldsymbol{x} = \lambda \boldsymbol{y} + (1 - \lambda)\boldsymbol{z}$.*

In Figure 10.10, we show three polyhedra. In each polyhedron, the vectors $\boldsymbol{v}_i$'s are extreme points, and the vector $\boldsymbol{w}$ is not an extreme point because $\boldsymbol{w}$ is a convex combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Definitions 10.2 and 10.3 are geometric, and hence intuitive. We need an equivalent definition that is algebraic, so that we can do computations. Before this, we need an intermediate concept for connecting geometry and algebra.

Definition 10.4 *If a vertex $\boldsymbol{x}^\star$ satisfies an inequality $\boldsymbol{a}^\mathsf{T}\boldsymbol{x} \geq b$ (or $\boldsymbol{a}^\mathsf{T}\boldsymbol{x} \leq b$) as an equality, i.e., $\boldsymbol{a}^\mathsf{T}\boldsymbol{x}^\star = b$, then we say that this inequality is active or binding at $\boldsymbol{x}^\star$.*

If $P \subset \mathbb{R}^n$ is a polyhedron defined by linear equality and inequality constraints, then $\boldsymbol{x}^\star \in \mathbb{R}^n$ may or may not be feasible with respect the constraints. Now, if $\boldsymbol{x}^\star \in \mathbb{R}^n$ is feasible (i.e., $\boldsymbol{x}^\star \in P$; satisfying all the constraints), then from Definition 10.4 all the equality constraints are active at $\boldsymbol{x}^\star$.

We have the following example to more illustrate Definition 10.4.

Example 10.17 The polyhedron shown in the middle of Figure 10.10 is expressed as

$$P = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0\}. \tag{10.6}$$

There are three constraints that are binding at each of the points $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$. Namely, the constraints $x_1+x_2+x_3 = 1$, $x_2 = 0$ and $x_3 = 0$ are active at $\boldsymbol{v}_1$, the constraints $x_1+x_2+x_3 = 1$, $x_1 = 0$ and $x_3 = 0$ are active at $\boldsymbol{v}_2$, and the constraints $x_1 + x_2 + x_3 = 1$, $x_1 = 0$ and $x_2 = 0$ are active at $\boldsymbol{v}_3$. Also, at the point $\boldsymbol{w}$, there are two constraints that are binding, which are $x_1 + x_2 + x_3 = 1$ and $x_3 = 0$. ■

If there are n constraints that are binding at a vector $\boldsymbol{x}^\star \in \mathbb{R}^n$, then $\boldsymbol{x}^\star$ satisfies a system of n linear equations in n unknowns. In view of Theorem 3.13, this system has a unique solution if and only if these n equations are linearly independent.

Now, we are ready to introduce the algebraic definition of a corner point.

Definition 10.5 *Let P be a polyhedron defined by linear equality and inequality constraints, and $\boldsymbol{x}^\star \in \mathbb{R}^n$. We say that the vector*

(*a*) $\boldsymbol{x}^\star$ *is a basic solution if the following two statements hold:*

   (*i*) *All equality constraints are active.*

   (*ii*) *Out of the constraints that are active at $\boldsymbol{x}^\star$, there are n of them that are linearly independent.*

(*b*) $\boldsymbol{x}^\star$ *is a basic feasible solution if it is a basic solution, and satisfies all of the constraints (i.e., $\boldsymbol{x}^\star \in P$).*

The following two examples illustrate Definition 10.5.

Example 10.18 In the polyhedron depicted in the middle of Figure 10.10, as represented in (10.6), we can identify the points $\boldsymbol{v}_i$'s as basic feasible solutions. However, point $\boldsymbol{o}$ fails to meet the equality constraint $x_1 + x_2 + x_3 = 1$, making it ineligible as a basic solution. On the other hand, point $\boldsymbol{w}$ is feasible but does not qualify as basic according to Definition 10.5. Nevertheless, if we replace the equality constraint $x_1 + x_2 + x_3 = 1$ with the inequality constraints $x_1 + x_2 + x_3 \le 1$, then $\boldsymbol{o}$ transforms into a basic feasible solution, as shown in Figure 10.11. ■

Example 10.19 In Figure 10.12, the points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$, and $\boldsymbol{g}$ all represent basic solutions since they each have two linearly independent constraints that are active. Specifically, points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}$, and $\boldsymbol{f}$ are considered basic feasible solutions as they fulfill all imposed constraints. ■

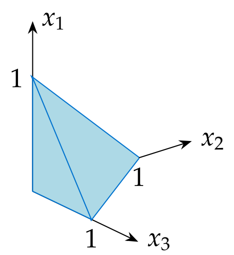


Figure 10.11: The polyhedron given in Example 10.18 with 4 corners.

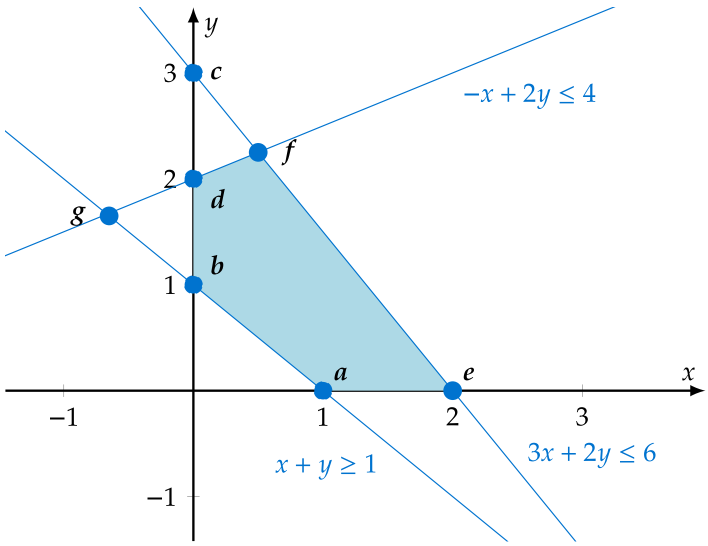


Figure 10.12: Basic solutions and basic feasible solutions.

We give, without proof, the following result in this context. For a proof, see, for example, Bertsimas and Tsitsiklis [1997].

Theorem 10.1 *Let $\boldsymbol{x}^\star$ be a point in a nonempty polyhedron P. Then the following are equivalent:*

(a) *$\boldsymbol{x}^\star$ is a vertex.*

(b) *$\boldsymbol{x}^\star$ is an extreme point.*

(c) *$\boldsymbol{x}^\star$ is a basic feasible solution.*

Definition 10.6 *Two distinct basic solutions to a set of linear constraints in $\mathbb{R}^n$ are called adjacent if there are $n-1$ linearly independent constraints that are binding at both of them.*

As an example, in Figure 10.12, the points $\boldsymbol{a}$ and $\boldsymbol{g}$ are adjacent to the point $\boldsymbol{b}$, and the points $\boldsymbol{d}$ and $\boldsymbol{e}$ are adjacent to $\boldsymbol{f}$.

In the subsequent development, we will see that we find an optimal corner point of a linear programming problem by moving from one basic feasible solution to an adjacent basic feasible solution that improves the objective function value, and so on, repeating this step until we cannot go to an adjacent basic feasible solution that improves the objective function value.

Let n and m be positive integers such that $m \leq n$. Let also $\boldsymbol{b} \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ with rank$(A) = m$ (i.e., A has a full-row rank). The set $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is called a polyhedron in standard form. Note that the number of equality constraints in P is m.

Finding basic feasible solutions

The question that arises now is, how to find basic solutions of polyhedra in standard form? The system $A\boldsymbol{x} = \boldsymbol{b}$ gives m linearly independent constraints as rank$(A) = m$. Consequently, we need $n - m$ more binding constraints from $\boldsymbol{x} \geq \boldsymbol{0}$ (this is n nonnegativity constraints: $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$). Which $n - m$ (out of those n) constraints to select for our purpose? We cannot choose any $(n - m)$ x_i's. Theorem 10.2 helps in this task. Before this theorem, we give some definitions.

As a matter of notation, we use ";" for adjoining vectors and matrices in a column, and use "," or ":" for adjoining them in a row.

We write A as $A = [\boldsymbol{a}_1 : \boldsymbol{a}_2 : \cdots : \boldsymbol{a}_n]$ where $\boldsymbol{a}_j$ is the jth column of A. Since rank$(A) = m$, there exists an invertible matrix

$$A_B \triangleq [\boldsymbol{a}_{B_1} : \boldsymbol{a}_{B_2} : \cdots : \boldsymbol{a}_{B_m}] \in \mathbb{R}^{m\times m}. \tag{10.7}$$

Let $B \triangleq \{B_1, B_2, \ldots, B_m\}$ and $N \triangleq \{1, 2, \ldots, n\} - B$. We can permute the columns of A so that $A = [A_B : A_N]$. We can write the system $A\boldsymbol{x} = \boldsymbol{b}$ as $A_B\boldsymbol{x}_B + A_N\boldsymbol{x}_N = \boldsymbol{b}$ where $\boldsymbol{x} = (\boldsymbol{x}_B; \boldsymbol{x}_N)$ (equivalently, $\boldsymbol{x}^\mathsf{T} = (\boldsymbol{x}_B^\mathsf{T}, \boldsymbol{x}_N^\mathsf{T})$).

Definition 10.7 *The $m \times m$ nonsingular matrix A_B is called a basis matrix. The vector $\boldsymbol{x}_B$ is called a basic solution (also called the vector of basic variables). The vector $\boldsymbol{x}_N$ is called a nonbasic solution (also called the vector of nonbasic variables).*

We are now ready to state the following theorem which will be given without proof. For a proof, see, for example, Bertsimas and Tsitsiklis [1997].

Theorem 10.2 *Let $\boldsymbol{b} \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ have linearly independent rows. Consider the constraints $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. A vector $\boldsymbol{x} \in \mathbb{R}^n$ is a basic solution if and only if we have*

(*a*) *The columns of A_B are linearly independent.*

(*b*) $\boldsymbol{x}_N = \boldsymbol{0}$.

Since A_B is nonsingular, we can solve the system of m linear equations $A\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{x}_B$. The solution is given by $\boldsymbol{x}_N = \boldsymbol{0}$ and $\boldsymbol{x}_B = A_B^{-1}\boldsymbol{b}$. The three-step procedure in the following workflow, followed by an example, will teach us to construct such basic solutions.

Workflow 10.4 *We construct all basic solutions to a standard form polyhedron by following three steps:*

(*i*) *Choose m linearly independent columns $\boldsymbol{a}_{B(1)}, \boldsymbol{a}_{B(2)}, \ldots, \boldsymbol{a}_{B(m)}$.*

(*ii*) *Set $\boldsymbol{x}_N = \boldsymbol{0}$.*

(*iii*) *Calculate $\boldsymbol{x}_B = A_B^{-1}\boldsymbol{b}$. If $\boldsymbol{x}_B \geq \boldsymbol{0}$, then the vector $\boldsymbol{x} = (\boldsymbol{x}_B; \boldsymbol{x}_N)$ is a basic feasible solution. Otherwise, $\boldsymbol{x} = (\boldsymbol{x}_B; \boldsymbol{x}_N)$ is a basic solution.*

It is clear that the maximum number of basic feasible solutions is $\binom{n}{m}$. Note that, generally, not all of $\binom{n}{m}$ choices of m columns may produce a basis (i.e., a nonsingular matrix A_B). Hence, the number of basic solutions may be smaller than $\binom{n}{m}$. Note also that not all of these $\binom{n}{m}$ bases may lead to basic feasible solutions.

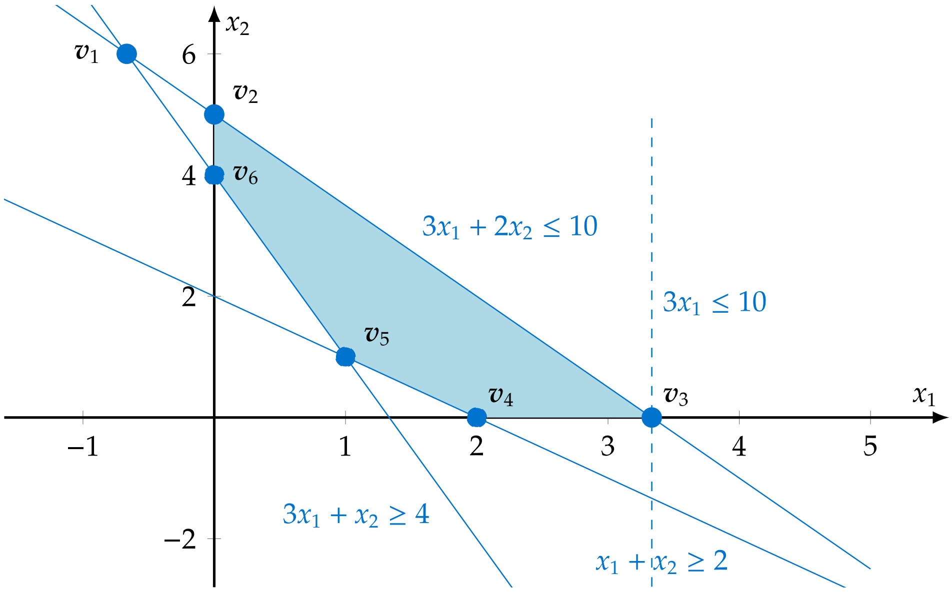


Figure 10.13: The polyhedron in Example 10.20.

In the following example, which is due to Krishnamoorthy [2023a], we have that $n = 5$ and $m = 3$, and that each of $\binom{5}{3} = 10$ choices produces a basic solution.

Example 10.20 Consider the linear system

$$
\begin{array}{rcrcl}
x_1 & + & x_2 & \geq & 2, \\
3x_1 & + & x_2 & \geq & 4, \\
3x_1 & + & 2x_2 & \leq & 10, \\
x_1, & & x_2 & \geq & 0.
\end{array} \tag{10.8}
$$

The resulting polyhedron is shown in Figure 10.13. In the standard form, we have

$$
\begin{array}{rcrcrcrcrcl}
x_1 & + & x_2 & - & x_3 & & & & & = & 2, \\
3x_1 & + & x_2 & & & - & x_4 & & & = & 4, \\
3x_1 & + & 2x_2 & & & & & + & x_5 & = & 10, \\
x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0.
\end{array}
$$

Consequently, the following arrays draw the resulting polyhedron.

$$
A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 3 & 1 & 0 & -1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}.
$$

The columns of A are

$$\boldsymbol{a}_1 = \begin{bmatrix}1\\3\\3\end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix}1\\1\\2\end{bmatrix}, \boldsymbol{a}_3 = \begin{bmatrix}-1\\0\\0\end{bmatrix}, \boldsymbol{a}_4 = \begin{bmatrix}0\\-1\\0\end{bmatrix}, \text{ and } \boldsymbol{a}_5 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Choosing $B = \{1, 2, 3\}$ (hence $N = \{4, 5\}$) gives

$$A_B = \begin{bmatrix}1 & 1 & -1\\3 & 1 & 0\\3 & 2 & 0\end{bmatrix}, \text{ and } \det(B) = -3 \neq 0 \text{ (hence } B \text{ is invertible).}$$

Let $\boldsymbol{x}_N = (x_4; x_5) = (0; 0)$. Finding A_B^{-1} and calculating $\boldsymbol{x}_B = A_B^{-1}\boldsymbol{b}$, we get

$$\boldsymbol{x}_B = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2/3\\6\\10/3\end{bmatrix}.$$

Vertex	B	$\det(A_B)$	The variable $\boldsymbol{x}$	Basic feasible solution?
v_1	$\{1,2,3\}$	-3	$(-2/3; 6; 10/3; 0; 0)$	✗
v_2	$\{2,3,4\}$	2	$(0; 5; 3; 1; 0)$	✓
v_3	$\{1,3,4\}$	3	$(10/3; 0; 4/3; 6; 0)$	✓
v_4	$\{1,4,5\}$	-1	$(2; 0; 0; 2; 4)$	✓
v_5	$\{1,2,5\}$	-2	$(1; 1; 0; 0; 5)$	✓
v_6	$\{2,3,5\}$	1	$(0; 4; 2; 0; 2)$	✓

Table 10.2: Correspondences between the basic feasible solutions in the standard form polyhedron and the vertices visualized in Figure 10.13.

This point is a basic solution, but it is not a basic feasible solution because not all entries are nonnegative. This point corresponds to the vertex $v_1 = (-2/3; 6)$.

Choosing $B = \{2, 3, 4\}$ (hence $N = \{1, 5\}$) gives

$$A_B = \begin{bmatrix}1 & -1 & 0\\1 & 0 & -1\\2 & 0 & 0\end{bmatrix}, \text{ and } \det(B) = 2 \neq 0 \text{ (hence } B \text{ is invertible).}$$

Let $\boldsymbol{x}_N = (x_1; x_5) = (0; 0)$. Finding A_B^{-1} and calculating $\boldsymbol{x}_B = A_B^{-1}\boldsymbol{b}$, we get

$$\boldsymbol{x}_B = \begin{bmatrix}x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}5\\3\\1\end{bmatrix}.$$

Thus $\boldsymbol{x} = (0;5;3;1;0)$. This point is a basic feasible solution because all the entries are nonnegative. This point corresponds to the vertex $\boldsymbol{v}_2 = (0;5)$. Table 10.2 summarizes the correspondences between the basic feasible solutions in the standard form polyhedron and the vertices visualized in Figure 10.13. ■

We want to emphasize that we can identify the basic feasible solution within the standard form polyhedron for each corner point through a straightforward examination. In simpler terms, there is no need to systematically go through all possible combinations, such as the $\binom{n}{m}$ choices for bases. Take, for instance, Example 10.20, where at vertex $\boldsymbol{v}_5$, the constraints $x_1 + x_2 \geq 2$ and $3x_1 + x_2 \geq 4$ are active, while $3x_1 + 2x_2 \leq 10$ is not. Consequently, in the corresponding basic feasible solutions within the standard form polyhedron, we have $x_3 = x_4 = 0$ and $x_5 > 0$. Additionally, both x_1 and x_2 are strictly positive. Therefore, $\boldsymbol{x}_B = (x_1;x_2;x_5)$ forms the corresponding basis.

Likewise, consider vertex $\boldsymbol{v}_2$ where the constraints $x_1 + x_2 \geq 2$ and $3x_1 + x_2 \geq 4$ are not binding, but the constraints $3x_1 + 2x_2 \leq 10$ and $x_1 \geq 0$ are active. Consequently, in the corresponding basic feasible solutions within the standard form polyhedron, we find that x_3 and x_4 are both greater than 0, while x_5 equals 0. Additionally, it is worth noting that x_5 is strictly positive. Therefore, $\boldsymbol{x}_B = (x_2;x_3;x_4)$ forms the corresponding basis.

Degeneracy At a basic solution, we must have n linearly independent active constraints. However, since no more than n constraints can be linearly independent in an nth-dimensional space, it is possible for more than n active constraints to exist at a basic solution. In such cases, this basic solution is referred to as degenerate.

Definition 10.8 *A basic solution $\boldsymbol{x} \in \mathbb{R}^n$ is called degenerate if more than n of the constraints are active at $\boldsymbol{x}$. In a nonempty polyhedron in standard form with $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{x}$ is a degenerate basic solution if more than $n - m$ of the components of $\boldsymbol{x}$ are zero.*

Example 10.21 (Example 10.20 revisited) Adding the constraint $x_1 \leq 10/3$ to System 10.8 results in three active constraints at $\boldsymbol{v}_3$ (see Figure 10.13). Therefore, $\boldsymbol{v}_3$ qualifies as a degenerate basic feasible solution. In the standard form, this constraint becomes $x_1 + x_6 = 10/3$, where x_6 represents the slack variable for the constraint $x_1 \leq 10/3$. In this case, with $n = 6$ and $m = 4$, and with $x_2 = x_5 = x_6 = 0$, we have more than $n - m = 2$ components of $\boldsymbol{x}$ equal to zero. ■

Degeneracy might not pose significant issues in small-scale problems, but it can introduce inefficiencies when dealing with large linear programming instances. In typical algorithms, the goal is to transition from one basic feasible solution to another nearby solution that either improves the objective function or, at the very least, maintains the current value. However, in the presence of degeneracy, the algorithm may cycle through several degenerate basic feasible solutions before finally reaching a vertex that genuinely enhances the objective function value. The extent of degeneracy largely hinges on how we represent the polyhedron. For instance, in Example 10.21, we could eliminate the constraint $x_1 \leq 10/3$ without altering the polyhedron, thereby resolving the degeneracy issue at $\boldsymbol{v}_3$. Additionally, if permissible, we could circumvent degeneracy by making minor adjustments to certain constraints, such as replacing $x_1 \leq 10/3$ with $x_1 \leq 10/3 + 0.001$. However, the feasibility of such modifications heavily relies on the specific problem application.

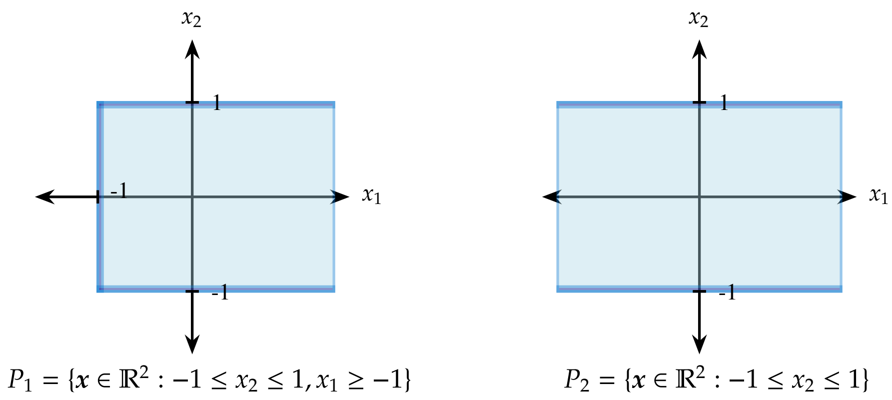


Figure 10.14: Pointed polyhedron (left) versus non-pointed polyhedron (right).

Pointedness

A polyhedron is pointed if it contains no lines (a line is a straight one-dimensional figure formed when two points are connected with minimum distance between them, and both the ends extended to infinity). Figure 10.14 shows two polyhedra, one of them (namely, P_1) is pointed but the other (namely, P_2) is non-pointed. Note that every nonempty polyhedron subset of a pointed polyhedron is pointed.

A good question to ask: Is every nonempty polyhedron pointed? We give the following theorem without proof. For a proof, see, for example, Bertsimas and Tsitsiklis [1997].

Theorem 10.3 *Assume that the polyhedron* $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_i^\top \boldsymbol{x} \ge b_i, i = 1, \ldots, m\}$ *is nonempty. Then the following are equivalent:*

(a) *The polyhedron P is pointed.*

(b) *The polyhedron P has at least one extreme point.*

(c) *There exist n vectors out of the family* $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$*, which are linearly independent.*

Note that, from Theorem 10.3, a bounded polyhedron is pointed. Similarly, the nonnegative orthant cone $\mathbb{R}^n_+ \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \ge \boldsymbol{0}\}$ is pointed. Since any standard form polyhedron is a subset of the nonnegative orthant cone, it is pointed too. The following two corollaries are now immediate.

Corollary 10.1 *Every nonempty bounded polyhedron has at least one basic feasible solution.*

Corollary 10.2 *Every nonempty polyhedron in standard form has at least one basic feasible solution.*

Note also that, from Theorem 10.3, every nonempty polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \ge \boldsymbol{b}\}$, with $A \in \mathbb{R}^{m \times n}$ and $m < n$, cannot have any basic feasible solution.

Optimality

In the above part, we have established the conditions for the existence of extreme points. In this part, we will see that if a nonempty polyhedron P has no corner points, then the linear programming problem of minimizing a linear objective function over P cannot have a unique optimal solution. The following theorem presents the contrapositive of this statement.

Theorem 10.4 *Consider the linear programming problem over a polyhedron P. If P has at least one extreme point and there exists an optimal solution, then there exists an extreme point of P which is optimal.*

Proof Let $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \geq \boldsymbol{b}\}$, and v be the optimal value of the cost $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ which we have assumed to be attained. Then $P_{\text{opt}} \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{c}^\mathsf{T}\boldsymbol{x} = v\}$ contains all optimal solutions in P. By assumption, P_{opt} is a nonempty polyhedron. From Theorem 10.3, P is pointed. Since $P_{\text{opt}} \subset P$, P_{opt} is pointed too. Using Theorem 10.3 again, P_{opt} has an extreme point, say $\boldsymbol{x}^\star$. Since $\boldsymbol{x}^\star \in P_{\text{opt}}$, we have $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}^\star = v$, i.e., $\boldsymbol{x}^\star$ is optimal. To complete the proof, it remains to show that $\boldsymbol{x}^\star$ is an extreme point of P.

Suppose, in the contrary, that $\boldsymbol{x}^\star$ is not an extreme point of P. Then, there exist $\boldsymbol{y}, \boldsymbol{z} \in P$ and a scalar $\lambda \in (0,1)$ such that $\boldsymbol{x}^\star = \lambda\boldsymbol{y} + (1-\lambda)\boldsymbol{z}$. Consequently, $v = \boldsymbol{c}^\mathsf{T}\boldsymbol{x}^\star = \lambda\boldsymbol{c}^\mathsf{T}\boldsymbol{y} + (1-\lambda)\boldsymbol{c}^\mathsf{T}\boldsymbol{z}$. Furthermore, since v is the optimal cost, $\boldsymbol{c}^\mathsf{T}\boldsymbol{y} \geq v$ and $\boldsymbol{c}^\mathsf{T}\boldsymbol{z} \geq v$. It follows that $\boldsymbol{c}^\mathsf{T}\boldsymbol{y} = \boldsymbol{c}^\mathsf{T}\boldsymbol{z} = v$, and therefore $\boldsymbol{y}, \boldsymbol{z} \in P_{\text{opt}}$. But this contradicts the fact that $\boldsymbol{x}^\star$ is an extreme point of P_{opt}. Thus, $\boldsymbol{x}^\star$ is an extreme point of P. The proof is complete. ■

A more general result than that in Theorem 10.4 is stated in the following theorem, which will be given without proof. For a proof, see, for example, Bertsimas and Tsitsiklis [1997].

Theorem 10.5 *Consider the linear programming problem over a polyhedron P. If P has at least one extreme point, then either the optimal cost is equal to $-\infty$, or there exists an extreme point of P which is optimal.*

Theorems 10.4 and 10.5 specifically address polyhedra under the condition that they possess at least one extreme point. But what about polyhedra that don not satisfy this condition? Interestingly, any linear programming problem, whether dealing with a polyhedron with or without extreme points, can be converted into an equivalent problem in standard form. This transformation enables us to apply Theorem 10.5, as highlighted in Corollary 10.2. This insight leads to the following corollary.

Corollary 10.3 *Consider the linear minimization problem over a nonempty polyhedron P. Then either the optimal cost is equal to $-\infty$, or there exists an optimal solution.*

Generally, Corollary 10.3 does not hold for nonlinear programming problems. For example, the nonlinear optimization problem

$$\begin{array}{ll} \min & 1/x \\ \text{s.t.} & x \geq 1, \end{array}$$

has no optimal solution, but the optimal cost is not $-\infty$.

10.5 The simplex method

We have introduced linear optimization problems and studied its geometry and duality. Now, we are ready to introduce the simplex method. The word "simplex" is a general term of LP feasible region. Simplex method is used to solve LPs with any number of variables and constraints. The idea behind this method is to move from one basic feasible solution to an adjacent basic feasible solution so that the objective function value improves.

The simplex method

We begin by outlining the simplex method for solving LP problems with a focus on maximization.

Simplex method for maximization The six-step procedure in Workflow 10.5, followed by Example 10.22, will teach us to apply the simplex method for solving the maximization problem. First, we need the following definition.

Definition 10.9 *Consider the standard form LP:*

$$\begin{array}{lrcl} \max & z = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\ \text{s.t.} & A\boldsymbol{x} & = & \boldsymbol{b}, \\ & \boldsymbol{x} & \geq & \boldsymbol{0}. \end{array} \tag{10.9}$$

The corresponding canonical form is

$$\begin{array}{rcl} z - \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & = & 0, \\ A\boldsymbol{x} & = & \boldsymbol{b}. \end{array}$$

For example, the canonical form corresponding to the standard form LP:

$$\begin{array}{llcl} \max & z = 2x_1 + 3x_2 & & \\ \text{s.t.} & x_1 + 2x_2 & = & 4, \\ & 2x_1 + x_2 + x_3 & = & 8, \\ & x_1, x_2, x_3 & \geq & 0, \end{array}$$

is the system

$$\begin{array}{rcl} z - 2x_1 - 3x_2 & = & 0, \\ x_1 + 2x_2 & = & 4, \\ 2x_1 + x_2 + x_3 & = & 8. \end{array}$$

Workflow 10.5 (The simplex method) *We solve a maximization LP problem by following five steps:*

(*i*) *Write the given LP in the standard form.*

(*ii*) *Convert the standard form to a canonical form.*

(*iii*) *Find a basic feasible solution for the canonical form.*

(*iv*) *If the current basic feasible solution is optimal, stop. If not, find which basic variable must become nonbasic and which nonbasic variable must become basic, and apply elementary row operations in order to move to an adjacent basic feasible solution with a higher value for the objective function.*

(*v*) *Go to Step (iv).*

For guidance on determining which variables should change from basic to nonbasic (or vice versa) and on assessing the optimality, refer to Remarks 10.1-10.3 below. The examples in this section, except the last two, are due to Krishnamoorthy [2023b].

Example 10.22 Use the simplex method to solve the following maximization LP.

$$\begin{array}{lllll} \max & z = 2x_1 + 3x_2 & & \\ \text{s.t.} & x_1 + 2x_2 & \leq & 6, \\ & 2x_1 + x_2 & \leq & 8, \\ & x_1, x_2 & \geq & 0. \end{array} \tag{10.10}$$

Solution We apply the steps in Workflow 10.5. Problem (10.10) in standard form is written as

$$\begin{array}{lllll} \max & z = 2x_1 + 3x_2 & & \\ \text{s.t.} & x_1 + 2x_2 + s_1 & = & 6, \\ & 2x_1 + x_2 + s_2 & = & 8, \\ & x_1, x_2, s_1, s_2 & \geq & 0. \end{array} \tag{10.11}$$

Problem (10.11) in the canonical form is written as

$$\begin{array}{ccccccccccc} z & - & 2x_1 & - & 3x_2 & & & & & = & 0, \\ & & x_1 & + & 2x_2 & + & s_1 & & & = & 6, \\ & & 2x_1 & + & x_2 & & & + & s_2 & = & 8. \end{array} \tag{10.12}$$

The canonical variables, which correspond to the unit columns, are the variables z, s_1 and s_2. Let BV denote the set of the basic variables. We select BV $= \{z, s_1, s_2\}$. Generally, we have $|\text{BV}| = m + 1$ ($m = 2$ in this example), and we choose $z \in$ BV always. Therefore, the BV contains the variable z plus m canonical variables. Let NBV denote the set of the nonbasic variables. Then NBV $= \{x_1, x_2\}$.

Fix $x_1 = x_2 = 0$. System (10.12) now reads $z = 0$, $s_1 = 6$ and $s_2 = 8$, which are a basic feasible solution.

Now, let us determine if the current basic feasible solution is optimal. An optimal solution is reached when we cannot further improve the value of z by increasing the value of any nonbasic variable (starting from zero). Currently, $z = 2x_1 + 3x_2 = 0$ as $x_1 = x_2 = 0$ (NBV $= \{x_1, x_2\}$). Increasing x_1 from 0 to 1 increases z from 0 to 2, while increasing x_2 from 0 to 1 increases z from 0 to 3 (we are increasing one variable at a time, while keeping the other nonbasic variable fixed at zero). It is more beneficial to increase x_2 here than x_1. Generally, we select the nonbasic variable with the largest positive coefficient in the z expression to enter the basis. In the canonical form, we choose the nonbasic variable with the most negative coefficient in row-0 to enter the basis. The following remark summarizes this discussion. We will move forward in solving Example 10.22 after this remark.

Remark 10.1 (Criterion for choosing the entering variable in maximization) *The entering variable in a maximization LP problem is the nonbasic variable having the most negative coefficient in the z-row.*

Considering Remark 10.1, we designate x_2 as the entering variable in this step.

With the entering variable identified, our next task is to find a new neighboring basic feasible solution by also determining a leaving variable. Note that we cannot increase the entering variable, x_2, without bounds. As x_2 increases, s_1 or s_2 may decrease, and we must ensure that they remain nonnegative to maintain feasibility.

From row 1 and row 2, with $x_1 = 0$, we get

$$\begin{array}{llll} \text{Row 1:} & 2x_2 + s_1 = 6, & \text{which implies} & s_1 = 6 - 2x_2 \geq 0, \\ \text{Row 2:} & x_2 + s_2 = 8, & \text{which implies} & s_2 = 8 - x_2 \geq 0. \end{array}$$

Note that s_1 and s_2 need to be nonnegative for feasibility. To keep $s_1 \geq 0$, we cannot increase x_2 beyond 6/2=3. To keep $s_2 \geq 0$, we cannot increase x_2 beyond 8/1=8.

Thus, we let $x_2 = 3$ which makes $s_1 = 0$. In this step, x_2 is called the entering variable, and s_1 is called the leaving variable.

The test in the following remark summarizes the above discussion. Applying this test guarantees that the basic solution remains feasible. We will move forward in solving Example 10.22 after this remark.

Remark 10.2 (Minimum ratio test for choosing the leaving variable) *For each constraint row that has a positive coefficient*[a] *for the entering variable, we compute the ratio:*

$$\frac{\textit{right-hand side of row}}{\textit{coefficient of entering variable in row}}.$$

Among all these ratios, the nonbasic variable with smallest nonnegative ratio is the leaving variable.

[a]We do not consider the row(s) with negative coefficients.

Note that the smallest among all the ratios computed in Remark 10.2 is the largest value that the entering variable can take. Going back to Example 10.22, the ratios are:

$$\begin{array}{ll} \text{Row 1:} & \frac{6}{2} = 3; \quad \longleftarrow \quad \text{The winner!} \\ \text{Row 2:} & \frac{8}{1} = 8. \end{array}$$

Therefore, s_1 leaves the basis, i.e., it becomes nonbasic, and the entering variable x_2 takes its place.

We use elementary row operations in order to make the entering variable basic in the row that the minimum ratio test meets the requirement outlined in Remark 10.2.

$$
\begin{array}{rcccccccccccc}
\hline
R_0: & z & - & 2x_1 & - & 3x_2 & & & & & = & 0, \\
R_1: & & & x_1 & + & 2x_2 & + & s_1 & & & = & 6, \\
R_2: & & & 2x_1 & + & x_2 & & & + & s_2 & = & 8; \\
\hline
\frac{3}{2}R_1 + R_0 \to R_0: & z & - & \frac{1}{2}x_1 & & & + & \frac{3}{2}s_1 & & & = & 9, \\
\frac{1}{2}R_1 \to R_1: & & & \frac{1}{2}x_1 & + & x_2 & + & \frac{1}{2}s_1 & & & = & 3, \\
\frac{-1}{2}R_1 + R_2 \to R_2: & & & \frac{3}{2}x_1 & & & - & \frac{1}{2}s_1 & + & s_2 & = & 5; \\
\hline
R_0 + \frac{1}{3}R_2 \to R_0: & z & & & & & + & \frac{4}{3}s_1 & + & \frac{1}{3}s_2 & = & \frac{32}{3}, \\
R_1 - \frac{1}{3}R_2 \to R_1: & & & & & x_2 & + & \frac{2}{3}s_1 & - & \frac{1}{3}s_2 & = & \frac{4}{3}, \\
\frac{2}{3}R_2 \to R_2: & & & x_1 & & & - & \frac{1}{3}s_1 & + & \frac{2}{3}s_2 & = & \frac{10}{3}. \\
\hline
\end{array}
$$

We will complete the resolution of Example 10.22 once we address the following remark.

Remark 10.3 (Criterion for optimality in maximization) *In a maximization LP problem, the optimum is reached at the iteration where all the z-row coefficient of the nonbasic variables are nonnegative.*

Note that in our example we cannot improve the value of z anymore (by making s_1 or s_2 basic). Hence, we have an optimal solution. The optimal solution is $(x_1, x_2) = (10/3, 4/3)$ with the optimal value $z = 32/3$. ■

The full tableau method

The full tableau method provides a more convenient approach for conducting the necessary calculations required by the simplex method.

Simplex tableau for maximization If we have a maximization problem, the structure of the simplex tableau is as follows:

z	$\boldsymbol{x}$	rhs	
1	$\boldsymbol{c}^\mathsf{T} - \boldsymbol{c}_B^\mathsf{T} A_B^{-1} A$	$-\boldsymbol{c}_B^\mathsf{T} A_B^{-1} \boldsymbol{b}$	
0	$A_B^{-1} A$	$A_B^{-1} \boldsymbol{b}$	$= \boldsymbol{x}_B$

Here A_B is defined in (10.7) and $\boldsymbol{c}_B$ is the cost vector corresponding to the basic variables. We keep maintaining and updating the above table till we reach the optimality. Example 10.23 resolves Example 10.22 using the simplex tableau method.

Example 10.23 (Example 10.22 revisited) Use the simplex tableau method to solve the following maximization problem.

$$
\begin{array}{lll}
\max & z = 2x_1 + 3x_2 & \\
\text{s.t.} & x_1 + 2x_2 & \leq \ 6, \\
 & 2x_1 + x_2 & \leq \ 8, \\
 & x_1, x_2 & \geq \ 0.
\end{array}
$$

Solution After introducing slack variables, we obtain the standard form problem given in (10.11). Note that $\boldsymbol{x} = (0, 0, 6, 8)$ is a basic feasible solution. Hence, we have the following initial tableau:

z	x_1	x_2	s_1	s_2	rhs	
1	−2	−3	0	0	0	
0	1	2	1	0	6	$= s_1$
0	2	1	0	1	8	$= s_2$

Since we are maximizing the objective function, we select a nonbasic variable with the greatest positive reduced cost to be the one that enters the basis. Indicating the pivot element with a circled number, we obtain the following tableau:

z	x_1	x_2	s_1	s_2	rhs	
1	−1/2	0	3/2	0	9	
0	1/2	1	1/2	0	3	$= x_2$
0	3/2	0	−1/2	1	5	$= s_2$

Note that we brought x_2 into the basis and s_1 exited. We then bring x_1 into the basis; s_2 exits and we obtain the following tableau:

z	x_1	x_2	s_1	s_2	rhs	
1	0	0	4/3	1/3	32/3	$= z$
0	0	1	2/3	−1/3	4/3	$= x_2$
0	1	0	−1/3	2/3	10/3	$= x_1$

The reduced costs in the zeroth row of the tableau are all nonnegative, so the current basic feasible solution is optimal. In terms of the original variables x_1 and x_2, this solution is $\boldsymbol{x} = (10/3, 4/3)$. The optimal value is $z = 32/3$.

The above series of tableaux can be combined in one single table as follows:

EROs	z	x_1	x_2	s_1	s_2	rhs	MR
R_0 :	1	−2	−3	0	0	0	
R_1 :	0	1	2	1	0	6	6/2
R_2 :	0	2	1	0	1	8	8/1
$R_0 + \frac{3}{2}R_1 \to R_0$:	1	−1/2	0	3/2	0	9	
$\frac{1}{2}R_1 \to R_1$:	0	1/2	1	1/2	0	3	3/0.5
$\frac{-1}{2}R_1 + R_2 \to R_2$:	0	3/2	0	−1/2	1	5	5/1.5
$R_0 + \frac{1}{3}R_2 \to R_0$:	1	0	0	4/3	1/3	32/3	
$R_1 - \frac{1}{3}R_2 \to R_1$:	0	0	1	2/3	−1/3	4/3	Optimal
$\frac{2}{3}R_2 \to R_2$:	0	1	0	−1/3	2/3	10/3	tableau!

■

Example 10.24 Use the simplex method to solve the following LP.

$$\begin{array}{llcll}
\max & z = 2x_1 - x_2 + x_3 & & & \\
\text{s.t.} & 3x_1 + x_2 + x_3 & \leq & 60, & (\text{adding } s_1) \\
& x_1 - x_2 + 2x_3 & \leq & 10, & (\text{adding } s_2) \\
& x_1 + x_2 - x_3 & \leq & 20, & (\text{adding } s_3) \\
& x_1, x_2, x_3 & \geq & 0. &
\end{array}$$

Solution We have the following tableaux:

EROs	z	x_1	x_2	x_3	s_1	s_2	s_3	rhs	MR
R_0 :	1	−2	1	−1	0	0	0	0	
R_1 :	0	3	1	1	1	0	0	60	60/3
R_2 :	0	1	−1	2	0	1	0	10	10/1
R_3 :	0	1	1	−1	0	0	1	20	20/1
$R_0 + 2R_2 \to R_0$:	1	0	−1	3	0	2	0	20	
$R_1 - 3R_2 \to R_1$:	0	0	4	−5	1	−3	0	30	30/4
$R_2 \to R_2$:	0	1	−1	2	0	1	0	10	
$-R_2 + R_3 \to R_3$:	0	0	2	−3	0	−1	1	10	10/2
$R_0 + \frac{1}{2}R_3 \to R_0$:	1	0	0	3/2	0	3/2	1/2	25	
$R_1 - 2R_3 \to R_1$:	0	0	0	1	1	−1	−2	10	Optimal
$R_2 + \frac{1}{2}R_3 \to R_2$:	0	1	0	1/2	0	1/2	1/2	15	tableau!
$\frac{1}{2}R_3 \to R_3$:	0	0	1	−3/2	0	−1/2	1/2	5	

The optimal solution is given by $(x_1, x_2, x_3) = (15, 5, 0)$, and the optimal value is $z = 25$. ■

Detecting the existence of alternative optimal solutions The simplex method can tell if alternative optimal solutions (i.e., infinitely many solutions) exist. The following remark signifies a pivotal insight into LP: A condition that opens the door to alternative optimal solutions that yield the same optimal value for the LP problem.

Remark 10.4 *If the coefficient of a nonbasic variable in the zeroth row of the tableau is zero, then the linear programming problem has alternative optimal solutions.*

We have the following example.

Example 10.25 Use the simplex tableau method to solve the following maximization problem.

$$\begin{array}{llcll}
\max & z = 4x_1 + x_2 & & & \\
\text{s.t.} & 8x_1 + 2x_2 & \leq & 16, & (\text{adding } s_1) \\
& 5x_1 + 2x_2 & \leq & 12, & (\text{adding } s_2) \\
& x_1, x_2 & \geq & 0. &
\end{array}$$

Solution We have the following tableaux:

EROs	z	x_1	x_2	s_1	s_2	rhs	MR
R_0 :	1	-4	-1	0	0	0	
R_1 :	0	8	2	1	0	16	$16/8 = 2$
R_2 :	0	5	2	0	1	12	$12/5 = 2.4$
$R_0 + \frac{1}{2}R_1 \to R_0$:	1	0	0	$1/2$	0	8	
$\frac{1}{8}R_1 \to R_1$:	0	1	$1/4$	$1/8$	0	2	This tableau
$\frac{-5}{8}R_1 + R_2 \to R_2$:	0	0	$3/4$	$-5/8$	1	2	is optimal!
$R_0 \to R_0$:	1	0	0	$1/2$	0	8	
$R_1 - \frac{1}{3}R_2 \to R_1$:	0	1	0	$1/3$	$-1/3$	$4/3$	This tableau
$\frac{4}{3}R_2 \to R_2$:	0	0	1	$-5/6$	$4/3$	$8/3$	is also optimal!

In view of Remark 10.4, we have alternative optimal solutions. An optimal solution is given by $(x_1, x_2) = (2, 0)$. Another optimal solution is given by $(x_1, x_2) = (4/3, 8/3)$. The optimal value is $z = 8$. As an exercise for the reader, use the graphical method to reach the same conclusion. ■

Detecting unboundedness The simplex method can be used to detect the unboundedness. The following remark tells us when we have an unbounded problem.

Remark 10.5 *If there is no candidate for the minimum ratio test, then the linear programming problem is unbounded.*

Example 10.26 Use the simplex tableau method to solve the following maximization problem.

$$\begin{array}{lrcll} \max & z = 2x_2 & & & \\ \text{s.t.} & x_1 - x_2 & \leq & 4, & \text{(adding } s_1) \\ & -x_1 + x_2 & \leq & 1, & \text{(adding } s_2) \\ & x_1, x_2 & \geq & 0. & \end{array}$$

Solution We have the following tableaux:

EROs	z	x_1	x_2	s_1	s_2	rhs	MR
R_0 :	1	0	-2	0	0	0	
R_1 :	0	1	-1	1	0	4	$16/8 = 2$
R_2 :	0	-1	1	0	1	1	$12/5 = 2.4$
$R_0 + 2R_2 \to R_0$:	1	-2	0	0	2	2	
$R_1 + R_2 \to R_1$:	0	0	0	1	1	5	The LP is
$R_2 \to R_2$:	0	-1	1	0	1	1	unbounded!

Note that there is no candidate for the minimum ratio test. In view of Remark 10.5, we have an unbounded linear programming problem. As an exercise for the reader, use the graphical method to reach the same conclusion. ■

Breaking ties The following remark tells us how to break ties for entering or leaving variables if any.

Remark 10.6 *If there are ties for entering or leaving, we can break them arbitrarily.*

Later in this section, we will delve into more determined strategies for resolving tie-breakers when it comes to the selection of nonbasic variables. More specifically, we will refer to other remarks (Remarks 10.10 and 10.11) that provide more guidance on breaking ties, not only for the selection of nonbasic variables but also for deciding which variables should enter or leave the set of basic variables. By exploring these tie-breaking strategies, we aim to enhance more clarity and effectiveness of the decision-making process in LP problem-solving.

Example 10.27 Use the simplex tableau method to solve the following maximization problem.

$$\begin{array}{lllll} \max & z = x_1 + x_2 & & & \\ \text{s.t.} & x_1 + x_2 + x_3 & \leq & 1, & (\text{adding } s_1) \\ & x_1 + 2x_3 & \leq & 1, & (\text{adding } s_2) \\ & x_1, x_2, x_3 & \geq & 0. & \end{array}$$

Solution We have the following tableaux:

EROs	z	x_1	x_2	x_3	s_1	s_2	rhs	MR
R_0 :	1	−1	−1	0	0	0	0	
R_1 :	0	1	1	1	1	0	1	1 (A candidate)
R_2 :	0	1	0	2	0	1	1	1 (Another candidate!)
$R_0 + R_1 \to R_0$:	1	0	0	1	1	0	1	Alternative
$R_1 \to R_1$:	0	1	1	1	1	0	1	optimal
$-R_1 + R_2 \to R_2$:	0	0	−1	1	−1	1	0	solutions!

We have alternative optimal solutions. An optimal solution is given by $(x_1, x_2, x_3) = (1, 0, 0)$. The optimal value is $z = 1$. ■

Simplex tableau for minimization Up to this point, our exploration has been centered on the simplex method as a means of tackling linear maximization problems. However, it is important to note that this method is versatile enough to be employed for solving linear minimization problems as well. When it comes to linear minimization LP problems, the guidelines pertaining to entering variables and determining optimality are diametrically opposite to those governing maximization LP problems. Further elaboration on these nuances is provided in the subsequent remarks.

Remark 10.7 (Criterion for choosing the entering variable in minimization) *The entering variable in a minimization LP problem is the nonbasic variable having the most positive coefficient in the z-row.*

Remark 10.8 (Criterion for optimality in minimization) *In a minimization LP problem, the optimum is reached at the iteration where all the z-row coefficient of the nonbasic variables are nonpositive.*

In essence, the above remarks underscore the fundamental differences in approach between solving maximization and minimization LP problems using the simplex method, particularly when it comes to determining which variables to add to or remove from the basis and when to declare a solution as optimal. We have the following examples.

Example 10.28 Use the simplex tableau method to solve the following minimization problem.

$$\begin{array}{llcll} \min & z = -x_1 - x_2 & & & \\ \text{s.t.} & x_1 - x_2 & \leq & 1, & (\text{adding } s_1) \\ & x_1 + x_2 & \leq & 2, & (\text{adding } s_2) \\ & x_1, x_2 & \geq & 0. & \end{array}$$

Solution We have the following tableaux:

EROs	z	x_1	x_2	s_1	s_2	rhs	MR
R_0 :	1	1	1	0	0	0	
R_1 :	0	1	−1	1	0	1	$1/1 = 1$
R_2 :	0	1	1	0	1	2	$2/1 = 2$
$R_0 - R_1 \to R_0$:	1	0	2	−1	0	−1	
$R_1 \to R_1$:	0	1	−1	1	0	1	
$-R_1 + R_2 \to R_2$:	0	0	2	−1	1	1	$1/2$
$R_0 - (1/2)R_2 \to R_0$:	1	0	0	0	−1	−2	
$R_1 + (1/2)R_2 \to R_1$:	0	1	0	1/2	1/2	3/2	*Optimal*
$(1/2)R_2 \to R_2$:	0	0	1	−1/2	1/2	1/2	*tableau!*

The reduced costs in the zeroth row of the tableau are all nonpositive, so the current basic feasible solution is optimal. In terms of the original variables x_1 and x_2, this solution is $\boldsymbol{x} = (3/2, 1/2)$. The optimal value is $z = -2$. In addition, since the coefficient of the nonbasic variable s_1 in the zeroth row of the last tableau is zero, we have alternative optimal solutions. ■

Example 10.29 Use the simplex tableau method to solve the following minimization problem.

$$\begin{array}{llcll} \min & z = -x_1 - x_2 & & & \\ \text{s.t.} & 2x_1 + x_2 & \leq & 4, & (\text{adding } s_1) \\ & 3x_1 + 5x_2 & \leq & 15, & (\text{adding } s_2) \\ & x_1, x_2 & \geq & 0. & \end{array}$$

Solution We have the following tableaux:

EROs	z	x_1	x_2	s_1	s_2	rhs	MR
$R_0:$	1	1	1	0	0	0	
$R_1:$	0	2	1	1	0	4	$4/2 = 2$
$R_2:$	0	3	5	0	1	15	$15/3 = 5$
$R_0 - \frac{1}{2}R_1 \to R_0:$	1	0	$1/2$	$-1/2$	0	-2	
$\frac{1}{2}R_1 \to R_1:$	0	1	$1/2$	$1/2$	0	2	$2/0.5 = 4$
$-\frac{3}{2}R_1 + R_2 \to R_2:$	0	0	$7/2$	$-3/2$	1	9	$9/3.5 \approx 2.57$
$R_0 - \frac{1}{7}R_2 \to R_0:$	1	0	0	$-2/7$	$-1/7$	$-23/7$	
$R_1 - \frac{1}{7}R_2 \to R_1:$	0	1	0	$5/7$	$-1/7$	$5/7$	Optimal
$\frac{2}{7}R_2 \to R_2:$	0	0	1	$-3/7$	$2/7$	$18/7$	tableau!

The optimal solution is $\boldsymbol{x} = (5/7, 18/7)$. The optimal value is $z = -23/7$. ■

Problems with nonpositive variables and/or free variables Up to this point, we have explored the simplex method as a means to address linear optimization problems featuring variables constrained to be nonnegative. Handling scenarios where variables are required to be nonpositive is relatively straightforward. The approach involves introducing a new nonnegative variable that represents the negation of the original variable. In essence, if we encounter a variable, let us call it x_j, such that $x_j \leq 0$, we substitute it with $-x'_j$ while including the constraint $x'_j \geq 0$.

Similarly, when dealing with problems that include unrestricted-in-sign variables (often referred to as free variables), the solution approach remains uncomplicated. Here, the strategy is to introduce two fresh nonnegative variables, x'_j and x''_j, with the constraint that their difference equals the original variable x_j. Consequently, if we encounter an unrestricted-in-sign variable, denoted as x_j, we replace it with $x'_j - x''_j$ and supplement the model with the constraints $x'_j, x''_j \geq 0$. Notably, only one of x'_j or x''_j can be part of the basis in a given tableau, but not both. To further elucidate this, an illustrative example follows.

Example 10.30 Use the simplex tableau method to solve the maximization LP:

$$
\begin{array}{lllll}
\max & z = 2x_1 + x_2 & & \\
\text{s.t.} & 3x_1 + x_2 & \leq & 6, \\
 & x_1 + x_2 & \leq & 4, \\
 & x_1 & \geq & 0.
\end{array}
$$

Solution Note that the variable x_2 is unrestricted-in-sign. An equivalent problem is

$$
\begin{array}{lllll}
\max & z = 2x_1 + x'_2 - x''_2 & & & \\
\text{s.t.} & 3x_1 + x'_2 - x''_2 & \leq & 6, & (\text{adding } s_1) \\
 & x_1 + x'_2 - x''_2 & \leq & 4, & (\text{adding } s_2) \\
 & x_1, x'_2, x''_2 & \geq & 0. &
\end{array}
$$

We then have the following tableaux:

EROs	z	x_1	x_2'	x_2''	s_1	s_2	rhs	MR
R_0 :	1	−2	−1	1	0	0	0	
R_1 :	0	3	1	−1	1	0	6	$6/3 = 2$
R_2 :	0	1	1	−1	0	1	4	$4/1 = 4$
$R_0 + \frac{2}{3}R_1 \to R_0$:	1	0	−1/3	1/3	2/3	0	4	
$\frac{1}{3}R_1 \to R_1$:	0	1	1/3	−1/3	1/3	0	2	$2/(1/3) = 6$
$-\frac{1}{3}R_1 + R_2 \to R_2$:	0	0	2/3	−2/3	−1/3	1	2	$2/(2/3) = 3$
$R_0 + \frac{1}{2}R_2 \to R_0$:	1	0	0	0	1/2	1/2	5	
$R_1 - \frac{1}{2}R_2 \to R_1$:	0	1	0	0	1/2	−1/2	1	Optimal
$\frac{3}{2}R_2 \to R_2$:	0	0	1	−1	−1/2	3/2	3	tableau!

An optimal solution is given by $x_1 = 1$ and $x_2 = x_2' - x_2'' = 3 - 0 = 3$. The optimal value is $z = 5$. We point out that the columns corresponding to x_2' and x_2'' are always identical but with opposite signs. ■

The big-M method

Until this point, our exploration of the simplex method has centered on resolving linear optimization problems with inequality constraints of the form "≤". An intriguing question that arises is how to adapt this method to address maximization and minimization problems featuring inequality constraints of "≥" or "=" nature. To tackle such problem types, a widely employed technique is known as the big-M method. Essentially, the big-M method extends the applicability of the simplex algorithm to encompass problems encompassing "greater-than" and/or "equal" constraints.

Problems with "greater-than" and/or "equal" constraints In cases where we solely encounter "≤" constraints, it is relatively straightforward to identify an initial basic feasible solution, typically involving the slack variables. However, a critical question emerges: how can we establish an initial basic feasible solution when confronted with "≥" and/or "=" constraints? The big-M method offers a solution to this conundrum by introducing artificial variables for each "≥" and "=" constraints, following the steps in Workflow 10.6.

Workflow 10.6 (The big-M method) *We solve problems with "greater-than" and/or "equal" constraints by following six steps:*

(*i*) *Modify constraints as needed so that all the right-hand side values are nonnegative.*

(*ii*) *Add an artificial variable, say a_i, for constraint i if it is a "≥" or "=" constraint. Then add the nonnegativity constraint $a_i \geq 0$.*

(*iii*) *Add $\pm M a_i$ to the objective function, where M is a big positive number, as follows:*

- *For a maximization LP problem, add $-Ma_i$.*
- *For a minimization LP problem, add $+Ma_i$.*

(iv) *Convert the resulting LP into the standard form by adding slack/excess variables.*

(v) *Convert the LP into the canonical form and make the coefficient of a_i in the zeroth row zero by using elementary row operations involving M.*

(vi) *Operate Steps (iii)-(vi) in Workflow 10.5.*

As a direct application of Workflow 10.6, we have the following example.

Example 10.31 Use the simplex tableau method to solve the following minimization problem.

$$\begin{array}{lll} \min & z = 2x_1 + 3x_2 & \\ \text{s.t.} & 2x_1 + x_2 & \geq \ 4, \\ & x_1 - x_2 & \geq \ -1, \\ & x_1, x_2 & \geq \ 0. \end{array}$$

Solution Our initial course of action entails the execution of the procedure outlined in Steps (i) through (v) as laid out in Workflow 10.6. In this sequence, our primary objective is to bring about a modification to the constraints of the problem. Our aim is to ensure that all the right-hand side values within these constraints are adjusted to be nonnegative. Consequently, we get

$$\begin{array}{lll} \min & z = 2x_1 + 3x_2 & \\ \text{s.t.} & 2x_1 + x_2 & \geq \ 4, \\ & -x_1 + x_2 & \leq \ 1, \\ & x_1, x_2 & \geq \ 0. \end{array}$$

Then, we add an artificial variable a_i for constraint i if it is a "≥" or "=" constraint. Then add $a_i \geq 0$. We also add Ma_i to the objective function, where M is a big positive number. This yields

$$\begin{array}{lll} \min & z = 2x_1 + 3x_2 + Ma_1 & \\ \text{s.t.} & 2x_1 + x_2 + a_1 & \geq \ 4, \\ & -x_1 + x_2 & \leq \ 1, \\ & x_1, x_2, a_1 & \geq \ 0. \end{array}$$

Next, we convert the resulting LP into the standard form to get

$$\begin{array}{lll} \min & z = 2x_1 + 3x_2 + Ma_1 & \\ \text{s.t.} & 2x_1 + x_2 + a_1 - e_1 & = \ 4, \\ & -x_1 + x_2 + s_1 & = \ 1, \\ & x_1, x_2, s_1, e_1, a_1 & \geq \ 0. \end{array}$$

Now, our next step involves the conversion of the LP problem into its canonical form. To achieve this, we employ elementary row operations to manipulate the coefficients of the variable a_i in the zeroth row, making sure that they become zero. Following this preliminary step, we proceed to execute the subsequent steps, specifically Steps (iii) through (vi), as delineated in Workflow 10.5.

This can be seen in the subsequent tableaux.

EROs	z	x_1	x_2	e_1	s_1	a_1	rhs	
$R_0:$	1	-2	-3	0	0	$-M$	0	Not in
$R_1:$	0	2	1	-1	0	1	4	canonical
$R_2:$	0	-1	1	0	1	0	1	form
$R_0 + MR_1 \to R_0:$	1	$2M-2$	$M-3$	$-M$	0	0	$4M$	In
$R_1 \to R_1:$	0	2	1	-1	0	1	4	canonical
$R_2 \to R_2:$	0	-1	1	0	1	0	1	form
$R_0 + (1-M)R_1 \to R_0:$	1	0	-2	-1	0	$-M+1$	4	
$\frac{1}{2}R_1 \to R_1:$	0	1	$1/2$	$-1/2$	0	$1/2$	2	Optimal
$\frac{1}{2}R_1 + R_2 \to R_2:$	0	0	$3/2$	$-1/2$	1	$1/2$	3	tableau!

The optimal solution is given by $(x_1, x_2) = (2, 0)$. The optimal value is given by $z = 4$. ■

Example 10.32 Use the simplex tableau method to solve the following minimization problem.

$$\begin{array}{lll} \min & z = 2x_1 - 3x_2 & \\ \text{s.t.} & x_1 + 3x_2 & \leq 9, \\ & 2x_1 + 5x_2 & \geq -6, \\ & x_2 & \geq 1, \\ & x_2 & \geq 0. \end{array}$$

Solution Note that the variable x_1 does not have any sign restrictions, making it what is known as unrestricted-in-sign. We have the following tableaux:

EROs	z	x_1'	x_1''	x_2	s_1	s_2	e_1	a_1	rhs	
$R_0:$	1	-2	2	3	0	0	0	$-M$	0	
$R_1:$	0	1	-1	3	1	0	0	0	9	
$R_2:$	0	-2	2	-5	0	1	0	0	6	
$R_3:$	0	0	0	1	0	0	-1	1	1	
$R_0 + MR_3 \to R_0:$	1	-2	2	$M+3$	0	0	$-M$	0	M	
$R_1 \to R_1:$	0	1	-1	3	1	0	0	0	9	
$R_2 \to R_2:$	0	-2	2	-5	0	1	0	0	6	
$R_3 \to R_3:$	0	0	0	1	0	0	-1	1	1	
$R_0 - (M+3)R_3 \to R_0:$	1	-2	2	0	0	0	3	$-M-3$	-3	
$R_1 - 3R_3 \to R_1:$	0	1	-1	0	1	0	3	-3	6	
$R_2 + 5R_3 \to R_2:$	0	-2	2	0	0	1	-5	5	11	
$R_3 \to R_3:$	0	0	0	1	0	0	-1	1	1	
$R_0 - R_1 \to R_0:$	1	-3	3	0	-1	0	0	$-M$	-9	
$\frac{1}{3}R_1 \to R_1:$	0	$1/3$	$-1/3$	0	$1/3$	0	1	-1	2	
$\frac{5}{3}R_1 + R_2 \to R_2:$	0	$-1/3$	$1/3$	0	$5/3$	1	0	0	21	
$\frac{1}{3}R_1 + R_3 \to R_3:$	0	$1/3$	$-1/3$	1	$1/3$	0	0	0	3	
$R_0 - 9R_2 \to R_0:$	1	0	0	0	-16	-9	0	$-M$	-198	O
$R_1 + R_2 \to R_1:$	0	0	0	0	2	3	1	-1	23	P
$3R_2 \to R_2:$	0	-1	1	0	5	3	0	0	63	T
$R_2 + R_3 \to R_3:$	0	0	0	1	2	1	0	0	24	M

We contemplated an equivalent problem to establish a version of the problem with equality constraints while ensuring that all variables involved are nonnegative. Our focus has shifted to the following form:

$$\begin{array}{lll} \min & z = 2(x_1' - x_1'') - 3x_2 + Ma_1 & \\ \text{s.t.} & (x_1' - x_1'') + 3x_2 + s_1 & = & 9, \\ & -2(x_1' - x_1'') - 5x_2 + s_2 & = & 6, \\ & x_2 - e_1 + a_1 & = & 1, \\ & x_1', x_1'', x_2, s_1, s_2, e_1, a_1 & \geq & 0. \end{array}$$

So, the optimal solution is given by $x_1 = x_1' - x_1'' = 0 - 63 = -63$, $x_2 = 24$ and $e_1 = 23$, hence $(x_1, x_2) = (-63, 24)$. The optimal value is $z = -198$. ■

Detecting infeasibility The big-M method can be used to detect the infeasibility. The following remark tells us when we have an infeasible problem.

Remark 10.9 *If any artificial variable is basic in the optimal tableau, i.e. $a_i > 0$ for some i, then the linear programming problem is infeasible.*

As a direct application, we have the following example.

Example 10.33 Use the simplex tableau method to solve the following minimization problem.

$$\begin{array}{llll} \min & z = 3x_1 & & \\ \text{s.t.} & 2x_1 + x_2 & \geq & 6, \\ & 3x_1 + 2x_2 & = & 4, \\ & x_1, x_2 & \geq & 0. \end{array}$$

Solution Considering an equivalent problem with equality constraints and nonnegative variables only, we are interested in a problem of the form

$$\begin{array}{llll} \min & z = 3x_1 + Ma_1 + Ma_2 & & \\ \text{s.t.} & 2x_1 + x_2 + a_1 - e_1 & = & 6, \\ & 3x_1 + 2x_2 + a_2 & = & 4, \\ & x_1, x_2, a_1, a_2, e_1 & \geq & 0. \end{array}$$

We then have the following tableaux:

EROs	z	x_1	x_2	e_1	s_1	a_1	rhs	
$R_0:$	1	3	0	0	$-M$	$-M$	0	
$R_1:$	0	2	1	-1	1	0	6	
$R_2:$	0	3	2	0	0	1	4	
$R_0 + MR_1 + MR_2 \to R_0:$	1	$5M - 3$	$3M$	$-M$	0	0	$10M$	
$R_1 \to R_1:$	0	2	1	-1	1	0	6	
$R_2 \to R_2:$	0	3	2	0	0	1	4	
$R_0 + (1 - \frac{5}{3}M)R_2 \to R_0:$	1	0	$-M/3 + 2$	$-M$	0	$-5M/3 + 1$	$10M/3 + 4$	O
$R_1 - \frac{2}{3}R_2 \to R_1:$	0	0	$-1/3$	-1	1	$-2/3$	$10/3$	P
$\frac{1}{3}R_2 \to R_2:$	0	1	$2/3$	0	0	$1/3$	$4/3$	T

The last tableau is optimal. Since $a_1 = 10/3 > 0$, the problem is infeasible. As an exercise for the reader, use the graphical method to reach the same conclusion. ■

Summary of the simplex method steps We now summarize the above description of the simplex method.

Workflow 10.7 (Overview of the simplex method) *We solve a linear optimization problem by operating the following steps:*

(*i*) *Modify constraints as needed so that all the right-hand side values are nonnegative.*

(*ii*) *Add an artificial variable, say a_i, for constraint i if it is a "$\geq$" or "$=$" constraint. Then add the nonnegativity constraint $a_i \geq 0$.*

(*iii*) *Add $\pm Ma_i$ to the objective function, where M is a big positive number.*

(*iv*) *Convert the resulting LP into the standard form by adding slack/excess variables.*

(*v*) *Convert the LP into the canonical form and make the coefficient of a_i in the zeroth row zero by using elementary row operations involving M.*

(*vi*) *Find a basic feasible solution for the canonical form.*

(*vii*) *If the current basic feasible solution is optimal, stop. If not, move to an adjacent basic feasible solution with a higher value for the objective function by applying elementary row operations and noting that*

- *If the coefficient of a nonbasic variable in the zeroth row of the tableau is zero, then the LP has alternative optimal solutions.*
- *If there is no candidate for the minimum ratio test, then the LP is unbounded.*
- *If any artificial variable is basic in the optimal tableau, i.e. $a_i > 0$ for some i, then the LP is infeasible.*

(*viii*) *Go to Step (vii).*

We end this part with the following theorem which gives the amount of computation per iteration (worst behavior) in terms of the size of the coefficient matrix. We give Theorem 10.6, without proof. For a proof, see, for example, Bertsimas and Tsitsiklis [Bertsimas and Tsitsiklis, 1997, Section 3].

Theorem 10.6 *Assume that the matrix A in Problem (10.9) is $m \times n$. The number of arithmetic operations in each iteration of the simplex tableau algorithm solving Problem (10.9) is $O(mn)$.*

Note that the estimate of the computational complexity in Theorem 10.6 refers to a single iteration. This complexity estimate is for both the worst-case time and the best-case time.

Anticycling

The simplex method may encounter a phenomenon known as cycling, where it struggles to make progress. To address this issue and ensure that the simplex method always terminates, two anticycling rules have been developed. These rules are the lexicographic rule and Bland's rule, named after Robert Bland, who discovered it in 1976. In this section, we will focus our discussion on Bland's rule. Part of Exercise 10.21 targets the lexicographic rule. However, there are a number of good references to learn this and other anticycling rules, see for example [Bertsimas and Tsitsiklis, 1997, Section 3.4].

Below we outline the pivoting rule for choosing the entering and leaving variables.

> **Remark 10.10 (Pivoting rule)** *In the ordinary pivoting rule, we choose the entering variable with the most negative c_j and choose the leaving variable according to the minimum ratio test. If there are ties, break them by picking the variable with the smallest index.*

Considering the above rule, we may get back to the starting tableau after some iterations in some LP problems, as in the following example attributed to Bertsimas and Tsitsiklis [1997].

Example 10.34 Consider the following LP problem.

$$\begin{array}{lllll} \max & z = \frac{3}{4}x_1 - 20x_2 + \frac{1}{2}x_3 - 6x_4 & & & \\ \text{s.t.} & \frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 & \leq & 0, & (\text{adding } x_5) \\ & \frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 & \leq & 0, & (\text{adding } x_6) \\ & x_3 + 6x_4 & \leq & 1, & (\text{adding } x_7) \\ & x_1, x_2, x_3, x_4 & \geq & 0. & \end{array} \tag{10.13}$$

If we use the simplex method to solve Problem 10.13 with the ordinary rule, we obtain the simplex tableau in Table 10.3.

Note that the ending tableau is identical to the starting tableau. This means that the simplex method is cycling here! ■

Example 10.34 will be revisited in order to avoid cycling after discussing Bland's rule.

Bland's rule Bland's rule (or the minimum index rule) is one of the algorithmic refinements of the simplex method to avoid cycling.

> **Remark 10.11 (Bland's rule)** *Bland's rule under which the simplex method for linear optimization terminates is as follows:*
>
> - *Choose the entering variable x_j such that j is the smallest index with $c_j < 0$.*
> - *Choose the leaving variable according to the minimum ratio test, and in the case of ties, choose the one with the smallest index.*

An illustrative example follows to elucidate Bland's rule.

Example 10.35 (Example 10.34 revisited) If we use the simplex method to solve Problem 10.13 with Bland's rule, we obtain the simplex tableau in Table 10.4.

Since we have applied Bland's rule, the simplex method has terminated. The last tableau is optimal, the optimal solution is $\boldsymbol{x} = (1; 0; 1; 0)$, and the optimal value is $z = 5/4$. ■

EROs	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
$R_0:$	1	$-3/4$	20	$-1/2$	6	0	0	0	0
$R_1:$	0	1/4	-8	-1	9	1	0	0	0
$R_2:$	0	1/2	-12	$-1/2$	3	0	1	0	0
$R_3:$	0	0	0	1	6	0	0	1	1
$R_0+3R_1 \to R_0:$	1	0	-4	$-7/2$	33	3	0	0	0
$4R_1 \to R_1:$	0	1	-32	-4	36	4	0	0	0
$-2R_1+R_2 \to R_2:$	0	0	4	3/2	-15	-2	1	0	0
$R_3 \to R_3:$	0	0	0	1	0	0	0	1	1
$R_0+R_2 \to R_0:$	1	0	0	-2	18	1	1	0	0
$R_1+8R_2 \to R_1:$	0	1	0	8	-84	-12	8	0	0
$\frac{1}{4}R_2 \to R_2:$	0	0	1	3/8	$-15/4$	$-1/2$	1/4	0	0
$R_3 \to R_3:$	0	0	0	1	0	0	0	1	1
$R_0+\frac{1}{4}R_1 \to R_0:$	1	1/4	0	0	-3	-2	3	0	0
$\frac{1}{8}R_1 \to R_1:$	0	1/8	0	1	$-21/2$	$-3/2$	1	0	0
$-\frac{3}{64}R_1+R_2 \to R_2:$	0	$-3/64$	1	0	3/16	1/16	$-1/8$	0	0
$-\frac{1}{8}R_1+R_3 \to R_3:$	0	$-1/8$	0	0	21/2	3/2	-1	1	1
$R_0+16R_2 \to R_0:$	1	$-1/2$	16	0	0	-1	1	0	0
$R_1+56R_2 \to R_1:$	0	$-5/2$	56	1	0	2	-6	0	0
$\frac{16}{3}R_2 \to R_2:$	0	$-1/4$	16/3	0	1	1/3	$-2/3$	0	0
$-56R_2+R_3 \to R_3:$	0	5/2	-56	0	0	-2	6	1	1
$R_0+\frac{1}{2}R_1 \to R_0:$	1	$-7/4$	44	1/2	0	0	-2	0	0
$\frac{1}{2}R_1 \to R_1:$	0	$-5/4$	28	1/2	0	1	-3	0	0
$-\frac{1}{6}R_1+R_2 \to R_2:$	0	1/6	-4	$-1/6$	1	0	1/3	0	0
$R_1+R_3 \to R_3:$	0	0	0	1	0	0	0	1	1
$R_0+6R_2 \to R_0:$	1	$-3/4$	20	$-1/2$	6	0	0	0	0
$R_1+9R_2 \to R_1:$	0	1/4	-8	-1	9	1	0	0	0
$3R_2 \to R_2:$	0	1/2	-12	$-1/2$	3	0	1	0	0
$R_3 \to R_3:$	0	0	0	1	6	0	0	1	1

Table 10.3: The simplex tableau of Example 10.34.

The question that remains now in this context is: How to prevent cycling when we solve linear maximization problems? One answer stems from the following remark.

Remark 10.12 *If you start with a minimization problem, say* $\min f(\boldsymbol{x})$ *subject to* $\boldsymbol{x} \in S$, *where* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *is a function and* S *is a set, then an equivalent maximization problem is* $\max -f(x)$ *subject to* $\boldsymbol{x} \in S$. *Similarly, if you start with a maximization problem, say* $\max f(\boldsymbol{x})$ *subject to* $\boldsymbol{x} \in S$, *then an equivalent minimization problem is* $\min -f(x)$ *subject to* $\boldsymbol{x} \in S$.

In essence, the conversion between minimization and maximization problems provides an essential equivalence within optimization. By simply negating the objective function and retaining the integrity of the constraints, whether starting from a minimization or maximization problem, an equivalent problem with the opposite optimization objective is established.

EROs	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
R_0 :	1	−3/4	20	−1/2	6	0	0	0	0
R_1 :	0	1/4	−8	−1	9	1	0	0	0
R_2 :	0	1/2	−12	−1/2	3	0	1	0	0
R_3 :	0	0	0	1	6	0	0	1	1
$R_0 + 3R_1 \to R_0$:	1	0	−4	−7/2	33	3	0	0	0
$4R_1 \to R_1$:	0	1	−32	−4	36	4	0	0	0
$-2R_1 + R_2 \to R_2$:	0	0	4	3/2	−15	−2	1	0	0
$R_3 \to R_3$:	0	0	0	1	0	0	0	1	1
$R_0 + R_2 \to R_0$:	1	0	0	−2	18	1	1	0	0
$R_1 + 8R_2 \to R_1$:	0	1	0	8	−84	−12	8	0	0
$\frac{1}{4}R_2 \to R_2$:	0	0	1	3/8	−15/4	−1/2	1/4	0	0
$R_3 \to R_3$:	0	0	0	1	0	0	0	1	1
$R_0 + \frac{1}{4}R_1 \to R_0$:	1	1/4	0	0	−3	−2	3	0	0
$\frac{1}{8}R_1 \to R_1$:	0	1/8	0	1	−21/2	−3/2	1	0	0
$-\frac{3}{64}R_1 + R_2 \to R_2$:	0	−3/64	1	0	3/16	1/16	−1/8	0	0
$-\frac{1}{8}R_1 + R_3 \to R_3$:	0	−1/8	0	0	21/2	3/2	−1	1	1
$R_0 + 16R_2 \to R_0$:	1	−1/2	16	0	0	−1	1	0	0
$R_1 + 56R_2 \to R_1$:	0	−5/2	56	1	0	2	−6	0	0
$\frac{16}{3}R_2 \to R_2$:	0	−1/4	16/3	0	1	1/3	−2/3	0	0
$-56R_2 + R_3 \to R_3$:	0	5/2	−56	0	0	−2	6	1	1
$R_0 + \frac{1}{5}R_3 \to R_0$:	1	0	24/5	0	0	−7/5	11/5	1/5	1/5
$R_1 + R_3 \to R_1$:	0	0	0	1	0	0	0	1	1
$R_2 + \frac{1}{10}R_3 \to R_2$:	0	0	−4/15	0	1	2/15	−11/15	1/10	1/10
$\frac{2}{5}R_3 \to R_3$:	0	1	−112/5	0	0	−4/5	12/5	2/5	2/5
$R_0 + \frac{21}{2}R_2 \to R_0$:	1	0	2	0	21/2	0	3/2	5/4	5/4
$R_1 \to R_1$:	0	0	0	1	0	0	0	1	1
$\frac{15}{2}R_2 \to R_2$:	0	0	−2	0	15/2	1	−1/2	3/4	3/4
$6R_@ + R_3 \to R_3$:	0	1	−24	0	6	0	2	1	1

Table 10.4: The simplex tableau of Example 10.35.

In consideration of Remark 10.12, to mitigate cycling in the context of maximizing $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ while adhering to certain constraints, we employ the simplex method and implement Bland's rule to minimize $-\boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ under the same constraints. This approach yields an identical optimal solution, albeit with the optimal value of the maximization problem being equal to the result of the minimization problem, multiplied by -1.

Complexity

Similar to any algorithmic method, the computational complexity of the simplex method is determined by the following two factors: (a) The computational complexity of each iteration. (b) The total number of iterations.

The following theorem is known to hold (we refer to Section 3.3 in Bertsimas and Tsitsiklis [1997]). It indicates that the amount of computation in each iteration of the full tableau method is propositional to the size of the tableau.

Theorem 10.7 *The number of arithmetic operations in each iteration of the full tableau method is $O(mn)$.*

In practice, the simplex method's advantage lies in the observation that it typically converges in just $O(m)$ iterations to discover an optimal solution. However, from a theoretical perspective, the method has its drawback, as this observation does not hold true for every LP problem. In fact, there exists a class of problems for which an exponential number of iterations is needed [Bertsimas and Tsitsiklis, 1997, Section 3.7]. This phenomenon arises because the count of extreme points within the feasible set can grow exponentially with an increase in the number of variables and constraints.

10.6 Duality in linear programming

Linear programming duality studies the relationships between pairs of linear programs and their solutions.

The linear programming problem in the primal standard form is defined as

$$\begin{array}{rrcl} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\ \text{s.t.} & A\,\boldsymbol{x} & = & \boldsymbol{b}, \\ & \boldsymbol{x} & \geq & \boldsymbol{0}, \end{array} \tag{P|LP}$$

where $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ constitute given data, and $\boldsymbol{x} \in \mathbb{R}^n$ is called the primal decision variable.

The linear programming problem in the dual standard form is the dual of (P|LP), which is defined as

$$\begin{array}{rrcl} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} & \leq & \boldsymbol{c}, \\ & \boldsymbol{y} & \text{urs}, & \end{array} \tag{D|LP}$$

where $\boldsymbol{y} \in \mathbb{R}^m$ is called the dual decision variable.

Lagrangian duality and LP duality

Problem (D|LP) can be derived from (P|LP) through the usual Lagrangian approach. The optimization problems are classified into two classes: Constrained optimization problems and unconstrained optimization problems. This classification is based on whether or not we have constraints on the variables. The Lagrangian approach is a technique by which a constrained optimization problem becomes an unconstrained optimization problem by adding Lagrangian multipliers for the equality constraints. The Lagrangian function is a function that combines the objective function being optimized with functions penalizing constraint violations linearly.

The Lagrangian function for (P|LP) is defined as

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) \triangleq \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{\lambda}^{\mathsf{T}}(A\,\boldsymbol{x} - \boldsymbol{b}) - \boldsymbol{\nu}^{\mathsf{T}}\boldsymbol{x}.$$

The vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ are called Lagrangian multipliers. The dual of (P|LP) has the objective function

$$q(\boldsymbol{\lambda}, \boldsymbol{\nu}) \triangleq \inf_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \boldsymbol{\lambda}^{\mathsf{T}}\boldsymbol{b} + \inf_{\boldsymbol{x}} \,(\boldsymbol{c} - A^{\mathsf{T}}\boldsymbol{\lambda} - \boldsymbol{\nu})^{\mathsf{T}}\boldsymbol{x}.$$

Primal	Minimum	Maximum	Dual
C	$\geq \boldsymbol{b}$	$\geq \mathbf{0}$	V
N	$\leq \boldsymbol{b}$	$\leq \mathbf{0}$	A
S	$= \boldsymbol{b}$	urs	R
V	$\geq \mathbf{0}$	$\leq \boldsymbol{c}$	C
A	$\leq \mathbf{0}$	$\geq \boldsymbol{c}$	N
R	urs	$= \boldsymbol{c}$	S

Table 10.5: Correspondence rules between primal and dual linear programs.

The dual problem is obtained by maximizing $q(\boldsymbol{\lambda}, \boldsymbol{\nu})$ subject to $\boldsymbol{\nu} \geq \mathbf{0}$.

If $\boldsymbol{c} - A^{\mathsf{T}}\boldsymbol{\lambda} - \boldsymbol{\nu} \neq \mathbf{0}$, the infimum is clearly $-\infty$. So we can exclude $\boldsymbol{\lambda}$ for which $\boldsymbol{c} - A^{\mathsf{T}}\boldsymbol{\lambda} - \boldsymbol{\nu} \neq \mathbf{0}$. When $\boldsymbol{c} - A^{\mathsf{T}}\boldsymbol{\lambda} - \boldsymbol{\nu} = \mathbf{0}$, the dual objective function is simply $\boldsymbol{\lambda}^{\mathsf{T}}\boldsymbol{b}$. Hence, we can write the dual problem as follows:

$$\begin{array}{lrcl} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{\lambda} & & \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{\lambda} + \boldsymbol{\nu} & = & \boldsymbol{c}, \\ & \boldsymbol{\nu} & \geq & \mathbf{0}. \end{array} \tag{10.14}$$

Replacing $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ in (10.14) by $\boldsymbol{x}$ and $\boldsymbol{z}$, respectively, we get (D|LP).

In the realm of linear optimization, it is essential to acknowledge that there exist diverse formulations beyond the standard forms (P|LP) and (D|LP). Linear optimization problems can take on various alternative forms to suit specific problem requirements and constraints. When dealing with LPs that adopt different formulations, Table 10.5 offers a valuable resource. This table provides a summary of the correspondence rules that establish the relationships between the primal and dual LPs. In other words, it outlines how the parameters and components of the primal and dual formulations are interrelated, facilitating the translation and understanding of LP problems in their various forms.

In light of Table 10.5, we have the following remark.

Remark 10.13 *The following are three typical pairs of primal and dual linear programming problems:*

$$\begin{array}{llrcl}
 & \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\
(P|LP) & \text{s.t.} & A\boldsymbol{x} & = & \boldsymbol{b}, \\
 & & \boldsymbol{x} & \geq & \mathbf{0};
\end{array}
\qquad
\begin{array}{lrcll}
\max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & & \\
\text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} & \leq & \boldsymbol{c}, & (D|LP) \\
 & & & &
\end{array}$$

$$\begin{array}{llrcl}
 & \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\
(\overline{P|LP}) & \text{s.t.} & A\boldsymbol{x} & \geq & \boldsymbol{b}; \\
 & & & &
\end{array}
\qquad
\begin{array}{lrcll}
\max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & & \\
\text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} & = & \boldsymbol{c}, & (\overline{D|LP}) \\
 & \boldsymbol{y} & \geq & \mathbf{0}, &
\end{array}$$

$$\begin{array}{llrcl}
 & \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\
(\widehat{P|LP}) & \text{s.t.} & A\boldsymbol{x} & \geq & \boldsymbol{b}, \\
 & & \boldsymbol{x} & \geq & \mathbf{0};
\end{array}
\qquad
\begin{array}{lrcll}
\max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & & \\
\text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} & \leq & \boldsymbol{c}, & (\widehat{D|LP}) \\
 & \boldsymbol{y} & \geq & \mathbf{0}. &
\end{array}$$

The dual of the dual is the primal (see Proposition 10.1), so it does not matter which problem is called the primal.

Example 10.36 The following is a pair of primal-dual linear programs.

$$
\begin{array}{llrcl}
\max & 5x_1 + 4x_2 - 3x_3 & & & \\
\text{s.t.} & x_1 \qquad - 5x_3 & \geq & 4, \\
 & 3x_1 + x_2 + 2x_3 & \leq & 5, \\
 & x_1 \geq 0, x_2 \text{ urs}, x_3 \geq 0; & & &
\end{array}
\qquad
\begin{array}{llrcl}
\min & 4y_1 + 5y_2 & & \\
\text{s.t.} & y_1 + 3y_2 & \geq & 5, \\
 & y_2 & = & 4, \\
 & -5y_1 + 2y_2 & \geq & -3, \\
 & y_1 \leq 0, y_2 \geq 0. & &
\end{array}
$$

If we take the dual of the dual, we get

$$
\begin{array}{llrcl}
\max & 5z_1 + 4z_2 - 3z_3 & & \\
\text{s.t.} & z_1 \qquad - 5z_3 & \geq & 4, \\
 & 3z_1 + z_2 + 2z_3 & \leq & 5, \\
 & z_1 \geq 0, z_2 \text{ urs}, z_3 \geq 0, & &
\end{array}
$$

which is the primal problem. ∎

The proof of the following proposition is left as an exercise for the reader.

Proposition 10.1 *The dual of the dual is the primal.*

The duality theorem

The duality theorem is a very powerful theoretical tool that is very useful in applications because it leads to an interesting class of optimization algorithms. In this part, we state and prove the weak and strong duality theorems for the primal-dual pair (P|LP) and (D|LP). All the results in this part are stated for the pair (P|LP) and (D|LP), but we indicate that all the results established in this section are satisfied for any primal-dual pair, including the pair $(\overline{\text{P|LP}})$ and $(\overline{\text{D|LP}})$ as well as the pair $(\widehat{\text{P|LP}})$ and $(\widehat{\text{D|LP}})$ outlined in Remark 10.13. We have the following definition.

Definition 10.10

(*a*) *An optimization problem is called feasible if it has at least one feasible point, and infeasible otherwise.*

(*b*) *An optimization problem is called unbounded if it is feasible and has unbounded optimal value. More specifically, a minimization (maximization) problem is called unbounded if it is feasible and has the optimal cost $-\infty$ (optimal cost $+\infty$).*

We state the weak duality property in Theorem 10.8.

Theorem 10.8 (Weak duality in LP) *Consider the primal-dual pair (P|LP) and (D|LP). Let (P|LP) and (D|LP) be both feasible. If $\boldsymbol{x}$ is a feasible solution to (P|LP) and $\boldsymbol{y}$ is a feasible solution to (D|LP), then $\mathbf{b}^\mathsf{T}\boldsymbol{y} \leq \boldsymbol{c}^\mathsf{T}\boldsymbol{x}$.*

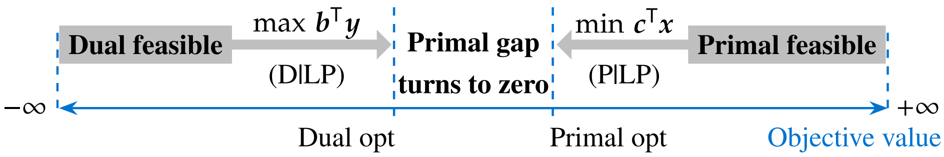


Figure 10.15: The duality gap between the primal and dual LP problems.

Proof Note that, in (D|LP), the constraint $A^\top y \le c$ can be written as $A^\top y + s = c$ with $s \ge 0$. It follows that $c^\top x - b^\top y = (A^\top y + s)^\top x - b^\top y = y^\top Ax + s^\top x - y^\top b = y^\top(Ax - b) + s^\top x = x^\top s \ge 0$, where the last equality follows from the constraint $Ax = b$ stated in (P|LP), and the inequality follows because $x \ge 0$ and $s \ge 0$. ■

The following corollary is now easy to obtain.

Corollary 10.4 *Consider the primal-dual pair (P|LP) and (D|LP).*

(*a*) *If (P|LP) is unbounded, then (D|LP) is infeasible.*

(*b*) *If (D|LP) is unbounded, then (P|LP) is infeasible.*

Proof If we prove item (a), item (b) immediately follows by a symmetrical argument. Suppose, on the contrary, that Problem (P|LP) is feasible, with the optimal cost $-\infty$, and that Problem (D|LP) is also feasible. Let w be the optimal cost in (D|LP). By weak duality, we have $w \le -\infty$. That is, $w \le r$ for all $r \in \mathbb{R}$, which is impossible. This means that (D|LP) cannot have a feasible solution. This proves item (a), and hence completes the proof. ■

In Figure 10.15, we show visually how the duality gap between the primal and dual LP problems turns to zero. That is, the difference $c^\top x - b^\top y$ becomes zero when x is an optimal solution to (P|LP) and y is an optimal solution to (D|LP). This is the essence of the strong duality property, which is stated below in Theorem 10.9.

Theorem 10.9 (Strong duality in LP) *Consider the primal-dual pair (P|LP) and (D|LP). Assume that (P|LP) and (D|LP) are both feasible. If one of (P|LP) or (D|LP) has a finite optimal solution, so does the other, and their optimal values are equal.*

Proof Let $\bar{x}$ and $\bar{y}$ be feasible solutions to Problems (P|LP) and (D|LP), respectively. Starting from the weak duality (as presented in Theorem 10.8), we have the inequality $b^\top \bar{y} \le c^\top \bar{x}$, which signifies that both Problems (P|LP) and (D|LP) are bounded. Let z and w represent the optimal values of (P|LP) and (D|LP), respectively. Using the weak duality once more, we have $w \le z$. To establish that $w = z$, we can consider a contradiction. Suppose, to the contrary, that $w < z$. In that case, there exists no y that satisfies the inequalities $A^\top y \le c$ and $b^\top y \ge z$, or equivalently

$$\begin{bmatrix} A^\top \\ -b^\top \end{bmatrix} y \le \begin{bmatrix} c \\ -z \end{bmatrix}. \tag{10.15}$$

Letting

$$\hat{A} \triangleq \begin{bmatrix} A \,\vdots\, -b \end{bmatrix}, \text{ and } \hat{c} \triangleq \begin{bmatrix} c \\ -z \end{bmatrix},$$

we can rewrite (10.15) as $\hat{A}^{\mathsf{T}}\boldsymbol{y} \leq \hat{\boldsymbol{c}}$. Using Farkas' lemma (Version II; see Theorem 3.16), there exists a vector $\hat{\boldsymbol{x}}$ satisfying

$$\hat{A}\hat{\boldsymbol{x}} = \boldsymbol{0},\ \hat{\boldsymbol{c}}^{\mathsf{T}}\hat{\boldsymbol{x}} < 0,\ \text{and}\ \hat{\boldsymbol{x}} \geq \boldsymbol{0}. \tag{10.16}$$

Note that the vector $\hat{\boldsymbol{x}}$ can be written as $\hat{\boldsymbol{x}} \triangleq (\boldsymbol{x}^{\mathsf{T}}, \alpha)^{\mathsf{T}}$ with $\alpha \neq 0$. This rewrites (10.16) as

$$\begin{bmatrix} A \,\vdots\, -\boldsymbol{b} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \alpha \end{bmatrix} = \boldsymbol{0},\ \begin{bmatrix} \boldsymbol{c} \\ -z \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \boldsymbol{x} \\ \alpha \end{bmatrix} < 0,\ \text{and}\ \begin{bmatrix} \boldsymbol{x} \\ \alpha \end{bmatrix} \geq \boldsymbol{0}. \tag{10.17}$$

To prove that $\alpha \neq 0$, suppose the contrary, i.e., $\alpha = 0$. Then, from (10.17), we have $A\boldsymbol{x} = \boldsymbol{0}$, $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} < 0$, and $\boldsymbol{x} \geq \boldsymbol{0}$. Applying Farkas' Lemma (Version II) once again, we find that there is no vector $\boldsymbol{y}$ satisfying $A^{\mathsf{T}}\boldsymbol{y} \leq \boldsymbol{c}$. This implies that Problem (D|LP) is infeasible, which is in contradiction with our initial assumption.

It is now evident that $\alpha \neq 0$, and further analysis shows that $\frac{1}{\alpha}\boldsymbol{x} \geq \boldsymbol{0}$. Additionally, $A\boldsymbol{x} - \alpha\boldsymbol{c} = \boldsymbol{0}$, which can be written as $A(\frac{1}{\alpha}\boldsymbol{x}) = \boldsymbol{c}$. This implies that the vector $\frac{1}{\alpha}\boldsymbol{x}$ is feasible for (P|LP). However, from (10.17), we have $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - \alpha z < 0$, so $\boldsymbol{c}^{\mathsf{T}}(\frac{1}{\alpha}\boldsymbol{x}) < z$. This contradicts the fact that z is the optimal value of (D|LP). Hence, it is confirmed that $w = z$. The proof is complete. ■

The following example, which is due to Nemhauser and Wolsey [1988], is a direct application of Theorem 10.9.

Example 10.37 Consider the following primal-dual pair of problems.

$$\begin{array}{llcl} \min & 7x_1 + 2x_2 & & \\ \text{s.t.} & -x_1 + 2x_2 & \leq & 4, \\ & 5x_1 + x_2 & \leq & 20, \\ & -2x_1 - 2x_2 & \leq & -7, \\ & x_1, x_2 & \leq & 0; \end{array} \qquad \begin{array}{llcl} \max & 4y_1 + 20y_2 - 7y_3 & & \\ \text{s.t.} & -y_1 + 5y_2 - 2y_3 & \geq & 7, \\ & 2y_1 + y_2 - 2y_3 & \geq & 2, \\ & y_1, y_2, y_3 & \geq & 0. \end{array}$$

Let $\boldsymbol{x}^\star \triangleq (\frac{36}{11}, \frac{40}{11})^{\mathsf{T}}$ and $\boldsymbol{y}^\star \triangleq (\frac{3}{11}, \frac{16}{11}, 0)^{\mathsf{T}}$. One can easily see that $\boldsymbol{x}^\star$ and $\boldsymbol{y}^\star$ are feasible in the primal and dual problems, respectively. One can also easily see that $\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}^\star = 30\frac{2}{11}$ and $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^\star = 30\frac{2}{11}$. Based on the strong duality property (Theorem 10.9), since $\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}^\star = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^\star$, we conclude that $\boldsymbol{x}^\star$ and $\boldsymbol{y}^\star$ are optimal in the primal and dual problems, respectively, and their optimal value is $30\frac{2}{11}$. ■

It is a natural question to ask: Can Problems (P|LP) and (D|LP) be both infeasible? The following example answers this question positively.

Example 10.38 The following primal-dual pair of problems are both infeasible.

$$\begin{array}{llcl} \min & x_1 + 2x_2 & & \\ \text{s.t.} & x_1 + x_2 & = & 2, \\ & 3x_1 + 3x_2 & = & 4; \end{array} \qquad \begin{array}{llcl} \max & 2y_1 + 4y_2 & & \\ \text{s.t.} & y_1 + 3y_2 & = & 1, \\ & y_1 + 3y_2 & = & 2. \end{array}$$

■

It is not hard now to establish the following corollary.

Corollary 10.5 *Consider the primal-dual pair (P|LP) and (D|LP).*

(*a*) *If (P|LP) is infeasible, then (D|LP) is either infeasible or unbounded.*

(*b*) *If (D|LP) is infeasible, then (P|LP) is either infeasible or unbounded.*

Proof Note that the possibility that Problems (P|LP) and (D|LP) could be both infeasible has been grounded in Example 10.38. To establish item (a), it remains to demonstrate that if (P|LP) is infeasible and (D|LP) is feasible, then (D|LP) must be unbounded. Assume that (P|LP) is infeasible, and let $\bar{\boldsymbol{y}}$ be a feasible solution for (D|LP). Since (P|LP) is infeasible, there exists no $\boldsymbol{x}$ satisfying $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. Applying Farkas' Lemma (Version I, as presented in Theorem 3.15), we can conclude that there exists a vector $\hat{\boldsymbol{y}}$ that satisfies $A^\top \hat{\boldsymbol{y}} \geq \boldsymbol{0}$ and $\boldsymbol{b}^\top \hat{\boldsymbol{y}} < 0$. Now, due to the feasibility of $\bar{\boldsymbol{y}}$ in (D|LP), we know that $A^\top \bar{\boldsymbol{y}} \leq \boldsymbol{c}$. Let us define $\boldsymbol{y}_\alpha \triangleq \bar{\boldsymbol{y}} - \alpha \hat{\boldsymbol{y}}$ for $\alpha \geq 0$. We can then observe that

$$A^\top \boldsymbol{y}_\alpha = A^\top (\bar{\boldsymbol{y}} - \alpha \hat{\boldsymbol{y}}) = A^\top \bar{\boldsymbol{y}} - \alpha A^\top \hat{\boldsymbol{y}} \leq \boldsymbol{c} - \alpha A^\top \hat{\boldsymbol{y}} \leq \boldsymbol{c}.$$

This demonstrates that $\boldsymbol{y}_\alpha$ is feasible in (D|LP). Furthermore, because $\boldsymbol{b}^\top \hat{\boldsymbol{y}} < 0$, it is clear that $\boldsymbol{b}^\top \boldsymbol{y}_\alpha$ tends toward infinity as α approaches infinity:

$$\boldsymbol{b}^\top \boldsymbol{y}_\alpha = \boldsymbol{b}^\top (\bar{\boldsymbol{y}} - \alpha \hat{\boldsymbol{y}}) = \boldsymbol{b}^\top \bar{\boldsymbol{y}} - \alpha \boldsymbol{b}^\top \hat{\boldsymbol{y}} \longrightarrow \boldsymbol{b}^\top \bar{\boldsymbol{y}} + \infty = \infty.$$

This implies that Problem (D|LP) is unbounded, successfully proving item (a). To prove item (b), we can apply a symmetrical argument similar to that of item (a) and utilize Farkas' lemma (Version II). We leave the proof of this part as an exercise for the reader (see Exercise 10.11). With this, we conclude the proof. ■

Corollary 10.6 is now obvious. See also Table 10.6.

Corollary 10.6 *There are only four possibilities for the primal-dual pair (P|LP) and (D|LP). Namely:*

(*a*) *Both (P|LP) and (D|LP) are feasible and their optimal values are finite and equal.*

(*b*) *(P|LP) is infeasible and (D|LP) is unbounded.*

(*c*) *(P|LP) is unbounded and (D|LP) is infeasible.*

(*d*) *Both (P|LP) and (D|LP) are infeasible.*

		(P|LP) Finite optimum	(P|LP) Unbounded	(P|LP) Infeasible
(D|LP)	Finite optimum	✓	✗	✗
(D|LP)	Unbounded	✗	✗	✓
(D|LP)	Infeasible	✗	✓	✓

Table 10.6: Possibilities for the primal and the dual linear programs.

Complementary slackness

Complementary slackness refers to the idea that for an optimal solution, the product of the decision variable in the primal problem and the corresponding slack variable in the dual problem is equal to zero. We have the following definition.

Definition 10.11 *A slack variable is a variable that is added to an inequality constraint to transform it into an equality. Likewise, an excess (also called surplus or negative slack) variable is a variable that is subtracted to an inequality constraint to transform it into an equality.*

Consider the pair ($\widehat{\text{P|LP}}$) and ($\widehat{\text{D|LP}}$) outlined in Remark 10.13.

$$(\widehat{\text{P|LP}}) \quad \begin{array}{llll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\ \text{s.t.} & A\boldsymbol{x} & \geq & \boldsymbol{b}, \\ & \boldsymbol{x} & \geq & \boldsymbol{0}; \end{array} \qquad \begin{array}{llll} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} & \leq & \boldsymbol{c}, \\ & \boldsymbol{y} & \geq & \boldsymbol{0}. \end{array} \quad (\widehat{\text{D|LP}})$$

Let $\boldsymbol{s} \triangleq \boldsymbol{c} - A^{\mathsf{T}}\boldsymbol{y} \geq \boldsymbol{0}$ be the vector of slack variables of ($\widehat{\text{P|LP}}$), and $\boldsymbol{e} \triangleq A\boldsymbol{x} - \boldsymbol{b} \geq \boldsymbol{0}$ be the vector of excess variables of ($\widehat{\text{D|LP}}$). Then ($\widehat{\text{P|LP}}$) and ($\widehat{\text{D|LP}}$) are written as

$$\begin{array}{llll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\ \text{s.t.} & A\boldsymbol{x} - \boldsymbol{e} & = & \boldsymbol{b}, \\ & \boldsymbol{x}, \boldsymbol{e} & \geq & \boldsymbol{0}; \end{array} \qquad \begin{array}{llll} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} & = & \boldsymbol{c}, \\ & \boldsymbol{y}, \boldsymbol{s} & \geq & \boldsymbol{0}. \end{array}$$

The complementary slackness conditions for linear programming are provided in the following theorem.

Theorem 10.10 (Complementary slackness) *Consider the primal-dual pair ($\widehat{\text{P|LP}}$) and ($\widehat{\text{D|LP}}$). If $\boldsymbol{x}^\star$ is an optimal solution to ($\widehat{\text{P|LP}}$) and $\boldsymbol{y}^\star$ is an optimal solution to ($\widehat{\text{D|LP}}$), then $x_i^\star s_i^\star = 0$ for all i, and $y_j^\star e_j^\star = 0$ for all j, where $\boldsymbol{e}^\star \triangleq A\boldsymbol{x}^\star - \boldsymbol{b}$ and $\boldsymbol{s}^\star \triangleq \boldsymbol{c} - A^{\mathsf{T}}\boldsymbol{y}^\star$.*

Proof Note that

$$\begin{array}{rcl} \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^\star & = & (A^{\mathsf{T}}\boldsymbol{y}^\star + \boldsymbol{s}^\star)^{\mathsf{T}}\boldsymbol{x}^\star \\ & = & \boldsymbol{y}^{\star\mathsf{T}}A\boldsymbol{x}^\star + \boldsymbol{s}^{\star\mathsf{T}}\boldsymbol{x}^\star \\ & = & \boldsymbol{y}^{\star\mathsf{T}}(\boldsymbol{b} + \boldsymbol{e}^\star) + \boldsymbol{s}^{\star\mathsf{T}}\boldsymbol{x}^\star \; = \; \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}^\star + \boldsymbol{y}^{\star\mathsf{T}}\boldsymbol{e}^\star + \boldsymbol{s}^{\star\mathsf{T}}\boldsymbol{x}^\star. \end{array}$$

Note also that the strong duality property (Theorem 10.9) implies that $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^\star = \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}^\star$. It follows that $\boldsymbol{y}^{\star\mathsf{T}}\boldsymbol{e}^\star + \boldsymbol{s}^{\star\mathsf{T}}\boldsymbol{x}^\star = 0$. Because $\boldsymbol{x}^\star, \boldsymbol{e}^\star, \boldsymbol{y}^\star$ and $\boldsymbol{s}^\star$ are all nonnegative vectors, we have $x_i^\star s_i^\star = 0$ for all i, and $y_j^\star e_j^\star = 0$ for all j. The proof is complete. ■

Example 10.39 (Example 10.37 revisited) To see how the complementary slackness conditions hold for the primal-dual pair in Example 10.37, note that the slack and excess variables are $\boldsymbol{s}^\star \triangleq (0, 0, 6\frac{9}{11})^{\mathsf{T}}$ and $\boldsymbol{e}^\star \triangleq (0, 0)^{\mathsf{T}}$, respectively. Clearly, $x_i^\star s_i^\star = 0$ for $i = 1, 2$, and $y_j^\star e_j^\star = 0$ for $j = 1, 2, 3$. ■

Complementary slackness conditions are not only used to verify optimality but also serve as a foundation for duality theory, which is a powerful tool for solving and interpreting LP problems in various real-world applications.

Optimal values of the dual variables

Recall that, from Theorem 10.9, if the primal and dual problems are both feasible and one of them has a finite optimal solution, so does the other, and their optimal values are equal. The question that emerges at this point is: How can one determine the optimal solution of a dual problem using the simplex tableau of the primal problem? We address this inquiry by presenting the following remark.

Remark 10.14 *If we are given the simplex tableau of a primal maximization problem, then*

$$\text{Optimal } y_i = \begin{cases} \text{Coefficient of } s_i \text{ in } R_0, & \text{if the ith constraint is} \text{“} \leq \text{"}; \\ -(\text{Coefficient of } e_i \text{ in } R_0), & \text{if the ith constraint is “} \geq \text{"}; \\ (\text{Coefficient of } a_i \text{ in } R_0) - M^{a}, & \text{if the ith constraint is “} = \text{"}. \end{cases}$$

If we are given the simplex tableau of a primal minimization problem, then

$$\text{Optimal } y_i = \begin{cases} \text{Coefficient of } s_i \text{ in } R_0, & \text{if the ith constraint is} \text{“} \leq \text{"}; \\ -(\text{Coefficient of } e_i \text{ in } R_0), & \text{if the ith constraint is “} \geq \text{"}; \\ (\text{Coefficient of } a_i \text{ in } R_0) + M, & \text{if the ith constraint is “} = \text{"}. \end{cases}$$

[a] This also holds when the ith constraint is “≥".

In the following example from Winston [1996], Remark 10.14 is applied.

Example 10.40 The dual problem of the maximization LP problem

$$\begin{array}{ll} \max & z = 30x_1 + 100x_2 \\ \text{s.t.} & x_1 + x_2 \leq 7, \\ & 4x_1 + 10x_2 \leq 40, \\ & 10x_1 \geq 30, \\ & x_1 \geq 0, x_2 \geq 0, \end{array} \tag{10.18}$$

is the minimization LP problem

$$\begin{array}{ll} \min & w = 7y_1 + 40y_2 + 30y_3 \\ \text{s.t.} & y_1 + 4y_2 + 10y_3 \geq 30, \\ & y_1 + 10y_2 \geq 100, \\ & y_1 \geq 0, y_2 \geq 0, y_3 \leq 0. \end{array} \tag{10.19}$$

Solving Problem (10.18) by the simplex tableau method, we obtain

EROs	z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
$R_0:$	1	-30	-100	0	0	0	M	0
$R_1:$	0	1	1	1	0	0	0	7
$R_2:$	0	4	10	0	1	0	0	40
$R_3:$	0	10	0	0	0	-1	1	30
$R_0 - MR_3 \to R_0:$	1	$-30-10M$	-100	0	0	M	0	$-30M$
$R_1 \to R_1:$	0	1	1	1	0	0	0	7
$R_2 \to R_2:$	0	4	10	0	1	0	0	40
$R_3 \to R_3:$	0	10	0	0	0	-1	1	30
$R_0 + (3+M)R_3 \to R_0:$	1	0	-100	0	0	-3	$M+3$	90
$R_1 - \frac{1}{10}R_3 \to R_1:$	0	0	1	1	0	$1/10$	$-1/10$	4
$R_2 - \frac{2}{5}R_3 \to R_2:$	0	0	10	0	1	$3/5$	$-3/5$	28
$\frac{1}{10}R_3 \to R_3:$	0	1	0	0	0	$-1/10$	$1/10$	3
$R_0 + 10R_1 \to R_0:$	1	0	0	0	10	1	$M-1$	370
$R_1 \to R_1:$	0	0	0	1	$-1/10$	$3/50$	$-3/50$	1.2
$R_2 \to R_2:$	0	0	1	0	$-1/10$	$1/25$	$-1/25$	2.8
$R_3 \to R_3:$	0	1	0	0	0	$-1/10$	$1/10$	3

The last tableau is optimal. The optimal value is $z = 370$, the primal optimal solution is $(x_1, x_2) = (3, 2.8)$. According to Remark 10.14, the dual optimal solution is:

$$
\begin{array}{lcl}
y_1 & = & \text{Coefficient of } s_1 \text{ in } R_0 & = & 0; \\
y_2 & = & \text{Coefficient of } s_2 \text{ in } R_0 & = & 10; \\
y_3 & = & -(\text{Coefficient of } e_3 \text{ in } R_0) & = & -1, \\
\text{or } y_3 & = & (\text{Coefficient of } a_3 \text{ in } R_0) - M & = & (M-1) - M = -1.
\end{array}
$$

To check this, note that the dual optimal value is:

$$w = 7y_1 + 10y_2 + 30y_3 = 370,$$

which is exactly the primal optimal value. ■

10.7 A homogeneous interior-point method

Interior-point methods Nesterov and Nemirovskii [1994] represent a class of highly efficient techniques designed to solve both linear and nonlinear programming problems. In stark contrast to the simplex method, interior-point methods excel by moving through the interior of the feasible region to reach an optimal solution. Multiple interior-point algorithms have been devised for linear programming problems, offering versatile tools for optimization [Bertsimas and Tsitsiklis, 1997, Chapter 9].

Among the interior-point methods, homogeneous self-dual algorithms stand out as a valuable method for solving both linear and nonlinear programming. This section introduces a homogeneous interior-point algorithm tailored for solving (P|LP) and (D|LP) problems outlined in Section 10.6. The content presented here draws from previous work found in Tucker [1957], Ye et al. [1994].

We define the following feasibility sets for the primal-dual pair (P|LP) and (D|LP).

$$
\begin{aligned}
\mathcal{F}_{\text{P|LP}} &\triangleq \{x \in \mathbb{R}^n : Ax = b,\ x \geq \mathbf{0}\}, \\
\mathcal{F}_{\text{D|LP}} &\triangleq \left\{(y, s) \in \mathbb{R}^m \times \mathbb{R}^n : A^\top y + s = c,\ s \geq \mathbf{0}\right\}, \\
\mathcal{F}^\circ_{\text{P|LP}} &\triangleq \{x \in \mathbb{R}^n : Ax = b,\ x > \mathbf{0}\}, \\
\mathcal{F}^\circ_{\text{D|LP}} &\triangleq \left\{(y, s) \in \mathbb{R}^m \times \mathbb{R}^n : A^\top y + s = c,\ s > \mathbf{0}\right\}, \\
\mathcal{F}^\circ_{\text{LP}} &\triangleq \mathcal{F}^\circ_{\text{P|LP}} \times \mathcal{F}^\circ_{\text{D|LP}}.
\end{aligned}
$$

We also make the following assumptions about the primal-dual pair (P|LP) and (D|LP).

Assumption 10.1 *The m rows of the matrix A are linearly independent.*

Assumption 10.2 *The set $\mathcal{F}^\circ_{LP}$ is nonempty.*

Assumption 10.1 is introduced for the sake of convenience. On the other hand, Assumption 10.2 imposes the requirement that both Problem (P|LP) and its dual counterpart (D|LP) must possess strictly feasible solutions. This condition serves as a guarantee, ensuring the existence of strong duality within the context of the linear programming problem.

The following primal-dual LP model provides sufficient conditions (but not always necessary) for an optimal solution of (P|LP) and (D|LP).

$$
\begin{aligned}
Ax &= b, \\
A^\top y + s &= c, \\
x^\top s &= 0, \\
x, s &\geq \mathbf{0}.
\end{aligned}
\tag{10.20}
$$

The homogeneous LP model for the pair (P|LP) and (D|LP) is as follows:

$$
\begin{array}{rrrrrcl}
Ax & & & -b\tau & & = & \mathbf{0}, \\
 & -A^\top y & -s & +c\tau & & = & \mathbf{0}, \\
-c^\top x & +b^\top y & & & -\kappa & = & 0, \\
x & & & & & \geq & \mathbf{0}, \\
 & & s & & & \geq & \mathbf{0}, \\
 & & & \tau & & \geq & 0, \\
 & & & & \kappa & \geq & 0.
\end{array}
\tag{10.21}
$$

The first two equations in (10.21), with $\tau = 1$, represent primal and dual feasibility (with $x, s \geq \mathbf{0}$) and reversed weak duality. So they, together with the third equation after forcing $\kappa = 0$, define primal and dual optimal solutions. Note that homogenizing τ (i.e., making it a variable) adds the required variable dual to the third equation, introducing the artificial variable κ achieves feasibility, and adding the third equation in (10.21) achieves self-duality.

One can show that $\boldsymbol{x}^\mathsf{T}\boldsymbol{s} + \tau\kappa = 0$ (see Exercise 10.22). The next theorem relates (10.20) to (10.21), and it is easily proved. Here, as defined previously, $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \boldsymbol{0}\}$ is the nonnegative orthant cone.

Theorem 10.11 *The primal-dual LP model (10.20) has a solution if and only if the homogeneous LP model (10.21) has a solution*

$$(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{s}^\star, \tau^\star, \kappa^\star) \in \mathbb{R}^n_+ \times \mathbb{R}^m \times \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+$$

such that $\tau^\star > 0$ *and* $\kappa^\star = 0$.

The main step at each iteration of the homogeneous interior-point algorithm for solving (P|LP) and (D|LP) is the computation of the search direction $(\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s})$ from the Newton equations defined by the following system.

$$\begin{array}{rcccccccl}
A\,\Delta\boldsymbol{x} & & & & -\boldsymbol{b}\,\Delta\tau & & & = & \eta\boldsymbol{r}_p, \\
& - & A^\mathsf{T}\,\Delta\boldsymbol{y} & - \;\Delta\boldsymbol{s} & +\boldsymbol{c}\,\Delta\tau & & & = & \eta\boldsymbol{r}_d, \\
-\boldsymbol{c}^\mathsf{T}\,\Delta\boldsymbol{x} & + & \boldsymbol{b}^\mathsf{T}\,\Delta\boldsymbol{y} & & & - & \Delta\kappa & = & \eta r_g, \\
& & & & \kappa\,\Delta\tau & + & \tau\,\Delta\kappa & = & \gamma\mu - \tau\kappa, \\
S\Delta\boldsymbol{x} & & & +\,X\Delta\boldsymbol{s} & & & & = & \gamma\mu\boldsymbol{1} - X\boldsymbol{s},
\end{array} \tag{10.22}$$

where $\boldsymbol{1}$ is a vector of ones with an appropriate dimension, η and γ are two parameters, $X \triangleq \mathrm{Diag}(\boldsymbol{x}) \in \mathbb{R}^{n\times n}$ is the diagonal matrix with the vector $\boldsymbol{x} \in \mathbb{R}^n$ on its diagonal, same for $S \triangleq \mathrm{Diag}(\boldsymbol{s}) \in \mathbb{R}^{n\times n}$, and

$$\begin{array}{rclcrcl}
\boldsymbol{r}_p & \triangleq & \boldsymbol{b}\tau - A\boldsymbol{x}, & \quad & \boldsymbol{r}_d & \triangleq & A^\mathsf{T}\boldsymbol{y} + \boldsymbol{s} - \tau\boldsymbol{c}, \\
r_g & \triangleq & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} - \boldsymbol{b}^\mathsf{T}\boldsymbol{y} + \kappa, & & \mu & \triangleq & \frac{1}{n+1}(\boldsymbol{x}^\mathsf{T}\boldsymbol{s} + \tau\kappa).
\end{array}$$

We state the homogeneous algorithm for (P|LP) and (D|LP) in Algorithm 10.1.

Algorithm 10.1: Generic homogeneous self-dual algorithm for LP

Input: Data in Problems (P|LP) and (D|LP) $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \tau, \kappa) \triangleq (\boldsymbol{1}, \boldsymbol{0}, \boldsymbol{1}, 1, 1)$
Output: An approximate optimal solution to Problem (P|LP)
while a stopping criterion is not satisfied **do**
  choose η, γ
  compute the solution $(\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s}, \Delta\tau, \Delta\kappa)$ of the linear system (10.22)
  compute a step length θ so that
  $\boldsymbol{x} + \theta\Delta\boldsymbol{x} > \boldsymbol{0}$
  $\boldsymbol{s} + \theta\Delta\boldsymbol{s} > \boldsymbol{0}$
  $\tau + \theta\Delta\tau > 0$
  $\kappa + \theta\Delta\kappa > 0$
  set the new iterate according to
  $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \tau, \kappa) \triangleq (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \tau, \kappa) + \theta(\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s}, \Delta\tau, \Delta\kappa)$
end

The following theorem is known to hold (see Ye et al. [1994]). It gives the computational complexity (worst behavior) of Algorithm 10.1 in terms of the dimension of the decision variable (n).

Theorem 10.12 *Let $\epsilon_0 > 0$ be the residual error at a starting point, and $\epsilon > 0$ be a given tolerance. Under Assumptions 10.2 and 10.1, if the pair (P|LP) and (D|LP) has a solution $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{s}^\star)$, then Algorithm 10.1 finds an ϵ-approximate solution (i.e., a solution with residual error less than or equal to ϵ) in at most*

$$O\left(\sqrt{n}\ln\left(\mathbf{1}^{\mathsf{T}}(\boldsymbol{x}^\star + \boldsymbol{s}^\star)\left(\frac{\epsilon_0}{\epsilon}\right)\right)\right) \text{ iterations.}$$

Theoretically, the advantage of this interior-point method is maintaining the iteration complexity of $O(\sqrt{n}\ln(L))$, where L is the data length of the underlying LP. Practically, the disadvantage of this method is the doubled dimension of the system of equations, which must be solved at each iteration.

Exercises

10.1 Choose the correct answer for each of the following multiple-choice questions/items.

(a) All LP problems may be solved using the graphical method.

(i) True. (ii) False.

(b) XYZ Inc. manufactures two varieties of paper towels, known as “regular” and “super-soaker”. The marketing department has established a requirement that the total monthly production of regular paper towels should not exceed twice the monthly production of super-soaker paper towels. In this context, let us denote by x_1 the quantity of regular paper towels produced per month and by x_2 the quantity of super-soaker paper towels produced per month. The relevant constraint(s) can be expressed as:

(i) $2x_1 \leq x_2$.
(ii) $x_1 \leq 2x_2$.
(iii) $x_1 \leq 0.5x_2$.
(iv) $x_1 - x_2 \leq 0$.
(v) $x_1 - 0.5x_2 \geq 0$.

(c) Problem A is a given formulation of an LP problem with an optimal solution. Problem B is a formulation obtained by multiplying the objective function of Problem A by a positive constant and leaving all other things unchanged. Problems A and B will have

(i) the same optimal solution and same objective function value.
(ii) the same optimal solution but different objective function values.
(iii) different optimal solutions but same objective function value.
(iv) different optimal solutions and different objective function values.

(*d*) Consider the following LP problem:

$$\begin{array}{lrcl} \max & 12x + 10y & & \\ \text{s.t.} & 4x + 3y & \leq & 480, \\ & 2x + 3y & \leq & 360, \\ & x, y & \geq & 0. \end{array}$$

Which of the following points (x, y) could be a feasible corner point?

(*i*) $(40, 48)$.
(*ii*) $(120, 0)$.
(*iii*) $(180, 120)$.
(*iv*) $(30, 36)$.
(*v*) None of the above.

(*e*) XYZ Inc. manufactures two categories of printers, which are labeled as "regular" and "high-speed". The regular printers utilize 2 units of recycled plastic per unit produced, while the high-speed printers consume 1 unit of recycled plastic per unit of production. The company has a monthly supply of 5,000 units of recycled plastic. To produce these printers, a critical machine is essential, with each unit of regular printers requiring 5 units of machine time and each unit of high-speed printers necessitating 3 units of machine time. The total available machine time per month amounts to 10,000 units. In this context, let us denote the number of units of regular printers produced per month as x_1 and the number of units of high-speed printers produced per month as x_2. The relevant constraint(s) can be expressed as:

(*i*) $2x_1 + x_2 = 5000$.
(*ii*) $2x_1 + x_2 \leq 5000$.
(*iii*) $5x_1 + 3x_2 \leq 10000$.
(*iv*) (a) and (c).
(*v*) (b) and (c).

(*f*) Problem A is a given formulation of an LP with an optimal solution and its constraint 1 is $\leq$ type. Problem B is a formulation obtained from Problem A by replacing the $\leq$ constraint by an equality constraint and leaving all other things unchanged. Problems A and B will have

(*i*) the same optimal solution and same objective function value.
(*ii*) the same optimal solution but different objective function values.
(*iii*) different optimal solutions but same objective function value.
(*iv*) same or different solution profile depending on the role of the constraints in the solutions.

(*g*) Consider the following LP problem:

$$\begin{array}{lrcl} \max & 12x + 10y & & \\ \text{s.t.} & 4x + 3y & \leq & 480, \\ & 2x + 3y & \leq & 360, \\ & x, y & \geq & 0. \end{array}$$

Which of the following points (x, y) is not in the feasible region?

(i) $(30, 60)$.
(ii) $(105, 0)$.
(iii) $(0, 110)$.
(iv) $(100, 10)$.
(v) None of the above.

(h) In any graphically solvable LP problem, if two feasible points exist, then any nonnegative weighted average of these points (with weights summing up to 1) is also feasible.

(i) True.
(ii) False.

(i) In a two-variable graphical LP problem, if the coefficient of one of the variables in the objective function is changed (while the other remains fixed), then the slope of the objective function expression will change.

(i) True.
(ii) False.

(j) XYZ Inc. engages in the production of two printer variants, denoted as “regular” and “high-speed”. The regular printers consume 2 units of recycled plastic per unit of production, while the high-speed printers utilize 1 unit of recycled plastic per unit manufactured. As part of its commitment to sustainability, XYZ ensures that a minimum of 5,000 units of recycled plastic are used each month. The manufacturing process requires a crucial machine, with each unit of regular printers demanding 5 units of machine time and each unit of high-speed printers necessitating 3 units of machine time. The total available machine time per month is limited to 10,000 units. Given this context, we can represent the number of units of regular printers produced per month as x_1 and the number of units of high-speed printers produced per month as x_2. By imposing these constraints, along with the non-negativity constraints, we can identify one of the feasible corner points as (assuming the first number in the parenthesis is x_1 and the second number in the parenthesis is x_2):

(i) $(0, 0)$.
(ii) $(2000, 0)$.
(iii) None exists.
(iv) $(0, 5000)$.
(v) $(2500, 0)$.

(k) If a graphically solvable LP problem is unbounded, then it can always be converted to a regular bounded problem by removing a constraint.

(i) True.
(ii) False.

(l) A point that satisfies all of a problem’s constraints simultaneously is a(n):

(i) optimal solution.
(ii) corner point.
(iii) intersection of the profit line and a constraint.
(iv) intersection of two or more constraints.
(v) None of the above.

(m) In a two-variable graphical LP problem, if the RHS of one of the constraints is changed (keeping all other things fixed) then the plot of the corresponding constraint will move in parallel to its old plot.

(i) True. (ii) False.

(n) Two models of a product - Regular (x) and Deluxe (y) - are produced by a company. An LP model is used to determine the production schedule. The formulation is as follows:

$$\begin{array}{llcll} \max & 50x + 60y & & & \text{(maximum profit)} \\ \text{s.t.} & 8x + 10y & \leq & 800 & \text{(labor hours),} \\ & x + y & \leq & 120 & \text{(total units demanded),} \\ & 4x + 5y & \leq & 500 & \text{(raw materials),} \\ & x, y & \geq & 0 & \text{(non-negativity).} \end{array}$$

The optimal solution is $x = 100, y = 0$. How many units of the labor hours must be used to produce this number of units?

(i) 400. (iii) 500. (v) None of the above.
(ii) 200. (iv) 5000.

(o) LP theory states that the optimal solution to any problem will lie at:

(i) the origin.
(ii) a corner point of the feasible region.
(iii) the highest point of the feasible region.
(iv) the lowest point in the feasible region.
(v) none of the above.

(p) The dual of an LP problem with maximized objective function, all $\leq$ constraints and non-negative variables, has minimized objective function, all $\geq$ constraints and nonnegative decision variables.

(i) True. (ii) False.

(q) The two objective functions (max $5x + 7y$, and min $-15x - 21y$) will produce the same solution to an LP problem.

(i) True. (ii) False.

(r) In order for an LP problem to have a unique solution, the solution must exist

(i) at the intersection of the non-negativity constraints.
(ii) at the intersection of the objective function and a constraint.
(iii) at the intersection of two or more constraints.

(*iv*) none of the above.

(*s*) If a minimization problem has an objective function of $2x_1 + 5x_2$, which of the following corner points is the optimal solution?

(*i*) $(0,2)$.
(*ii*) $(0,3)$.
(*iii*) $(3,3)$.
(*iv*) $(1,1)$.
(*v*) $(2,0)$.

(*t*) In an LP problem with a nonempty feasible region, when the objective function is parallel to one of the constraints, then

(*i*) the solution is not optimal.
(*ii*) multiple optimal solutions may exist.
(*iii*) a single corner point solution exists.
(*iv*) no feasible solution exists.
(*v*) none of the above.

(*u*) An LP problem cannot have

(*i*) no optimal solutions.
(*ii*) exactly two optimal solution.
(*iii*) as many optimal solutions as there are decision variables.
(*iv*) an infinite number of optimal solutions.
(*v*) none of the above.

10.2 A homemaker intends to create a blend of two food types, denoted as F_1 and F_2, with the objective of ensuring that the vitamin composition of the mixture contains a minimum of 8 units of vitamin A and 11 units of vitamin B. The cost of Food F_1 is 60 per kilogram, while the cost of Food F_2 is 80 per kilogram. Food F_1 contains 3 units per kilogram of vitamin A and 5 units per kilogram of vitamin B, while Food F_2 contains 4 units per kilogram of vitamin A and 2 units per kilogram of vitamin B. Formulate this problem as an LP problem with the objective of minimizing the cost of the mixture.

10.3 A baker possesses 30 ounces of flour and 5 packages of yeast. For each loaf of bread, 5 ounces of flour and 1 package of yeast are required. The baker can sell each loaf of bread for 30 cents. Additionally, the baker has the option to purchase extra flour at a rate of 4 cents per ounce or sell any remaining flour at the same price. Formulate this an LP problem that can be used to assist the baker in maximizing profits, which are calculated as revenues minus costs.

10.4 A farmer owns a 126-acre farm and cultivates Radish, Onion, and Potato. When he sells his entire harvest in the market, he earns 5 per kilogram for Radish, 4 per kilogram for Onion, and 5 per kilogram for Potato. The average yield on his farm is 1,500 kilograms of Radish per acre, 1,800 kilograms of Onion per acre, and 1,200 kilograms of Potato per acre. To grow 100 kilograms of Radish, 100 kilograms of Onion, and 80 kilograms of Potato, he needs to spend 12.5 on water. The labor requirement to cultivate each crop is 6 man-days per acre for Radish and Potato and 5 man-days per acre for Onion. He has a total of 500 man-days of labor available at a rate of 40 per man-day. Write an LP model that can help the farmer maximize his total profit.

10.5 Use the graphical method to solve the following LP problem.

$$\begin{aligned} \min\ z = {} & 15x_1 + 10x_2 \\ \text{s.t.}\quad & 0.25x_1 + x_2 \leq 65, \\ & 1.25x_1 + 0.5x_2 \leq 90, \\ & x_1 + x_2 \leq 85, \\ & x_1,\ x_2 \geq 0. \end{aligned}$$

10.6 A company manufactures two products, denoted as X and Y, with a combined daily production capacity of 9 tons. Both products, X and Y, necessitate the same production capacity. The company has a standing contract to deliver a minimum of 2 tons of X and 3 tons of Y per day to another business. The production of one ton of X consumes 20 machine hours, while one ton of Y requires 50 machine hours. The maximum daily available machine hours amount to 360. The company can sell all its output, and it earns a profit of JD 80 per ton for X and JD 120 per ton for Y.

(*a*) Formulate this as an LP problem that can be used to maximize the total profit.

(*b*) Solve this optimization problem graphically.

10.7 A small paint company produces two paint types, labeled as P_1 and P_2, using two raw materials, denoted as M_1 and M_2. The table shown below contains the essential data for this scenario.

According to a market survey, the highest daily demand for product P_2 is limited to 2 tons. Additionally, the daily demand for product P_1 should not surpass that of P_2 by more than 1 ton.

Tons of raw material per ton of paints produced			
	P_1	P_2	Availability
M_1	6	4	24
M_2	1	2	6
Profit per ton (in \$)	500	400	

(*a*) Write an LP formulation for the problem.

(*b*) Solve the LP model obtained in item (a) using the graphical method.

(*c*) If the number of tons to be produced for P_2 is restricted to be integer-valued, the problem obtained in item (a) is called a mixed integer program. Sketch its feasible region and solve it graphically.

(*d*) If the number of tons to be produced for P_1 and P_2 are both restricted to be integer-valued, the problem obtained in item (a) becomes a pure integer program. Sketch its feasible region and solve it graphically.

10.8 Use the graphical method to solve the following optimization problems.

(a) $$\begin{array}{ll} \min & 5x + 7y \\ \text{s.t.} & x + 3y \geq 6, \\ & 5x + 2y \geq 10, \\ & y \leq 4, \\ & x,\ y \geq 0. \end{array}$$

(b) $$\begin{array}{ll} \max & 5x + 4y \\ \text{s.t.} & 6x + 4y \leq 24, \\ & x + 2y \leq 6, \\ & -x + y \leq 1, \\ & y = 2, \\ & x,\ y \geq 0. \end{array}$$

(c) $$\begin{array}{ll} \max & x_2 \\ \text{s.t.} & -x_1 + x_2 \leq 1, \\ & 3x_1 + 2x_2 \leq 12, \\ & 2x_1 + 3x_2 \geq 12, \\ & x_1,\ x_2 \geq 0, \\ & x_1,\ x_2 \in \mathbb{Z}. \end{array}$$

10.9 Consider the following LP problem.

$$\begin{array}{ll} \max & z = 5x_1 + 4x_2 \\ \text{s.t.} & 3x_1 + 2x_2 \leq 12, \\ & x_1 + 2x_2 \leq 6, \\ & -x_1 + x_2 \leq 1, \\ & x_2 \leq 2, \\ & x_1,\ x_2 \geq 0. \end{array}$$

Sketch the feasible region and solve it graphically for each of the following cases:

(a) The variable x_2 is restricted to be integer-valued; in this case the problem becomes a mixed integer program.

(b) The variables x_1 and x_2 are both restricted to be integer-valued; in this case the problem becomes a pure integer program.

10.10 Transform the following LP into the standard form.

$$\begin{array}{ll} \min & z = 2x_1 - 4x_2 + 5x_3 - 30 \\ \text{s.t.} & 3x_1 + 2x_2 - x_3 \geq 10, \\ & -2x_1 + 4x_3 \leq 35, \\ & 4x_1 - x_2 \leq 20, \\ & x_1 \leq 6, \quad x_2 \leq 8, \quad x_3 \leq 10. \end{array}$$

10.11 Prove item (b) in Corollary 10.5.

10.12 Find the feasible solution (x_1, x_2) for the original LP problem in Example 10.16 given the feasible solution $(x_1, x_2^+, x_2^-, x_3) = (4, 0, 1/3, 2/3)$ to the same problem in the standard form.

10.13 Choose the correct answer for each of the following multiple-choice questions/items.

(*a*) A two-variable LP problem cannot be solved by the simplex method.

(*i*) True. (*ii*) False.

(*b*) If, when we are using a simplex table to solve a maximization problem, we find that the ratios for determining the pivot row are all negative, then we know that the solution is:

(*i*) unbounded. (*iii*) degenerate. (*v*) none of the above.
(*ii*) feasible. (*iv*) optimal.

(*c*) In converting a greater-than-or-equal constraint for use in a simplex table, we must add:

(*i*) an artificial variable.
(*ii*) a slack variable.
(*iii*) a slack and an artificial variable.
(*iv*) an excess and an artificial variable.
(*v*) a slack and an excess variable.

(*d*) For a minimization problem using a simplex table, we know we have reached the optimal solution when the row R_0

(*i*) has no numbers in it. (*iv*) has no nonzero numbers in it.
(*ii*) has no positive numbers in it.
(*iii*) has no negative numbers in it. (*v*) none of the above.

(*e*) A feasible solution requires that all artificial variables are

(*i*) greater than zero. (*ii*) less than zero. (*iii*) equal to zero.

(*iv*) there are no special requirements on artificial variables; they may take on any value.
(*v*) none of the above.

(*f*) If the right-hand side of a constraint is changed, the feasible region will not be affected and will remain the same.

(*i*) True. (*ii*) False.

(*g*) With Bland's rule, the simplex algorithm solves feasible linear minimization problems without cycling when

(*i*) we choose the rightmost nonbasic column with a negative cost to select the entering variable.
(*ii*) we choose the rightmost nonbasic column with a negative cost to select the leaving variable.
(*iii*) we choose the leftmost nonbasic column with a negative cost to select the entering variable.

(*iv*) we choose the leftmost nonbasic column with a negative cost to select the leaving variable.

10.14 Use the simplex tableau method to solve the following maximization problems.

(*a*)

$$\begin{aligned} \max \quad & z = x_1 + 1.5x_2 \\ \text{s.t.} \quad & 2x_1 + 4x_2 \le 12, \\ & 3x_1 + 2x_2 \le 10, \\ & x_1, x_2 \ge 0. \end{aligned}$$

(*b*)

$$\begin{aligned} \max \quad & z = 3x_1 + 5x_2 + 4x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 \le 8, \\ & 2x_2 + 5x_3 \le 10, \\ & 3x_1 + 2x_2 + 4x_3 \le 15, \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

(*c*)

$$\begin{aligned} \max \quad & z = 2x_1 - x_2 + x_3 \\ \text{s.t.} \quad & 3x_1 + x_2 + x_3 \le 6, \\ & x_1 + x_2 + 2x_3 \le 1, \\ & x_1 + x_2 - x_3 \le 2, \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

(*d*)

$$\begin{aligned} \max \quad & z = 60x_1 + 30x_2 + 20x_3 \\ \text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \le 48, \\ & 4x_1 + 2x_2 + 1.5x_3 \le 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 \le 8, \\ & x_2 \le 5,\ x_1, x_2, x_3 \ge 0. \end{aligned}$$

10.15 Consider the maximization problem presented by the following tableau. The parameters a and b are unknown.

x_1	x_2	s_1	s_2	s_3	rhs
0	0	17	$-3+2a$	0	10
1	0	3	-1	0	2
0	1	4	a	0	2
0	0	1	b	1	6

For each of the following cases, explicitly discuss how many optimal solutions, if any, there are to the LP problem. (If the LP is unbounded state that).

(*a*) $a = -2$ and $b = 0$. (*b*) $a = 2$ and $b = -1$. (*c*) $a = 3/2$ and $b = 1$.

10.16 Consider the following tableau of the simplex method for a maximization LP problem

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
1	0	0	0	c_1	c_2	c_3	$z^\star$
0	0	-2	a_3	a_5	-1	0	0
0	1	a_1	0	-3	0	a_7	2
0	0	a_2	a_4	-4	a_6	a_8	b

(*a*) There have to be three basic variables. Find them and give conditions on (all or some of) the unknowns $c_1, c_2, c_3, a_1, a_2, \ldots, a_8$ that make these variables basic.

(*b*) Give a condition on b that makes the LP feasible and conditions on c_1, c_2 and c_3 that make the LP optimal.

(*c*) Do we have alternative optimal solutions? Justify your answer.

10.17 Consider the following optimization problem:

$$\begin{array}{ll} \max & z = 5x_1 - x_2 \\ \text{s.t.} & x_1 - 3x_2 \leq 1, \\ & x_1 - 4x_2 \leq 3, \\ & x_1, \quad x_2 \geq 0. \end{array}$$

Use the simplex algorithm to show that this LP is an unbounded LP problem.

10.18 Consider the following primal-dual pair of problems.

$$\begin{array}{llll} \min & 13x_1 + 10x_2 + 6x_3 & & \\ \text{s.t.} & 5x_1 + x_2 + 3x_3 & = & 8, \\ & 3x_1 + x_2 & = & 3, \\ & x_1, x_2, x_3 & \geq & 0; \end{array} \qquad \begin{array}{llll} \max & 8y_1 + 3y_2 & & \\ \text{s.t.} & 5y_1 + 3y_2 & \leq & 13, \\ & y_1 + y_2 & \leq & 10, \\ & 3y_1 & \leq & 6. \end{array}$$

Show that $\boldsymbol{x}^\star \triangleq (1, 0, 1)^\mathsf{T}$ and $\boldsymbol{y}^\star \triangleq (2, 1)^\mathsf{T}$ are optimal in the primal and dual problems, respectively, and find the corresponding optimal values.

10.19 In Example 10.38, we give a pair of problems with the property that the primal and dual problems are both infeasible. Give an example of another pair with this property.

10.20 Consider the following LP problem.

$$\begin{array}{ll} \min & z = 5x_1 + 3x_2 - 2x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 \geq 4, \\ & 2x_1 + 3x_2 - x_3 \geq 9, \\ & x_2 + 3x_3 \leq 5, \\ & x_1, \quad x_2, \quad x_3 \geq 0. \end{array}$$

(*a*) Write down the corresponding dual LP problem.

(*b*) Suppose that the simplex method has been applied directly to the primal problem, and the resulting optimal tableau is:

z	x_1	x_2	x_3	e_1	a_1	e_2	a_2	s_3	rhs
1	−2.5	0	0	0	$-M$	−1.25	$1.25 - M$	−0.75	7.5
0	−0.5	0	1	0	0	0.25	−0.25	0.75	1.5
0	0.5	1	0	0	0	−0.25	0.25	0.25	3.5
0	−1	0	0	1	−1	0	0	1	1

(*i*) Deduce the optimal solution to the primal problem and the optimal value.

(*ii*) Deduce the optimal solution to the corresponding dual problem.

10.21 In this exercise, you are required to implement both the revised simplex method and the tableau simplex method using Octave/Matlab or another programming tool of your preference. Subsequently, you should conduct performance comparisons between your implemented algorithms and established standard optimization software. To evaluate your programs, you will apply them to a selection of LP problems for which you must generate random data. Finally, you are expected to present a well-organized and structured solution for this assignment.

(*a*) Write an Octave function capable of solving an LP in standard form using the revised simplex method. This function should accept the constraint matrix A, the right-hand side vector $\boldsymbol{b}$, and the cost vector $\boldsymbol{c}$ as input and provide as output an optimal solution vector $\boldsymbol{x}$ along with the optimal cost. In cases where the LP is unbounded or infeasible, the function should appropriately indicate this. Additionally, the number of simplex pivots or iterations employed should be part of the function's output. The function should offer flexibility in selecting both entering and leaving variables, with the following options available:

- For choosing the entering variable, the function should provide the choice to implement the following options.
    - Smallest value rule: After calculating all reduced costs, choose the variable with the smallest value (i.e., the most negative reduced cost) to enter the basis. This should be the default option.
    - Smallest index rule/ Bland's rule: Calculate the reduced costs one at a time and choose the variable that first gives a negative reduced cost to enter. In this option, you must not calculate all reduced costs.
- For choosing the leaving variable, the function should implement the following rule:
    - Smallest index rule: From among the candidates, the variable x_j with the smallest index j leaves. This should be the default option.

(*b*) Write an Octave function capable of solving an LP in standard form using the tableau simplex method. This function should accept the constraint matrix A, the right-hand side vector $\boldsymbol{b}$, and the cost vector $\boldsymbol{c}$ as input and provide as output an optimal solution vector $\boldsymbol{x}$ along with the optimal cost. In cases where the LP is unbounded or infeasible, the function should appropriately indicate this. Additionally, the number of simplex pivots or iterations employed should be part of the function's output. The function should offer flexibility in selecting both entering and leaving variables, with the following options available:

- For choosing the entering variable, the function should provide the choice to implement the following options:
    - Smallest value rule: After calculating all reduced costs, choose the variable with the smallest value (i.e., the most negative reduced cost) to enter the basis. This should be the default option.
    - Smallest index rule/ Bland's rule: After calculating all reduced costs, choose the variable with the smallest index with a negative reduced cost to enter the basis.

- For choosing the leaving variable, the function should provide the following options.
  - Smallest index rule: From among the candidates, the variable x_j with the smallest index j leaves. This should be the default option.
  - Lexicographic rule: The leaving variable corresponds to the lexicographically[1] smallest row, after scaling (see [Bertsimas and Tsitsiklis, 1997, Section 3.4]).

10.22 Use (10.21) to show that $\boldsymbol{x}^{\mathsf{T}}\boldsymbol{s} + \tau\kappa = 0$.

Notes and sources

The history of optimization and linear programming is a blend of ancient and modern influences. The origins of "optimization" can be traced back to ancient civilizations, where early mathematicians formulated and solved various optimization problems. Early references to optimization can be found in the works of ancient mathematicians like Euclid and Archimedes, who sought to maximize or minimize certain geometric quantities. The term "calculus of variations" was introduced in the 18th century, with pioneers like Leonhard Euler making substantial contributions to the field. However, the formalization of "linear programming", a specific branch of optimization, emerged in the mid-20th century. George Dantzig is often credited with pioneering linear programming during World War II, when he developed the simplex method for solving linear programming problems (refer to Dantzig [2016]). His work, along with the contributions of John von Neumann and Leonid Kantorovich, marked a significant turning point in the history of optimization and linear programming.

In linear optimization problems, we optimize a linear function subject to linear equality and inequality constraints. In this chapter, we began our study of linear optimization with the graphical method. We delved into the intricacies of the geometry of linear programming. Subsequently, our focus shifted to the study of the simplex method, which is the most prevalent algorithm for solving linear optimization problems. After that, we delved into an exploration of the duality in linear programming. As we neared the conclusion of this chapter, we addressed the linear programming problems that extended beyond the scope of the simplex method by investigating an interior-point method.

As we conclude this chapter, it is worth noting that the cited references and others, such as Boyd et al. [2004], Chong and Zak [2013], Nocedal and Wright [2006], Panik [1996], Roos et al. [1998], Schrijver [1999], Mitchell et al. [2006], Ferris et al. [2007], Griva et al. [2008], Aggarwal [2020], Vavasis [1999], Luenberger [1973], Bazaraa et al. [2005], Solow [2008], Chandru and Rao [1999], Golub and Bartels [2007], Pan [2023], Taha [1971], Hillier and Lieberman [2001], Peressini et al. [2012], Ackoff and Sasieni [1968], also serve as valuable sources of information pertaining to the subject matter covered in this chapter. Exercise 10.21 is due to Krishnamoorthy [2023a].

[1] The lexicographic order, also referred to as lexical order or dictionary order, extends the concept of alphabetical ordering found in dictionaries to sequences of arranged symbols or, in a broader context, to elements within a totally ordered set.

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CHAPTER 11

SECOND-ORDER CONE PROGRAMMING

Chapter overview: In second-order cone programming (SOCP) problems, we optimize a linear objective function over the intersection of an affine set and the product of second-order cones. Within this chapter, we explore the SOCP problems and their associated interior-point methods, providing a comprehensive understanding of this important class of optimization problems. After studying the algebraic structure of the second-order cone, we delve into the theory and applications of SOCP. Additionally, we shed light on the interior-point methods developed for solving SOCP problems efficiently. The chapter concludes with a set of exercises to encourage readers to apply their knowledge and enrich their understanding.

Keywords: Second-order cone, SOCP, Interior-point methods

Contents

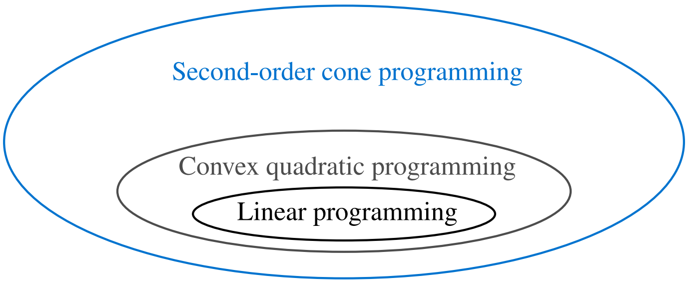


Figure 11.1: A Venn diagram of different classes of optimization problems.

In Chapter 10, we have studied linear programming. In modern convex optimization, the class of optimization that is an immediate conic enlargement of linear programming is not quadratic programming, but rather the so-called second-order cone programming. Second-order cone programming (SOCP for short) problems, which include linear programming problems and quadratic programming problems as special cases, are a class of convex optimization problems in which the variable is not a vector whose each of its components is required to be nonnegative, but rather a block vector whose each of its subvectors is required to reside in a second-order cone (see Definition 11.1).

Figure 11.1 shows graphical relationships among different classes of optimization problems. So in an SOCP problem, we optimize a linear function over the intersection of an affine linear manifold[1] with the Cartesian product of second-order cones. This chapter is devoted to studying SOCP problems. We also refer to Alizadeh and Goldfarb [2003] for an excellent survey paper on this topic.

11.1 The second-order cone and its algebraic structure

This section aims to introduce algebraic tools needed to study the SOCP problems. We start this by introducing notations that will be used throughout this chapter. As in earlier chapters, we use "," for adjoining scalars, vectors and matrices in a row, and use ";" for adjoining them in a column. For example, a vector $\boldsymbol{x} \in \mathbb{R}^n$ can be written as $\boldsymbol{x} = (x_1; x_2; \ldots; x_n)$. For each vector $\boldsymbol{x} \in \mathbb{R}^n$ whose first entry is indexed with 0, we write $\widetilde{\boldsymbol{x}}$ for the subvector consisting of entries 1 through $n-1$; therefore $\boldsymbol{x} = (x_0; \widetilde{\boldsymbol{x}}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. We let $\mathbb{E}^n$ denote the nth-dimensional real Euclidean space $\mathbb{R} \times \mathbb{R}^{n-1}$ whose elements $\boldsymbol{x}$ are indexed from 0.

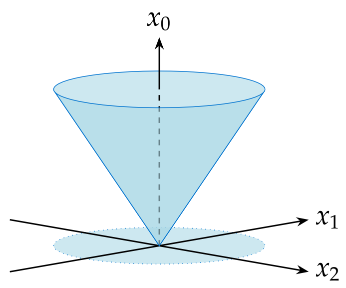
Definition 11.1 (The second-order cone) *The nth-dimensional second-order cone is defined as*

$$\mathbb{E}^n_+ \triangleq \{\boldsymbol{x} \in \mathbb{E}^n : x_0 \geq \|\widetilde{\boldsymbol{x}}\|\},$$

where $\|\cdot\|$ *denotes the Euclidean norm. The interior of this cone is the set* $\text{int}\ \mathbb{E}^n_+ \triangleq \{\boldsymbol{x} \in \mathbb{E}^n : x_0 > \|\widetilde{\boldsymbol{x}}\|\}$.

[1] A manifold, in mathematics, can be viewed as a generalization and abstraction of the notion of a curved surface.

The graph to the right shows the 3rd-dimensional second-order cone $\mathbb{E}^3_+$. The cone $\mathbb{E}^n_+$ is closed, pointed (i.e., it does not contain a pair of opposite nonzero vectors) and convex with nonempty interior in $\mathbb{R}^n$. The cone $\mathbb{E}^n_+$ is one of the well-known examples of of the so-called symmetric cones (or self-scaled cones, see Schmieta and Alizadeh [2003], Nesterov and Todd [1998] for definitions).



It is known that the space $\mathbb{E}^n$ under the bilinear map $\circ : \mathbb{E}^n \times \mathbb{E}^n \longrightarrow \mathbb{E}^n$ defined as

$$x \circ y \triangleq \begin{bmatrix} x^{\mathsf{T}} y \\ x_0 \bar{y} + y_0 \bar{x} \end{bmatrix}$$

forms a Euclidean Jordan algebra (see Faraut [1994], Schmieta and Alizadeh [2003] for definitions) equipped with the standard inner product $\langle x, y \rangle \triangleq x^{\mathsf{T}} y$.

The spectral factorization of a given element is a factorization of this element into eigenvectors together with eigenvalues.

Property 11.1 (Spectral decomposition in $\mathbb{E}^n$) *Any $x \in \mathbb{E}^n$ can be expressed in exactly one way as a product:*

$$x = \underbrace{(x_0 + \|\widetilde{x}\|)}_{\lambda_1(x)} \underbrace{\left(\frac{1}{2}\right) \begin{bmatrix} 1 \\ \dfrac{\widetilde{x}}{\|\widetilde{x}\|} \end{bmatrix}}_{c_1(x)} + \underbrace{(x_0 - \|\widetilde{x}\|)}_{\lambda_2(x)} \underbrace{\begin{bmatrix} 1 \\ -\dfrac{\widetilde{x}}{\|\widetilde{x}\|} \end{bmatrix}}_{c_2(x)}. \tag{11.1}$$

The following definition is based on Property 11.1.

Definition 11.2 (Eigenvalues and eigenvectors) *The decomposition in (11.1) is called the spectral decomposition of $x \in \mathbb{E}^n$. The values $\lambda_{1,2}(x) \triangleq x_0 \pm \|\widetilde{x}\|$ are called the eigenvalues of x, and the vectors $c_{1,2}(x) \triangleq (1; \pm\widetilde{x}/\|\widetilde{x}\|)$ are called the eigenvectors of x.*

Having eigenvalues, we can define spectral notions such as trace and determinant.

Definition 11.3 (Trace and determinant) *The trace and determinant of $x \in \mathbb{E}^n$ are defined as* $\operatorname{trace}(x) \triangleq \lambda_1(x) + \lambda_2(x) = 2x_0$ *and* $\det(x) \triangleq \lambda_1(x)\lambda_2(x) = x_0^2 - \|\widetilde{x}\|^2$.

Note that $x \in \mathbb{E}^n_+$ ($x \in \operatorname{int} \mathbb{E}^n_+$) if and only if $\det(x) \geq 0$ ($\det(x) > 0$). Note also that $x \bullet y = \frac{1}{2}\operatorname{trace}(x \circ y) = x^{\mathsf{T}} y$.

We call $e_n \triangleq (1; \mathbf{0}) \in \mathbb{E}^n$ the identity vector of $\mathbb{E}^n$. The logarithmic barrier function is defined on $\operatorname{int} \mathbb{E}^n_+$ as $x \longmapsto -\ln \det(x)$. This map will be significant in our upcoming discussions.

One can show that the eigenvectors $c_1(x)$ and $c_2(x)$ satisfy the properties:

$$\begin{aligned} c_1(x) \circ c_2(x) &= \mathbf{0}, \\ c_1(x) \circ c_1(x) &= c_1(x), \\ c_2(x) \circ c_2(x) &= c_2(x), \\ c_1(x) + c_2(x) &= e_n. \end{aligned}$$

Any continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined on $\mathbb{E}^n$ as

$$f(x) \triangleq f(\lambda_1(x))c_1(x) + f(\lambda_2(x))c_2(x).$$

In particular, x^n for $n \geq 2$, which is defined recursively as $x^n \triangleq x \circ x^{n-1}$, can be redefined as

$$x^n \triangleq \lambda_1^n(x)c_1(x) + \lambda_2^n(x)c_2(x).$$

Note that $x^{-1} \circ x = e_n$. The vector x is called invertible if x^{-1} is defined, and noninvertible otherwise. Note also that every positive definite element is invertible and its inverse is also positive definite.

The Frobenius norm of $x \in \mathbb{E}^n$ is defined as $||x||_F \triangleq \sqrt{\lambda_1^2(x) + \lambda_2^2(x)} = 2||x||$.

It can be shown that, for any $x, y \in \mathbb{E}^n$, we have

$$||x \circ y||_F \leq ||x||\, ||y||_F \leq ||x||_F\, ||y||_F \ \text{ and } \ ||x + y||_F^2 = ||x||_F^2 + ||y||_F^2 + 4x^{\mathsf{T}} y. \tag{11.2}$$

Let $x \in \mathbb{E}^n$. The arrow-shaped matrix is defined so that $x \circ y = \operatorname{Arw}(x)y$ for every $y \in \mathbb{E}^n$.

Definition 11.4 (Arrow-shaped matrix) *Associated with each vector $x \in \mathbb{E}^n$, the arrow-shaped matrix of x is defined as*

$$Arw(x) \triangleq \begin{bmatrix} x_0 & \widetilde{x}^{\mathsf{T}} \\ \widetilde{x} & x_0 I \end{bmatrix}.$$

Note that $x \in \mathbb{E}_+^n$ ($x \in \operatorname{int} \mathbb{E}_+^n$) if and only if $\operatorname{Arw}(x)$ is positive semidefinite ($\operatorname{Arw}(x)$ is positive definite).

Definition 11.5 (Quadratic representation matrix) *Associated with each vector $x \in \mathbb{E}^n$, the quadratic representation matrix of x is defined as*

$$Q_x \triangleq 2Arw^2(x) - Arw(x^2) = \begin{bmatrix} ||x||^2 & 2x_0\widetilde{x}^{\mathsf{T}} \\ 2x_0\widetilde{x} & \det(x)I + 2\widetilde{x}\widetilde{x}^{\mathsf{T}} \end{bmatrix}.$$

Note that Q_x is also a linear operator in $\mathbb{E}^n$. Note also that $\operatorname{Arw}(x)e = x, \operatorname{Arw}(x)x = x^2$, $\operatorname{Arw}(e) = \mathrm{Q}_e = I,$ $\operatorname{trace}(e) = 2$ and $\det(e) = 1$ (since each of the two eigenvalues of e equals one).

We say that two vectors $x, y \in \mathbb{E}^n$ are simultaneously decomposed if they share a Jordan frame, i.e., $x = \lambda_1(x)\, c_1 + \cdots + \lambda_r(x)\, c_r$ and $y = \lambda_1(y)\, c_1 + \cdots + \lambda_r(y)\, c_r$ for a Jordan frame $\{c_1, \ldots, c_r\}$ (hence, we have $c_i(x) = c_i(y)$ for each $i = 1, \ldots, r$).

We say that two vectors x and y operator commute if for all z, we have that $x \circ (y \circ z) = y \circ (x \circ z)$.

It is known that two elements of $\mathbb{E}^n$ are simultaneously decomposed if and only if they operator commute [Schmieta and Alizadeh, 2003, Theorem 27].

Table 11.1 summarizes the notions associated with the algebra of the second-order cone.

Algebraic notions	Algebra of second-order cone
Space	$\mathbb{E}^n \triangleq \left\{ \boldsymbol{x} \triangleq \begin{pmatrix} x_0 \\ \bar{x} \end{pmatrix} : \boldsymbol{x} \in \mathbb{R} \times \mathbb{R}^{n-1} \right\}$
Second-order cone	$\mathbb{E}^n_+ \triangleq \{\boldsymbol{x} \in \mathbb{E}^n : x_0 \geq \|\bar{x}\|\}$
Dimension	$\dim(\mathbb{E}^n_+) = n$
Rank	$\operatorname{rank}(\mathbb{E}^n, \circ) = 2$
Inner product	$\boldsymbol{x} \bullet \boldsymbol{y} \triangleq x_0 y_0 + \bar{x}^\mathsf{T} \bar{y} = \boldsymbol{x}^\mathsf{T} \boldsymbol{y}$
Bilinear map	$\boldsymbol{x} \circ \boldsymbol{y} \triangleq \begin{bmatrix} \boldsymbol{x}^\mathsf{T} \boldsymbol{y} \\ x_0 \bar{y} + y_0 \bar{x} \end{bmatrix}$
Algebra	$(\mathbb{E}^n, \circ)$
Identity	$\boldsymbol{e}_n \triangleq \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix}$
Reflection matrix	$R \triangleq \begin{bmatrix} 1 & \boldsymbol{0}^\mathsf{T} \\ \boldsymbol{0} & -I_{n-1} \end{bmatrix}$
Eigenvalues	$\lambda_1(\boldsymbol{x}) \triangleq x_0 + \|\bar{x}\|^2, \quad \lambda_2(\boldsymbol{x}) \triangleq x_0 - \|\bar{x}\|^2$
Eigenvectors	$\boldsymbol{c}_1(\boldsymbol{x}) \triangleq \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\bar{x}}{\|\bar{x}\|} \end{bmatrix}, \quad \boldsymbol{c}_2(\boldsymbol{x}) \triangleq \frac{1}{2} \begin{bmatrix} 1 \\ \frac{-\bar{x}}{\|\bar{x}\|} \end{bmatrix}$
Spectral decomposition	$\boldsymbol{x} = \lambda_1(\boldsymbol{x}) \boldsymbol{c}_2(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}) \boldsymbol{c}_2(\boldsymbol{x})$
Trace	$\operatorname{trace}(\boldsymbol{x}) := 2x_0$
Determinant	$\det(\boldsymbol{x}) := x_0^2 - \|\bar{x}\|^2$
Square	$\boldsymbol{x}^2 \triangleq \boldsymbol{x} \circ \boldsymbol{x} = \begin{bmatrix} \|\boldsymbol{x}\|^2 \\ 2\, x_0 \bar{x} \end{bmatrix}$
Inverse	$\boldsymbol{x}^{-1} \triangleq \frac{1}{\lambda_1(\boldsymbol{x})} \boldsymbol{c}_2(\boldsymbol{x}) + \frac{1}{\lambda_2(\boldsymbol{x})} \boldsymbol{c}_2(\boldsymbol{x}) = \frac{1}{\det(\boldsymbol{x})} R\boldsymbol{x}$
Frobenius norm	$\|\boldsymbol{x}\|_F \triangleq \sqrt{2}\ \sqrt{\boldsymbol{x} \bullet \boldsymbol{x}} = \sqrt{2}\, \|\boldsymbol{x}\|$
Arrow-shaped matrix	$\operatorname{Arw}(\boldsymbol{x}) \triangleq \begin{bmatrix} x_0 & \bar{x}^\mathsf{T} \\ \bar{x} & x_0 I_{n-1} \end{bmatrix}$
Quadratic representation	$Q_{\boldsymbol{x}} \triangleq \begin{bmatrix} \|\boldsymbol{x}\|^2 & 2x_0 \bar{x}^\mathsf{T} \\ 2x_0 \bar{x} & \det(\boldsymbol{x}) I_{n-1} + 2\bar{x}\bar{x}^\mathsf{T} \end{bmatrix}$
Logarithmic barrier	$-\ln \det(\boldsymbol{x})$
Gradient of log barrier	$-\boldsymbol{x}^{-1}$
Hessian of log barrier	$-Q_{\boldsymbol{x}^{-1}} = -\frac{1}{\det^2(\boldsymbol{x})} \begin{bmatrix} \|\boldsymbol{x}\|^2 & -2x_0 \bar{x}^\mathsf{T} \\ -2x_0 \bar{x} & \det(\boldsymbol{x}) I_{n-1} + 2\bar{x}\bar{x}^\mathsf{T} \end{bmatrix}$

Table 11.1: The algebraic notions and concepts associated with the second-order cone.

All the above notions are also used in the block sense as follows: Let $\boldsymbol{x} \triangleq (\boldsymbol{x}_1; \boldsymbol{x}_2; \ldots; \boldsymbol{x}_r)$, $\boldsymbol{y} \triangleq (\boldsymbol{y}_1; \boldsymbol{y}_2; \ldots; \boldsymbol{y}_r)$, and $\boldsymbol{x}_i, \boldsymbol{y}_i \in \mathbb{E}^{n_i}$ for $i = 1, 2, \ldots, r$. Then

(a) $\mathbb{E}^n_r \triangleq \mathbb{E}^{n_1} \times \mathbb{E}^{n_2} \times \cdots \times \mathbb{E}^{n_r}$;

(b) $\mathbb{E}^n_{r+} \triangleq \mathbb{E}^{n_1}_+ \times \mathbb{E}^{n_2}_+ \times \cdots \times \mathbb{E}^{n_r}_+$;

(c) $\operatorname{int} \mathbb{E}^n_{r+} \triangleq \operatorname{int} \mathbb{E}^{n_1}_+ \times \operatorname{int} \mathbb{E}^{n_2}_+ \times \cdots \times \operatorname{int} \mathbb{E}^{n_r}_+$;

(d) $\boldsymbol{x} \circ \boldsymbol{y} \triangleq (\boldsymbol{x}_1 \circ \boldsymbol{y}_1; \ldots; \boldsymbol{x}_r \circ \boldsymbol{y}_r)$;

(e) $\boldsymbol{x}^{\mathsf{T}}\boldsymbol{y} \triangleq \sum_{i=1}^r \boldsymbol{x}_i{}^{\mathsf{T}}\boldsymbol{y}_i$;

(f) $\det(\boldsymbol{x}) \triangleq \prod_{i=1}^r \det(\boldsymbol{x}_i)$;

(g) $\operatorname{trace}(\boldsymbol{x}) \triangleq \sum_{i=1}^r \operatorname{trace}(\boldsymbol{x}_i)$;

(h) $\operatorname{Arw}(\boldsymbol{x}) \triangleq \bigoplus_{i=1}^r \operatorname{Arw}(\boldsymbol{x}_i)$[2];

(i) $Q_{\boldsymbol{x}} \triangleq \bigoplus_{i=1}^r Q_{\boldsymbol{x}_i}$;

(j) $\|\boldsymbol{x}\|_F^2 \triangleq \sum_{i=1}^r \|\boldsymbol{x}_i\|_F^2$;

(k) $f(\boldsymbol{x}) \triangleq (f(\boldsymbol{x}_1); f(\boldsymbol{x}_2); \ldots; f(\boldsymbol{x}_r))$. In particular, $\boldsymbol{x}^{-1} \triangleq (\boldsymbol{x}_1{}^{-1}; \boldsymbol{x}_2{}^{-1}; \ldots; \boldsymbol{x}_r{}^{-1})$;

(l) $\boldsymbol{e} \triangleq (\boldsymbol{e}_{n_1}; \boldsymbol{e}_{n_2}; \ldots; \boldsymbol{e}_{n_r})$ is the identity vector of $\mathbb{E}^n_r$;

(m) The logarithmic barrier of $\boldsymbol{x} \in \operatorname{int} \mathbb{E}^n_{r+}$ is defined as $\boldsymbol{x} \longmapsto -\ln\det(\boldsymbol{x})$;

(n) $\boldsymbol{x}$ and $\boldsymbol{y}$ operator commute iff $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$ operator commute for each $i = 1, 2, \ldots, r$.

Note that $\boldsymbol{x}$ has $2r$ eigenvalues (including multiplicities) comprised of the union of the eigenvalues of each $\boldsymbol{x}_i$ for $i = 1, 2, \ldots, r$.

We end this section by introducing some more notations that will be used throughout this chapter. We write $\boldsymbol{x} \succeq \mathbf{0}$ to mean that $\boldsymbol{x} \in \mathbb{E}^n_+$ (i.e., $\boldsymbol{x}$ is a positive semidefinite vector). We also write $\boldsymbol{x} \succ \mathbf{0}$ to mean that $\boldsymbol{x} \in \operatorname{int}\mathbb{E}^n_+$ (i.e., $\boldsymbol{x}$ is a positive definite vector), and write $\boldsymbol{x} \succeq \boldsymbol{y}$ ($\boldsymbol{x} \succ \boldsymbol{y}$) to mean that $\boldsymbol{x} - \boldsymbol{y} \succeq \mathbf{0}$ ($\boldsymbol{x} - \boldsymbol{y} \succ \mathbf{0}$). Note that $\boldsymbol{x} \succeq \mathbf{0}$ ($\boldsymbol{x} \succ \mathbf{0}$) if and only if $\lambda_i(\boldsymbol{x}) \geq 0$ ($\lambda_i(\boldsymbol{x}) > 0$) for $i = 1, 2, \ldots, r$.

We use $\mathbb{S}^n$ to denote the set of symmetric matrices of order n, and $\mathbb{S}^{n+}$ to denote the set of symmetric positive semidefinite matrices of order n. The set $\mathbb{S}^{n+}$ is a convex cone and optimization problems over this cone will be studied in Chapter 12. Throughout this and next chapters, we write $X \succeq 0$ to mean that $X \in \mathbb{S}^n_+$ (i.e., $\boldsymbol{x}$ is a positive semidefinite matrix). We also write $X \succ 0$ to mean that $X \in \operatorname{int}\mathbb{S}^n_+$ (i.e., X is a positive definite matrix), and write $X \succeq Y$ ($X \succ Y$) to mean that $X - Y \succeq 0$ ($X - Y \succ 0$).

11.2 Second-order cone programming formulation

In this section, we introduce the SOCP problem and formulate known class of optimization problems as SOCPs.

[2] The direct sum of two square matrices A and B is the block diagonal matrix $A \oplus B \triangleq \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

Problem formulation

Let $r \geq 1$ be an integer. For each $i = 1, 2, \ldots, r$, let m, n, n_i be positive integers such that $n = \sum_{i=1}^{r} n_i$. Let also $\boldsymbol{x}, \boldsymbol{c}$ and $\boldsymbol{z}$ be vectors in $\mathbb{R}^n$, $\boldsymbol{y}$ and $\boldsymbol{b}$ be vectors in $\mathbb{R}^m$, and A be a matrix in $\mathbb{R}^{m \times n}$ such that they are all conformally partitioned as

$$\begin{aligned} \boldsymbol{x} &\triangleq (\boldsymbol{x}_1; \boldsymbol{x}_2; \ldots; \boldsymbol{x}_r), \\ \boldsymbol{s} &\triangleq (\boldsymbol{s}_1; \boldsymbol{s}_2; \ldots; \boldsymbol{s}_r), \\ \boldsymbol{c} &\triangleq (\boldsymbol{c}_1; \boldsymbol{c}_2; \ldots; \boldsymbol{c}_r), \\ A &\triangleq (A_1, A_2, \ldots, A_r), \end{aligned}$$

where $\boldsymbol{x}_i, \boldsymbol{s}_i, \boldsymbol{c}_i \in \mathbb{E}^{n_i}$ and $A_i \in \mathbb{R}^{m_i \times n_i}$ for $i = 1, 2, \ldots, r$. The SOCP problem and its dual in multi-block structures are defined as

$$\text{(P|SOCP}_i\text{)} \quad \begin{array}{ll} \min & \boldsymbol{c}_1^{\mathsf{T}} \boldsymbol{x}_1 + \cdots + \boldsymbol{c}_r^{\mathsf{T}} \boldsymbol{x}_r \\ \text{s.t.} & A_1 \boldsymbol{x}_1 + \cdots + A_r \boldsymbol{x}_r = \boldsymbol{b}, \\ & \boldsymbol{x}_i \succeq \boldsymbol{0}, \ i = 1, \ldots, r; \end{array} \qquad \text{(D|SOCP}_i\text{)} \quad \begin{array}{ll} \max & \boldsymbol{b}^{\mathsf{T}} \boldsymbol{y} \\ \text{s.t.} & {A_i}^{\mathsf{T}} \boldsymbol{y} + \boldsymbol{s}_i = \boldsymbol{c}_i, i = 1, \ldots, r, \\ & \boldsymbol{s}_i \succeq \boldsymbol{0}, \ i = 1, \ldots, r. \end{array}$$

The pair (P|SOCP$_i$, D|SOCP$_i$) can be compactly rewritten as

$$\text{(P|SOCP)} \quad \begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}} \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \succeq \boldsymbol{0}; \end{array} \qquad \text{(D|SOCP)} \quad \begin{array}{ll} \max & \boldsymbol{b}^{\mathsf{T}} \boldsymbol{y} \\ \text{s.t.} & A^{\mathsf{T}} \boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c}, \\ & \boldsymbol{s} \succeq \boldsymbol{0}. \end{array}$$

Formulating problems as SOCPs

In this part, we formulate three general classes of optimization problems as SOCPs. We start with linear optimization.

Linear programming The linear optimization in the standard form is the problem

$$\begin{array}{ll} \min & c_1 x_1 + \cdots + c_r x_r \\ \text{s.t.} & x_1 \boldsymbol{a}_1 + \cdots + x_r \boldsymbol{a}_r = \boldsymbol{b}, \\ & x_i \geq 0, \ i = 1, \ldots, r. \end{array} \tag{11.3}$$

Clearly, the linear optimization problem (11.3) is Problem (P|SOCP$_i$) with $n_1 = n_2 = \cdots = n_r = 1$. In other words, when the vector $\boldsymbol{x} \in \mathbb{R}^r$ resides the following Cartesian product of second-order cones:

$$\underbrace{\mathbb{E}_+^1 \times \mathbb{E}_+^1 \times \cdots \times \mathbb{E}_+^1}_{r\text{-times}},$$

we have $\boldsymbol{x} \succeq \boldsymbol{0}$ (as, by definition, $\mathbb{E}_+^1 \triangleq \{t \in \mathbb{R} : t \geq 0\}$), and hence the SOCP problem reduces to a linear programming problem.

Convex quadratic programming In convex quadratic optimization problems, we minimize a strictly convex quadratic function subject to affine constraint functions:

$$\begin{array}{ll} \min & q(\boldsymbol{x}) \triangleq \boldsymbol{x}^{\mathsf{T}} Q \boldsymbol{x} + \boldsymbol{c}^{\mathsf{T}} \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array} \tag{11.4}$$

Since Problem (11.4) is strictly convex, the matrix Q must be a symmetric positive definite matrix (i.e., $Q = Q^{\mathsf{T}}$ and $Q \succ O$). It follows that there exists another positive definite matrix (hence nonsingular), say $Q^{\frac{1}{2}}$, whose square is Q. Note that

$$q(\boldsymbol{x}) = \boldsymbol{x}^{\mathsf{T}} Q \boldsymbol{x} + \boldsymbol{c}^{\mathsf{T}} \boldsymbol{x} = \left\| Q^{\frac{1}{2}} \left(\boldsymbol{x} + \frac{1}{2} Q^{-1} \boldsymbol{c} \right) \right\|^2 - \frac{1}{4} \boldsymbol{c}^{\mathsf{T}} Q^{-1} \boldsymbol{c} = \left\| \bar{\boldsymbol{y}} \right\|^2 - \frac{1}{4} \boldsymbol{c}^{\mathsf{T}} Q^{-1} \boldsymbol{c},$$

where $\bar{\boldsymbol{y}} \triangleq Q^{\frac{1}{2}} \left(\boldsymbol{x} + \frac{1}{2} Q^{-1} \boldsymbol{c} \right)$. Therefore, the quadratic optimization problem (11.4) can be formulated as the SOCP problem

$$\begin{array}{ll} \min & y_0 \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \bar{\boldsymbol{y}} - \boldsymbol{x} = \frac{1}{2} Q^{-1} \boldsymbol{c}, \\ & \boldsymbol{y} \succeq \boldsymbol{0}, \ \boldsymbol{x} \geq \boldsymbol{0}, \end{array} \tag{11.5}$$

where the underlying cone is the $(n+1)$st-dimensional second-order cone $\mathbb{E}^n$. Note that Problems (11.4) and (11.5) have the same minimizer but their optimal objective values are equal up to the constant $\frac{1}{4} \boldsymbol{c}^{\mathsf{T}} Q^{-1} \boldsymbol{c}$. We also point out that, more generally, convex quadratically constrained quadratic optimization problems can also be formulated as SOCP problems (see Alizadeh and Goldfarb [2003]).

Rotated quadratic cone programming Let n be a positive integer and M be a nonsingular matrix of order $n-2$. The nth dimensional rotated quadratic cone is defined as

$$\mathcal{K}^n \triangleq \left\{ \boldsymbol{x} = (x_0; x_1; \hat{\boldsymbol{x}}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} : 2x_0 x_1 \geq \|\hat{\boldsymbol{x}}\|^2, \ x_0 \geq 0, x_1 \geq 0 \right\}.$$

One can see that the rotated quadratic cone is obtained by rotating the second-order cone through an angle of $\pi/6$ in the $x_0 x_1$-plane. We call the constraint on $\boldsymbol{x}$ that satisfies the inequality $2x_0 x_1 \geq \|\hat{\boldsymbol{x}}\|^2$ the hyperbolic constraint.

In rotated quadratic cone optimization problems, a linear objective function is minimized subject to linear constraints and hyperbolic constraints. In fact, a rotated quadratic cone optimization problem can be expressed as an SOCP problem because the hyperbolic constraint is equivalent to a second-order cone constraint, and this can be easily deduced by observing that

$$\begin{array}{rcl} \boldsymbol{x} = (x_0; x_1; \hat{\boldsymbol{x}}) \in \mathcal{K}^n & \Longleftrightarrow & 2x_0 x_1 \geq \|\hat{\boldsymbol{x}}\|^2 \\ & \Longleftrightarrow & 4x_0 x_1 \geq -4x_0 x_1 + 4\|\hat{\boldsymbol{x}}\|^2 \\ & \Longleftrightarrow & (2x_0 + x_1)^2 \geq \|(2x_0 - x_1; 2\hat{\boldsymbol{x}})\|^2 \\ & \Longleftrightarrow & (2x_0 + x_1; 2x_0 - x_1; 2\hat{\boldsymbol{x}}) \in \mathbb{E}^n_+. \end{array} \tag{11.6}$$

The next section contains an application with hyperbolic constraints.

11.3 Applications in engineering and finance

In this section, we describe three applications of SOCP in geometry and finance. Namely, Euclidean facility location problem, the portfolio optimization problem with loss risk constraints, and the optimal covering ellipsoid problem. The material of this section has appeared in Lobo et al. [1998] (see also Alzalg [2012]). For more applications of SOCP, we refer the reader to Lobo et al. [1998], Alizadeh and Goldfarb [2003], Benson and Ümit Seğlam [2013], Alzalg and Alioui [2022].

Euclidean facility location problem

Facility location problems (FLPs) involve strategically placing new facilities while minimizing their proximity to existing ones. FLPs manifest in various real-world scenarios, encompassing the establishment of airports, regional campuses, the deployment of wireless communication towers, and similar infrastructure planning endeavors.

FLPs can be categorized in two ways. Firstly, based on the number of new facilities added, we have single FLPs when only one new facility is introduced, and multiple FLPs when multiple new facilities are added simultaneously. Secondly, the classification can also be based on the distance measurement employed. When the Euclidean distance is used, they are referred to as Euclidean FLPs, while if the rectilinear distance is utilized, they are termed rectilinear FLPs.

In single Euclidean FLP, we are given r existing facilities represented by the fixed points $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_r$ in $\mathbb{R}^n$, and we plan to place a new facility represented by $\boldsymbol{x}$ so that we minimize the weighted sum of the distances between $\boldsymbol{x}$ and each of the points $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_r$. To elaborate, we tackle the following problem:

$$\min \quad \textstyle\sum_{i=1}^{r} w_i \, \|\boldsymbol{x} - \boldsymbol{a}_i\|$$

or, alternatively, to the SOCP model:

$$\begin{array}{ll} \min & \sum_{i=1}^{r} w_i \, t_i \\ \text{s.t.} & (t_i; \boldsymbol{x} - \boldsymbol{a}_i) \succeq \mathbf{0}, \ i = 1, 2, \ldots, r, \end{array}$$

where w_i is the weight associated with the ith existing facility and the new facility for $i = 1, 2, \ldots, r$.

Let us consider the scenario where we need to incorporate not just one, but a total of m new facilities, denoted as $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n$. In this context, we have two distinct cases, contingent on whether or not there exists any interaction among these new facilities within the underlying model. In the absence of any interaction between these new facilities, our primary objective revolves around the minimization of weighted sums, representing the distances between each new facility and the existing ones. To elaborate, we tackle the following SOCP model:

$$\begin{array}{ll} \min & \sum_{j=1}^{m} \sum_{i=1}^{r} w_{ij} \, t_{ij} \\ \text{s.t.} & (t_{ij}; \boldsymbol{x}_j - \boldsymbol{a}_i) \succeq \mathbf{0}, \ \ i = 1, 2, \ldots, r, \ j = 1, 2, \ldots, m, \end{array}$$

where w_{ij} is the weight associated with the ith existing facility and the jth new facility for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, r$.

When there is interaction among the new facilities, our objective extends beyond the previous requirements. In such cases, we aim to minimize the collective sum of Euclidean distances between every pair of new facilities. This leads to a model formulated as follows:

$$\begin{array}{ll} \min & \sum_{j=1}^{m} \sum_{i=1}^{r} w_{ij} t_{ij} + \sum_{j=1}^{m} \sum_{j'=1}^{j-1} \hat{w}_{jj'} \hat{t}_{jj'} \\ \text{s.t.} & (t_{ij}; \boldsymbol{x}_j - \boldsymbol{a}_i) \succeq \mathbf{0}, \ \ i = 1,2,\dots,r, \ j = 1,2,\dots,m, \\ & (\hat{t}_{js}; \boldsymbol{x}_j - \boldsymbol{x}_s) \succeq \mathbf{0}, \ \ j = 1,2,\dots,m-1, \ s = j+1, j+2, \dots, m, \end{array}$$

where $\hat{w}_{jj'}$ is the weight associated with the new facilities j' and j for $j' = 1,2,\dots,j-1$ and $j = 1,2,\dots,m$.

Portfolio optimization with loss risk constraints

We consider the problem of maximizing the expected return subject to loss risk constraints, a renowned challenge in portfolio optimization.

Consider a portfolio problem encompassing a collection of n assets or stocks over a predefined time horizon. In this context, we introduce the variable x_i, representing the quantity of asset i held at the commencement of, and consistently throughout, the specified time frame. Additionally, we utilize p_i to denote the alteration in price for asset i over this duration. Consequently, we can represent the price vector $\boldsymbol{p} \in \mathbb{R}^n$ as the set of prices during this period. For the sake of simplicity, we assume that $\boldsymbol{p}$ follows a Gaussian distribution with a known mean vector $\bar{\boldsymbol{p}}$ and a covariance matrix Σ. As a result, the return over this timeframe is modeled as a Gaussian random variable denoted by $r = \boldsymbol{p}^\top \boldsymbol{x}$, characterized by its mean $\bar{r} = \bar{\boldsymbol{p}}^\top \boldsymbol{x}$ and variance $\sigma = \boldsymbol{x}^\top \Sigma \boldsymbol{x}$, where $\boldsymbol{x} = (x_1; x_2; \dots; x_n)$. It is worth noting, as emphasized in Lobo et al. [1998], that the selection of the portfolio variable $\boldsymbol{x}$ inherently involves the classical Markowitz tradeoff between the mean return and the variance of this random return.

The optimization variables are the portfolio vectors $\boldsymbol{x} \in \mathbb{R}^n$. For this portfolio vector, we take the simplest assumption $\boldsymbol{x} \geq \mathbf{0}$ (i.e., no short position Lobo et al. [1998]) and $\sum_{i=1}^{n} x_i = 1$ (i.e., unit total budget Lobo et al. [1998]).

Given an undesired return level α and a maximum acceptable probability β over the time period, our objective is to ascertain the allocation of asset i (represented as x_i), denoted as $\boldsymbol{x}$, which maximizes the expected return over this duration. This optimization must adhere to the loss risk constraint, ensuring that the probability $P(r \leq \alpha) \leq \beta$ is met throughout the specified period.

As noted in Lobo et al. [1998], the constraint $P(r \leq \alpha) \leq \beta$ is equivalent to the second-order cone constraint

$$\left(\alpha - \bar{r};\ \Phi^{-1}(\beta)(\Sigma^{\frac{1}{2}} \boldsymbol{x})\right) \succeq \mathbf{0},$$

provided $\beta \leq 1/2$ (i.e.,$\Phi^{-1}(\beta) \leq 0$), where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$$

is the cumulative normal distribution function of a zero mean unit variance Gaussian random variable. To prove this (see also Lobo et al. [1998]), notice that the constraint $P(r \leq \alpha) \leq \beta$ can be written as

$$P\left(\frac{r - \bar{r}}{\sqrt{\sigma}} \leq \frac{\alpha - \bar{r}}{\sqrt{\sigma}}\right) \leq \beta.$$

Since $(r - \bar{r})/\sqrt{\sigma}$ is a zero mean unit variance Gaussian random variable, the probability above is simply $\Phi((\alpha - \bar{r})/\sqrt{\sigma})$, thus the constraint $P(r \leq \alpha) \leq \beta$ can be expressed as $\Phi((\alpha - \bar{r})/\sqrt{\sigma}) \leq \beta$ or $((\alpha - \bar{r})/\sqrt{\sigma}) \leq \Phi^{-1}(\beta)$, or equivalently $\bar{r} + \Phi^{-1}(\beta)\sqrt{\sigma} \geq \alpha$. Since

$$\sqrt{\sigma} = \sqrt{\boldsymbol{x}^{\mathsf{T}}\Sigma\,\boldsymbol{x}} = \sqrt{(\Sigma^{1/2}\,\boldsymbol{x})^{\mathsf{T}}(\Sigma^{1/2}\boldsymbol{x})} = \|\Sigma^{1/2}\boldsymbol{x}\|,$$

the constraint $P(r \leq \alpha) \leq \beta$ is equivalent to the second-order cone constraint $\bar{r}+\Phi^{-1}(\beta)\|\Sigma^{1/2}\,\boldsymbol{x}\| \geq \alpha$ or equivalently

$$\left(\alpha - \bar{r};\ \Phi^{-1}(\beta)(\Sigma^{\frac{1}{2}}\boldsymbol{x})\right) \succeq \boldsymbol{0}.$$

Our goal is to determine the amount of the asset i (which is x_i over the given period, i.e., determine $\boldsymbol{x}$, such that the expected return over this period is maximized. This problem can be cast as an SOCP as follows:

$$\begin{array}{ll} \max & \bar{\boldsymbol{p}}^{\mathsf{T}}\,\boldsymbol{x} \\ \text{s.t.} & \left(\alpha - \bar{\boldsymbol{p}}^{\mathsf{T}}\boldsymbol{x}; \Phi^{-1}(\beta)(\Sigma^{1/2}\boldsymbol{x})\right) \succeq \boldsymbol{0}, \\ & \boldsymbol{1}^{\mathsf{T}}\boldsymbol{x} = 1, \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

The simple problem described above has many extensions (see Lobo et al. [1998]). One of these extensions is imposing several loss risk constraints, i.e., the constraints $P(r \leq \alpha_i) \leq \beta_i,\ i = 1, 2, \ldots, k$ (where $\beta_i \leq 1/2,$ for $i = 1, 2, \ldots, k$), or equivalently

$$\left(\alpha_i - \bar{r}; \Phi^{-1}(\beta_i)(\Sigma^{1/2}\boldsymbol{x})\right) \succeq \boldsymbol{0},\ \text{for } i = 1, 2, \ldots, k,$$

to be satisfied over the period. So our problem becomes the SOCP model:

$$\begin{array}{ll} \max & \bar{\boldsymbol{p}}^{\mathsf{T}}\,\boldsymbol{x} \\ \text{s.t.} & \left(\alpha_i - \bar{r}; \Phi^{-1}(\beta_i)(\Sigma^{1/2}\boldsymbol{x})\right) \succeq \boldsymbol{0},\ i = 1, 2, \ldots, k, \\ & \boldsymbol{1}^{\mathsf{T}}\boldsymbol{x} = 1, \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

Optimal covering ellipsoid problem

We consider the minimum-volume covering ellipsoid problem, a renowned challenge in geometric optimization.

Suppose we have k existing ellipsoids:

$$\mathcal{E}_i \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^{\mathsf{T}}H_i\boldsymbol{x} + 2\boldsymbol{g}_i^{\mathsf{T}}\boldsymbol{x} + v_i \leq 0\} \subset \mathbb{R}^n, i = 1, 2, \ldots, k,$$

where $H_i \in \mathbb{S}^n_+$, $\boldsymbol{g}_i \in \mathbb{R}^n$ and $v_i \in \mathbb{R}$ are given data for $i = 1, 2, \ldots, k$. We need to determine a ball that contains all k existing ellipsoids. We adopt the same assumptions as in Alzalg [2012], where we assume that the cost associated with selecting a ball comprises two elements:

- The center cost, which is directly proportional to the Euclidean distance between the center and the origin;
- The radius cost, which is directly proportional to the square of the radius.

The objective is to minimize the overall cost by determining both the center and radius of the ball. In Alzalg [2012], the author provides a concrete stochastic version of this general application. However, it is important to note that all the data we are considering in our description is deterministic. In this concrete version, we assume that $n = 2$. Within this context, there are fixed ellipsoids that contain targets that must be destroyed. Fighter aircraft take off from an origin point with a pre-planned circular coverage area that encompasses these fixed ellipsoids. The task at hand is to optimize the size and position of this coverage circle to minimize the total cost.

Our goal is to determine $\bar{x} \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ such that the ball $\mathcal{B}$ defined by

$$\mathcal{B} \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^\mathsf{T}\boldsymbol{x} - 2\bar{\boldsymbol{x}}^\mathsf{T}\boldsymbol{x} + \gamma \leq 0\}$$

contains the existing ellipsoids $\mathcal{E}_i$ for $i = 1, 2, \ldots, k$. Notice that the center of the ball $\mathcal{B}$ is $\bar{x}$, its radius is $r \triangleq \sqrt{\bar{x}^\mathsf{T}\bar{x} - \gamma}$, and the distance from the origin to its center is $\sqrt{\bar{x}^\mathsf{T}\bar{x}}$.

We introduce the constraints $d_1{}^2 \geq \bar{x}^\mathsf{T}\bar{x}$ and $d_2 \geq r^2 = \bar{x}^\mathsf{T}\bar{x} - \gamma$. That is, d_1 is an upper bound on the distance between the center of the ball $\mathcal{B}$ and the origin, $\sqrt{\bar{x}^\mathsf{T}\bar{x}}$, and d_2 is an upper bound on square of the radius of the ball $\mathcal{B}$.

In order to proceed, we need the following lemma which is due to Sun and Freund [2004].

Lemma 11.1 *Suppose that we are given two ellipsoids $\mathcal{E}_i \subset \mathbb{R}^n$, $i = 1, 2$, defined by $\mathcal{E}_i \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^\mathsf{T}H_i\boldsymbol{x} + 2\boldsymbol{g}_i^\mathsf{T}\boldsymbol{x} + v_i \leq 0\}$, where $H_i \in \mathbb{S}^n_+$, $\boldsymbol{g}_i \in \mathbb{R}^n$ and $v_i \in \mathbb{R}$ for $i = 1, 2$, then $\mathcal{E}_1$ contains $\mathcal{E}_2$ if and only if there exists $\tau \geq 0$ such that the linear matrix inequality*

$$\begin{bmatrix} H_1 & g_1 \\ g_1^\mathsf{T} & v_1 \end{bmatrix} \preceq \tau \begin{bmatrix} H_2 & g_2 \\ g_2^\mathsf{T} & v_2 \end{bmatrix}$$

holds.

In view of Lemma 11.1 and the requirement that the ball $\mathcal{B}$ contains the existing ellipsoids $\mathcal{E}_i$ for $i = 1, 2, \ldots, k$, we accordingly add the following constraints:

$$\begin{bmatrix} I & -\bar{x} \\ -\bar{x}^\mathsf{T} & \gamma \end{bmatrix} \preceq \tau_i \begin{bmatrix} H_i & g_i \\ g_i{}^\mathsf{T} & v_i \end{bmatrix}, \quad i = 1, 2, \ldots, k,$$

or equivalently

$$M_i \succeq 0, \ \ \forall i = 1, \ldots, k, \ \ \text{where} \ \ M_i \triangleq \begin{bmatrix} \tau_i H_i - I & \tau_i g_i + \bar{x} \\ \tau_i g_i{}^\mathsf{T} + \bar{x}^\mathsf{T} & \tau_i v_i - \gamma \end{bmatrix},$$

for each $i = 1, \ldots, k$.

Since we are looking to minimizing d_2, where d_2 is an upper bound on the square of the radius of the ball $\mathcal{B}$, we can write the constraint $d_2 \geq \bar{x}^\mathsf{T}\bar{x} - \gamma$ as $d_2 = \bar{x}^\mathsf{T}\bar{x} - \gamma$. So, the matrix M_i can be then written as

$$M_i = \begin{bmatrix} \tau_i H_i - I & \tau_i g_i + \bar{x} \\ \tau_i g_i{}^\mathsf{T} + \bar{x}^\mathsf{T} & \tau_i v_i + d_2 - \bar{x}^\mathsf{T}\bar{x} \end{bmatrix}.$$

Now, let $H_i \triangleq \Xi_i \Lambda_i \Xi_i^\mathsf{T}$ be the spectral decomposition of H_i, where $\Lambda_i \triangleq \text{Diag}(\lambda_{i1}; \ldots; \lambda_{ik})$, and let $\boldsymbol{u}_i \triangleq \Xi_i^\mathsf{T}(\tau_i g_i + \bar{x})$.

Then, for each $i = 1, \ldots, k$, we have

$$\bar{M}_i \triangleq \begin{bmatrix} \Xi_i^{\mathsf{T}} & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} M_i \begin{bmatrix} \Xi_i & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} = \begin{bmatrix} \tau_i \Lambda_i - I & \boldsymbol{u}_i \\ \boldsymbol{u}_i^{\mathsf{T}} & \tau_i v_i + d_2 - \bar{\boldsymbol{x}}^{\mathsf{T}} \bar{\boldsymbol{x}} \end{bmatrix}.$$

Consequently, $M_i \succeq 0$ if and only if $\bar{M}_i \succeq 0$ for $i = 1, 2, \ldots, k$. Now our formulation of the problem in SOCP depends on the following lemma (see also Lobo et al. [1998]).

Lemma 11.2 *For each $i = 1, 2, \ldots, k$, $\bar{M}_i \succeq 0$ if and only if $\tau_i \lambda_{\min}(H_i) \geq 1$ and $\bar{\boldsymbol{x}}^{\mathsf{T}} \bar{\boldsymbol{x}} \leq d_2 + \tau_i v_i + \mathbf{1}^{\mathsf{T}} \boldsymbol{s}_i$, where $\boldsymbol{s}_i = (s_{i1}; s_{i2}; \ldots; s_{in})$, $s_{ij} = u_{ij}^2 / (\tau_i \lambda_{ij} - 1)$ for all j such that $\tau_i \lambda_{ij} > 1$, and $s_{ij} = 0$ otherwise.*

The proof of Lemma 11.2 is left as an exercise for the reader (see Exercise 11.1).

Now, since we are minimizing d_2, then for all $j = 1, 2, \ldots, n$, we can relax the definitions of s_{ij} by replacing them by $u_{ij}^2 \leq s_{ij}(\tau_i \lambda_{ij} - 1), i = 1, 2, \ldots, k$. Let $\bar{c} > 0$ denote the cost per unit of the Euclidean distance between the center of the ball $\mathcal{B}$ and the origin, and let $\alpha > 0$ be the cost per unit of the square of the radius of $\mathcal{B}$.

We now define the decision variable $\boldsymbol{x} \triangleq (d_1; d_2; \bar{\boldsymbol{x}}; \gamma; \boldsymbol{\tau})$. Then, by introducing the unit cost vector $\boldsymbol{c} \triangleq (\bar{c}; \alpha; \mathbf{0}; 0; \mathbf{0})$, and combining all of the above, we get the model:

$$\begin{array}{lrcl} \min & \boldsymbol{c}^{\mathsf{T}} \boldsymbol{x} & + & \\ \text{s.t.} & \boldsymbol{u}_i & = & \Xi_i^{\mathsf{T}} (\tau_i \boldsymbol{g}_i + \bar{x}), i = 1, 2, \ldots, k, \\ & u_{ij}^2 & \leq & s_{ij}(\tau_i \lambda_{ij} - 1), i = 1, 2, \ldots, k, \; j = 1, 2, \ldots, n, \\ & \bar{\boldsymbol{x}}^{\mathsf{T}} \bar{\boldsymbol{x}} & \leq & \sigma, \\ & \sigma & \leq & d_2 + \tau_i v_i - \mathbf{1}^{\mathsf{T}} \boldsymbol{s}_i, i = 1, 2, \ldots, k, \\ & \tau_i & \geq & 1/\lambda_{\min}(H_i), i = 1, 2, \ldots, k, \\ & \bar{\boldsymbol{x}}^{\mathsf{T}} \bar{\boldsymbol{x}} & \leq & d_1^2, \end{array}$$

which includes only linear and hyperbolic constraints.

11.4 Duality in second-order cone programming

The duality in SOCP requires the application of the Karush-Kuhn-Tucker theorem which we state in Theorem 11.1. We first need the following definition.

Definition 11.6 *The Lagrangian function of the general minimization problem*

$$\begin{array}{ll} \min & f(\boldsymbol{x}) \\ \text{s.t.} & g_i(\boldsymbol{x}) = 0, \; i = 1, 2, \ldots, p, \\ & h_j(\boldsymbol{x}) \leq 0, \; j = 1, 2, \ldots, q, \end{array} \tag{11.7}$$

is defined as

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}) \triangleq f(\boldsymbol{x}) + \sum_{i=1}^{p} u_i g_i(\boldsymbol{x}) + \sum_{j=1}^{q} v_j h_j(\boldsymbol{x}),$$

for $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{u} \in \mathbb{R}^p_+$ *and* $\boldsymbol{v} \in \mathbb{R}^q$.

The Karush-Kuhn-Tucker conditions[3] (KKT conditions for short) are also called first-order necessary conditions for a point to be a local minimizer.

Theorem 11.1 (KKT Theorem) *Let* $\boldsymbol{x}$ *be a local minimizer for Problem (11.7), then there exist* $\boldsymbol{u} \in \mathbb{R}^p$ *and* $\boldsymbol{v} \in \mathbb{R}^q$ *such that all the following KKT conditions are satisfied.*

$$
\begin{array}{ll}
\nabla_x \mathcal{L}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{0}, & \textit{(stationary);} \\
u_i g_i(\boldsymbol{x}) = 0,\ \forall i, & \textit{(complementary slackness);} \\
g_i(\boldsymbol{x}) = 0, h_j(\boldsymbol{x}) \leq 0,\ \forall i,\ \forall j, & \textit{(primal feasibility);} \\
u_i \geq 0,\ \forall i, & \textit{(dual feasibility).}
\end{array}
$$

The following example is attributed to Chong and Zak [2013].

Example 11.1 Consider the following problem.

$$
\begin{array}{ll}
\min & x_1 x_2 \\
\text{s.t.} & x_1 + x_2 \geq 2, \\
& -x_1 + x_2 \geq 0.
\end{array}
$$

The Lagrangian function is

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v}) &= x_1x_2 + v_1(2 - x_1 - x_2) + v_2(x_1 - x_2) \\
&= x_1x_2 - (v_1 - v_2)x_1 - (v_1 + v_2)x_2 + 2v_2,
\end{aligned}
$$

for $v_1, v_2 \in \mathbb{R}$. It follows that

$$
\nabla_x \mathcal{L}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v}) = \begin{bmatrix} x_2 - v_1 + v_2 \\ x_1 - v_1 - v_2 \end{bmatrix}.
$$

The KKT conditions are

$$
\begin{array}{rcl}
x_2 - v_1 + v_2 &=& 0, \\
x_1 - v_1 - v_2 &=& 0, \\
v_1(2 - x_1 - x_2) + v_2(x_1 - x_2) &=& 0, \\
x_1 - x_2 &\leq& 0, \\
2 - x_1 - x_2 &\leq& 0, \\
v_1, v_2 &\geq& 0.
\end{array}
$$

Note that the points $\boldsymbol{x}^\star = (1; 1)$ and $\boldsymbol{v}^\star = (1; 0)$ satisfy the KKT conditions, therefore $\boldsymbol{x}^\star$ is a candidate for being a minimizer. ■

[3] The KKT conditions were derived independently by William Karush in 1939 and by Harold Kuhn and Albert Tucker in 1951.

We point out that, in Example 11.1, there is no guarantee that $\boldsymbol{x}^\star$ is indeed a minimizer, because the KKT conditions are, in general, only necessary. As a central result in convex programming, if Problem (11.7) is a convex program, then the KKT conditions are not only necessary, but also sufficient for optimality.

Within this section, we embark on an exploration of the duality theory intricately linked with SOCP. We lay the foundation for this expedition by introducing the concept of the complementarity condition, a pivotal facet of SOCP's optimality criteria. The KKT conditions have been introduced because they are needed for the establishment of various duality-related findings. Much, but not all, of the duality theory for SOCP is very similar to the duality theory for linear programming.

Recall that a regular cone is self-dual if it equals its dual cone (see Definition 3.22). Now we need the following lemma.

Lemma 11.3 *The second-order cone $\mathbb{E}^n_+$ is self-dual.*

Proof We verify that the second-order cone $\mathbb{E}^n_+$ equals its dual cone, which is defined as

$$(\mathbb{E}^n_+)^\star \triangleq \left\{\boldsymbol{x} \in \mathbb{E}^n : \boldsymbol{x}^\mathsf{T}\boldsymbol{y} \geq 0 \text{ for all } \boldsymbol{y} \in \mathbb{E}^n_+\right\}.$$

We first show that $\mathbb{E}^n_+ \subseteq (\mathbb{E}^n_+)^\star$. Let $\boldsymbol{x} = (x_0; \widetilde{\boldsymbol{x}}) \in \mathbb{E}^n_+$, we need to show $\boldsymbol{x} \in (\mathbb{E}^n_+)^\star$. For any $\boldsymbol{y} \in \mathbb{E}^n_+$, we have

$$\boldsymbol{x}^\mathsf{T}\boldsymbol{y} = x_0 y_0 + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} \geq \left\|\widetilde{\boldsymbol{x}}\right\| \left\|\widetilde{\boldsymbol{y}}\right\| + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} \geq \left|\widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}}\right| + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} \geq 0,$$

where the first inequality follows from the fact that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n_+$, and the second inequality follows from Cauchy-Schwartz inequality. This implies that $\boldsymbol{x} \in (\mathbb{E}^n_+)^\star$ and hence $\mathbb{E}^n_+ \subseteq (\mathbb{E}^n_+)^\star$.

To prove the reverse inclusion, let $\boldsymbol{y} \in (\mathbb{E}^n_+)^\star$, we need to show that $\boldsymbol{y} \in \mathbb{E}^n_+$, which is trivial if $\widetilde{\boldsymbol{y}} = 0$. If $\widetilde{\boldsymbol{y}} \neq \boldsymbol{0}$, let $\boldsymbol{x} \triangleq (\|\widetilde{\boldsymbol{y}}\|; -\widetilde{\boldsymbol{y}}) \in \mathbb{E}^n_+$. Then, we have

$$\boldsymbol{x}^\mathsf{T}\boldsymbol{y} = x_0 y_0 + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} = y_0 \left\|\widetilde{\boldsymbol{y}}\right\| - \widetilde{\boldsymbol{y}}^\mathsf{T}\widetilde{\boldsymbol{y}} = y_0 \left\|\widetilde{\boldsymbol{y}}\right\| - \left\|\widetilde{\boldsymbol{y}}\right\|^2,$$

where we used Cauchy-Schwartz inequality, where the equality is attained, to obtain the last equality. Since $\boldsymbol{x}$ belongs to the second-order cone $\mathbb{E}^n_+$ and $\boldsymbol{y}$ belongs to its dual, it follows that

$$0 \leq \boldsymbol{x}^\mathsf{T}\boldsymbol{y} = \left\|\widetilde{\boldsymbol{y}}\right\| \left(y_0 - \left\|\widetilde{\boldsymbol{y}}\right\|\right).$$

As $\widetilde{\boldsymbol{y}} \neq \boldsymbol{0}$, this implies that $y_0 \geq \|\widetilde{\boldsymbol{y}}\|$. That is, $\boldsymbol{y} \in \mathbb{E}^n_+$ and hence $(\mathbb{E}^n_+)^\star \subseteq \mathbb{E}^n_+$. The proof is complete. ■

Now consider Problem (P|SOCP). Note that the variable $\boldsymbol{x}$ is the primal variable, and the variables $\boldsymbol{y}$ and $\boldsymbol{s}$ are the dual variables. We call $\boldsymbol{x} \in \mathbb{E}^n$ primal feasible if $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \succeq \boldsymbol{0}$. Similarly, $(\boldsymbol{s}, \boldsymbol{y}) \in \mathbb{E}^n \times \mathbb{R}^m$ is called dual-feasible if $A^\mathsf{T}\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c}$ and $\boldsymbol{s} \succeq \boldsymbol{0}$. Note that the matrix A is defined to map $\mathbb{E}^n$ into $\mathbb{R}^m$, and its transpose A^T is defined to map $\mathbb{R}^m$ into $\mathbb{E}^n$ such that $\boldsymbol{x}^\mathsf{T}(A^\mathsf{T}\boldsymbol{y}) = (A\boldsymbol{x})^\mathsf{T}\boldsymbol{y}$.

We state and prove the following weak duality property.

Lemma 11.4 (Weak duality in SOCP) *If x and (y, s) are primal and dual feasible solutions in (P|SOCP) and (D|SOCP), respectively, then the duality gap is*

$$c^\mathsf{T} x - b^\mathsf{T} y = x^\mathsf{T} s \geq 0.$$

Proof Consider Problems (P|SOCP) and (D|SOCP). Due to their constraints, we can replace c with $A^\mathsf{T} y + s$ and b with Ax and get

$$c^\mathsf{T} x - b^\mathsf{T} y = \left(A^\mathsf{T} y + s\right)^\mathsf{T} x - (Ax)^\mathsf{T} y = \left(A^\mathsf{T} y\right)^\mathsf{T} x + s^\mathsf{T} x - (Ax)^\mathsf{T} y = x^\mathsf{T} s.$$

Note that $x, s \in \mathbb{E}^n$. By the self duality of the second-order cone $\mathbb{E}^n_+$, it concludes that $x^\mathsf{T} s \geq 0$ and this completes the proof. ■

The paper Nesterov and Nemirovskii [1994] shows that the strong duality property can fail in general conic optimization. However, despite this, a slightly weaker property can be always shown in conic optimization. Now, we give conditions for such a slightly weaker property to hold in SOCP. We say that the primal problem is strictly feasible if there exists a primal feasible point $\hat{x}$ such that $\hat{x} \succ \mathbf{0}$. We make the following assumption for convenience.

Assumption 11.1 *The m rows of the matrix A are linearly independent.*

Using the KKT conditions (see Theorem 11.1), we state and prove the following semi-strong duality result.

Lemma 11.5 (Semi-strong duality in SOCP) *Consider the primal–dual pair (P|SOCP) and (D|SOCP). If the primal problem is strictly feasible and solvable, then the dual problem is solvable and their optimal values are equal.*

Proof Given the lemma's underlying assumption, which assures the strict feasibility and solvability of the primal problem, we proceed with considering an optimal solution denoted as x, where the application of the KKT conditions becomes applicable. Consequently, this entails the existence of Lagrange multiplier vectors y and s, yielding a satisfying solution (x, y, s) that adheres to the following set of conditions:

$$\begin{aligned}
&Ax = b,\\
&A^\mathsf{T} y + s = c,\\
&x^\mathsf{T} s = 0,\\
&x, s \succeq \mathbf{0}.
\end{aligned}$$

This entails that the pair (y, s) indeed qualifies as a feasible solution for the dual problem. Now, when considering any feasible solution pair (v, z) for the dual problem, we can establish the inequality $b^\mathsf{T} v \leq c^\mathsf{T} x = s^\mathsf{T} x + b^\mathsf{T} y = b^\mathsf{T} y$, utilizing the weak duality to derive the inequality. Subsequently, by invoking the complementary slackness, we arrive at the ultimate equality $c^\mathsf{T} x = b^\mathsf{T} y$. Thus, confirming that (y, s) stands as an optimal solution for the dual problem, and we attain the desired equality. This concludes the proof. ■

The following strong duality result can be obtained by applying the duality relations to our problem formulation.

Theorem 11.2 (Strong duality in SOCP) *Consider the primal–dual pair (P|SOCP) and (D|SOCP). If both the primal and dual problems have strictly feasible solutions, then they both have optimal solutions* $\boldsymbol{x}^\star$ *and* $(\boldsymbol{y}^\star, \boldsymbol{s}^\star)$*, respectively, and*

$$p^\star \triangleq \boldsymbol{c}^\mathsf{T}\boldsymbol{x}^\star = d^\star \triangleq \boldsymbol{b}^\mathsf{T}\boldsymbol{y}^\star (i.e.,\ {\boldsymbol{x}^\star}^\mathsf{T}\boldsymbol{s}^\star = 0).$$

The following lemma describes the complementarity condition as one of the optimality conditions of SOCP.

Lemma 11.6 (Complementarity condition in SOCP) *If* $\boldsymbol{x}, \boldsymbol{s} \succeq \boldsymbol{0}$*, then*

$$\boldsymbol{x}^\mathsf{T}\boldsymbol{s} = 0 \iff \boldsymbol{x} \circ \boldsymbol{s} = \boldsymbol{0}.$$

Proof For any $\boldsymbol{x}, \boldsymbol{s} \in \mathbb{E}^n$ having $\boldsymbol{x}, \boldsymbol{s} \succeq \boldsymbol{0}$, we need to show that $\boldsymbol{x}^\mathsf{T}\boldsymbol{s} = 0$ if and only if $\boldsymbol{x}\circ\boldsymbol{s} = \boldsymbol{0}$, or equivalently,

(i) $\boldsymbol{x}^\mathsf{T}\boldsymbol{s} = x_0 s_0 + \langle\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{s}}\rangle = 0$; (ii) $x_0\widetilde{\boldsymbol{s}} + s_0\widetilde{\boldsymbol{x}} = \boldsymbol{0}$.

The direction from right to left is straightforward since the requirement is inherently included within the initial assumptions. To prove the other direction, assume that $\boldsymbol{x}^\mathsf{T}\boldsymbol{s} = 0$ (which is again (i) itself), it is enough to show that (ii) is satisfied. If $x_0 = 0$ or $s_0 = 0$, the result is clearly trivial. So, we only consider the cases when $x_0 > 0$ and $s_0 > 0$. Using Cauchy-Schwartz inequality and the fact that $\boldsymbol{x}, \boldsymbol{s} \in \mathbb{E}^n$, we have

$$\langle\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{s}}\rangle = \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{s}} = \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{s}} \geq -\left\|\widetilde{\boldsymbol{x}}\right\|\left\|\widetilde{\boldsymbol{s}}\right\| \geq -x_0 s_0, \tag{11.8}$$

Note that $\boldsymbol{x}^\mathsf{T}\boldsymbol{s} = 0$ if and only if $\langle\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{s}}\rangle = -x_0 s_0$, therefore the inequalities in (11.8) are satisfied as equalities. This holds if and only if either $\boldsymbol{x} = \boldsymbol{0}$ or $\boldsymbol{s} = \boldsymbol{0}$, in which case (i) and (ii) trivially hold, or $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{s} \neq \boldsymbol{0}$, $\widetilde{\boldsymbol{x}} = -\alpha\widetilde{\boldsymbol{s}}$, where $\alpha > 0$, and $x_0 = \left\|\widetilde{\boldsymbol{x}}\right\| = \alpha\left\|\widetilde{\boldsymbol{s}}\right\| = \alpha s_0$, that is $\widetilde{\boldsymbol{x}} + \alpha\widetilde{\boldsymbol{s}} = \widetilde{\boldsymbol{x}} + (x_0/s_0)\widetilde{\boldsymbol{s}} = \boldsymbol{0}$. This implies that $x_0\widetilde{\boldsymbol{s}} + s_0\widetilde{\boldsymbol{x}} = \boldsymbol{0}$. The proof is complete. ∎

As a result of Lemma 11.6, the complementarity slackness condition for the primal and dual SOCP problems (P|SOCP) and (D|SOCP) can be equivalently represented by the equation $\boldsymbol{x} \circ \boldsymbol{s} = \boldsymbol{0}$. From the above results, we get the following corollary.

Corollary 11.1 (Optimality conditions in SOCP) *Consider the primal–dual pair (P|SOCP) and (D|SOCP). Assume that both the primal and dual problems are strictly feasible, then* $(\boldsymbol{x}, (\boldsymbol{y}, \boldsymbol{s})) \in \mathbb{E}^n \times \mathbb{R}^m \times \mathbb{E}^n$ *is a pair of optimal solutions to the SOCP (P|SOCP) and (D|SOCP) if and only if*

$$\begin{aligned} A\boldsymbol{x} &= \boldsymbol{b},\\ A^\mathsf{T}\boldsymbol{y} + \boldsymbol{s} &= \boldsymbol{c},\\ \boldsymbol{x} \circ \boldsymbol{s} &= \boldsymbol{0},\\ \boldsymbol{x}, \boldsymbol{s} &\succeq \boldsymbol{0}. \end{aligned} \tag{11.9}$$

We have established the duality relations in SOCP. The focus in the remaining part of this chapter is to solve SOCP algorithmically.

11.5 A primal-dual path-following algorithm

As we mentioned earlier, interior-point methods reach a best solution by traversing the interior of the feasible region. There are several Interior-point algorithms for SOCPs; see for example Lobo et al. [1998], Alizadeh and Goldfarb [2003], Alzalg [2020, 2018, 2011b, 2014b], Alzalg and Pirhaji [2017b], Alzalg et al. [2019], Alzalg [2014a], Alzalg and Pirhaji [2017a] and the references contained therein.

In this section, we present a primal-dual path-following algorithm for solving SOCP problems. The material presented in this section is based on, and similar to, any primal-dual path-following algorithm proposed for SOCP (see for instance Alzalg [2018]). The fundamental outline of the path-following algorithms designed for solving SOCP can be summarized as follows: We commence by associating the perturbed problems with second-order cone programming problems (P|SOCP) and (D|SOCP), and then drawing a path of the centers defined by the perturbed KKT optimality conditions. Following this path, we employ Newton's method to address the corresponding perturbed equations, aiming to derive a search direction that facilitates descent towards the optimal solution.

Let $\mu > 0$ be a barrier parameter. The perturbed primal problem corresponding to the primal problem (P|SOCP) is

$$
(\text{P|SOCP}_\mu) \qquad \begin{array}{ll} \min & f_\mu(\boldsymbol{x}) \triangleq \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - \mu \ln\det(\boldsymbol{x}) + r\mu\ln\mu \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \succ \boldsymbol{0}, \end{array}
$$

and the perturbed dual problem corresponding to the dual problem (D|SOCP) is

$$
(\text{D|SOCP}_\mu) \qquad \begin{array}{ll} \max & g_\mu(\boldsymbol{y},\boldsymbol{s}) \triangleq \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} + \mu \ln\det(\boldsymbol{s}) - r\mu\ln\mu \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c}, \\ & \boldsymbol{s} \succ \boldsymbol{0}. \end{array}
$$

Now, we define the following feasibility sets:

$$
\begin{array}{lcl}
\mathcal{F}_{\text{P|SOCP}} & \triangleq & \{\boldsymbol{x} \in \mathbb{E}^n : A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \succeq \boldsymbol{0}\}, \\
\mathcal{F}_{\text{D|SOCP}} & \triangleq & \left\{(\boldsymbol{y};\boldsymbol{s}) \in \mathbb{R}^m \times \mathbb{E}^n : A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c},\ \boldsymbol{s} \succeq \boldsymbol{0}\right\}, \\
\mathcal{F}^\circ_{\text{P|SOCP}} & \triangleq & \{\boldsymbol{x} \in \mathbb{E}^n : A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \succ \boldsymbol{0}\}, \\
\mathcal{F}^\circ_{\text{D|SOCP}} & \triangleq & \left\{(\boldsymbol{y};\boldsymbol{s}) \in \mathbb{R}^m \times \mathbb{E}^n : A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c},\ \boldsymbol{s} \succ \boldsymbol{0}\right\}, \\
\mathcal{F}^\circ_{\text{SOCP}} & \triangleq & \mathcal{F}^\circ_{\text{P|SOCP}} \times \mathcal{F}^\circ_{\text{D|SOCP}}.
\end{array}
$$

We also make the following assumption about the primal-dual pair (P|SOCP) and (D|SOCP).

Assumption 11.2 *The set $\mathcal{F}^\circ_{SOCP}$ is nonempty.*

Assumption 11.2 requires that Problem (P|SOCP$_\mu$) and its dual (D|SOCP$_\mu$) have strictly feasible solutions, which guarantees strong duality for the second-order cone programming problem. Note that the feasible region for Problems (P|SOCP$_\mu$) and (D|SOCP$_\mu$) is described implicitly by $\mathcal{F}^\circ_{\text{SOCP}}$. Due to the coercivity of the function f_μ on the feasible set of (P|SOCP$_\mu$), Problem (P|SOCP$_\mu$) has an optimal solution.

The following lemma proves the convergence of the optimal solution of Problem (P|SOCP$_\mu$) to the optimal solution of Problem (P|SOCP) when μ approaches zero.

Lemma 11.7 *Let $\bar{x}_\mu$ be an optimal primal solution of Problem (P|SOCP$_\mu$), then $\bar{x} = \lim_{\mu\to 0}\bar{x}_\mu$ is an optimal solution of Problem (P|SOCP).*

Proof Let $f_\mu(x) \triangleq f(x,\mu)$ and $f(x) \triangleq f(x,0)$. Due to the coercivity of the function f_μ on the feasible set of (P|SOCP$_\mu$), Problem (P|SOCP$_\mu$) has an optimal solution, say $\bar{x}_\mu$, such that $\nabla_x f_\mu(\bar{x}_\mu) = \nabla_x f(\bar{x}_\mu,\mu) = \mathbf{0}$. Then, for all $x \in \mathcal{F}^\circ_{\text{P|SOCP}}$, we have that

$$\begin{aligned} f(x) &\geq f(\bar{x}_\mu,\mu) + (x-\bar{x}_\mu)^{\mathsf T}\nabla_x f(\bar{x}_\mu,\mu) + (0-\mu)\frac{\partial}{\partial\mu} f(\bar{x}_\mu,\mu) \\ &\geq f(\bar{x}_\mu,\mu) + \mu\ln\det\bar{x}_\mu - r\mu\ln\mu - r\mu \\ &\geq c^{\mathsf T}\bar{x}_\mu - \mu\ln\det\bar{x}_\mu + r\mu\ln\mu + \mu\ln\det\bar{x}_\mu - r\mu\ln\mu - r\mu \ \geq\ c^{\mathsf T}\bar{x}_\mu - r\mu. \end{aligned}$$

Since x was arbitrary in $\mathcal{F}^\circ_{\text{P|SOCP}}$, this implies that $\min_{x\in\mathcal{F}^\circ_{\text{P|SOCP}}} f(x) \geq c^{\mathsf T}\bar{x}_\mu - r\mu \geq c^{\mathsf T}\bar{x}_\mu = f(\bar{x}_\mu)$. On the other side, we have $f(\bar{x}_\mu) \geq \min_{x\in\mathcal{F}^\circ_{\text{P|SOCP}}} f(x)$. As μ goes to 0, it immediately follows that $f(\bar{x}) = \min_{x\in\mathcal{F}^\circ_{\text{P|SOCP}}} f(x)$. Thus, $\bar{x}$ is an optimal solution of Problem (P|SOCP). The proof is complete. ■

Newton's method and commutative directions

As we mentioned, the objective function of Problem (P|SOCP$_\mu$) is strictly convex, hence the KKT conditions are necessary and sufficient to characterize an optimal solution to Problem (P|SOCP$_\mu$). Consequently, the points $\bar{x}_\mu$ and $(\bar{y}_\mu, \bar{s}_\mu)$ are optimal solutions of (P|SOCP$_\mu$) and (D|SOCP$_\mu$) respectively if and only if they satisfy the perturbed nonlinear system

$$\begin{array}{rclrcl} Ax &=& b, & x &\succ& \mathbf{0}, \\ A^{\mathsf T}y + s &=& c, & s &\succ& \mathbf{0}, \\ x\circ s &=& \mu e, & \mu &>& 0, \end{array} \tag{11.10}$$

where $e \triangleq (e_{n_1}; e_{n_2}; \ldots; e_{n_r})$ is the identity vector of $\mathbb{E}^n_r$.

We call the set of all solutions of system (11.10), denoted by $(x_\mu; y_\mu; s_\mu)$ with $\mu > 0$, the central path. We say that a point (x, y, s) is near to the central path if it belongs to the set $\mathcal{N}_{\text{SOCP}}(\mu)$, which is defined as

$$\mathcal{N}_{\text{SOCP}}(\mu) \triangleq \left\{(x;y;s) \in \mathcal{F}^\circ_{\text{P|SOCP}} \times \mathcal{F}^\circ_{\text{D|SOCP}} : d_{\text{SOCP}}(x,s) \leq \theta\mu, \theta \in (0,1)\right\},$$

where

$$d_{\text{SOCP}}(x,s) \triangleq \|Q_{x^{1/2}}s - \mu e\|_F.$$

Now, we can apply Newton's method to system (11.11) and obtain the following linear system

$$\begin{aligned} A\Delta x &= \mathbf{0}, \\ A^{\mathsf T}\Delta y + \Delta s &= \mathbf{0}, \\ x\circ\Delta s + \Delta x\circ s &= \sigma\mu e - x\circ s. \end{aligned} \tag{11.11}$$

where $(\Delta x; \Delta y; \Delta s) \in \mathbb{E}_r^n \times \mathbb{R}^m \times \mathbb{E}_r^n$ is the search direction, $\mu = \frac{1}{r} x^{\mathsf{T}} s$ is the normalized duality gap corresponding to $(x; y; s)$, and $\sigma \in (0, 1)$ is the centering parameter.

Note that the strict second-order cone inequalities $x, s \succ 0$ imply that $d_{\text{SOCP}}(x, s) \leq \|x \circ s - \mu e\|_F$ with equality holds when x and s operator commute [Schmieta and Alizadeh, 2003, Lemma 30]. In fact, it is known that many interesting properties become apparent for the analysis of interior-point methods when x and s operator commute.

Denote by $C(x, s)$ the set of all elements so that the scaled elements operator commute. That is,

$$C(x, s) \triangleq \left\{ p \in \mathbb{E}_r^n : p^{-1} \text{ exists, and } Q_p x \,\&\, Q_{p^{-1}} s \text{ operator commute} \right\}. \tag{11.12}$$

From [Schmieta and Alizadeh, 2003, Lemma 28], the equality $x \circ s = \mu e$ holds if and only if the equality $(Q_p x) \circ (Q_{p^{-1}} s) = \mu e$ holds, for any nonsingular vector p in $\mathbb{E}_r^n$. Therefore, for any given nonsingular vector $p \in \mathbb{E}_r^n$, the system (11.10) is equivalent to the system

$$\begin{array}{rclcrcl} Ax & = & b, & & x & \succ & 0, \\ A^{\mathsf{T}} y + s & = & c, & & s & \succ & 0, \\ (Q_p x) \circ (Q_{p^{-1}} s) & = & \mu e, & & \mu & > & 0. \end{array} \tag{11.13}$$

Let $v \in \mathbb{E}_r^n$. With respect to a nonsingular vector $p \in \mathbb{E}_r^n$, we define the scaling vectors $\overline{v}$ and $\underline{v}$ and the scaling matrix $\underline{A}$ as

$$\overline{v} \triangleq Q_p v, \quad \underline{v} \triangleq Q_{p^{-1}} v, \quad \text{and} \quad \underline{A} \triangleq Q_p A. \tag{11.14}$$

The definitions in (11.14) are valid for both the single and multiple block cases.

Using this change of variables and the fact that $Q_p(\mathbb{E}_r^n) = \mathbb{E}_r^n$, we conclude that the system (11.11) is equivalent to the following Newton system

$$\begin{array}{rcl} \underline{A}\overline{\Delta x} & = & b - \underline{A}\overline{x}, \\ \underline{A}^{\mathsf{T}} \Delta y + \underline{\Delta s} & = & \underline{c} - \underline{s} - \underline{A}^{\mathsf{T}} y, \\ \overline{x} \circ \underline{\Delta s} + \overline{\Delta x} \circ \underline{s} & = & \sigma \mu e - \overline{x} \circ \underline{s}. \end{array} \tag{11.15}$$

Here, the normalized duality gap is $\mu = \frac{1}{r} \overline{x}^{\mathsf{T}} \underline{s} = \frac{1}{r} x^{\mathsf{T}} s$. In fact,

$$\overline{x}^{\mathsf{T}} \underline{s} = \left(Q_p x \right)^{\mathsf{T}} Q_{p^{-1}} s = x^{\mathsf{T}} Q_p Q_{p^{-1}} s = x^{\mathsf{T}} s. \tag{11.16}$$

Solving the scaled Newton system (11.15) yields the search direction $(\overline{\Delta x}; \Delta y; \underline{\Delta s})$. Then, we apply the inverse scaling to $(\overline{\Delta x}; \underline{\Delta s})$ to obtain the Newton direction $(\Delta x; \Delta s)$. Note that the search direction $(\overline{\Delta x}; \Delta y; \underline{\Delta s})$ belongs to the so-called the MZ family of directions (due Monteiro [1997], Zhang [1998]).

Clearly, the set $C(x, s)$ defined in (11.12) is a subclass of the MZ family of search directions. Our focus is on vectors $p \in C(x, s)$. We discuss the following three choices of p (see [Schmieta and Alizadeh, 2003, Section 3]):

- The first one is to choose $p = x^{-1/2}$, which gives $\overline{x} = e$.

- The second one is to choose $p = s^{1/2}$, which gives $\underline{s} = e$.

- The third choice of $\boldsymbol{p}$ is given by $\boldsymbol{p} = (\mathrm{Q}_{\boldsymbol{x}^{1/2}}(\mathrm{Q}_{\boldsymbol{x}^{1/2}}\boldsymbol{s})^{-1/2})^{-1/2}$, which yields $\mathrm{Q}_{\boldsymbol{p}}^2\boldsymbol{x} = \boldsymbol{s}$, and therefore $\underline{\boldsymbol{s}} = \mathrm{Q}_{\boldsymbol{p}^{-1}}\boldsymbol{s} = \mathrm{Q}_{\boldsymbol{p}}\boldsymbol{x} = \overline{\boldsymbol{x}}$.

The first two directional choices are respectively referred to as the HRVW/KSH/M direction (attributed to Helmberg et al. Helmberg et al. [1996], Monteiro [1997], Kojima et al. [1997]), and the dual HRVW/KSH/M direction. The third direction option is known as the NT direction (credited to Nesterov and Todd [1998]).

Path-following algorithm

The path-following algorithm for solving SOCP problem is formally stated in Algorithm 11.1.

Algorithm 11.1: Path-following algorithm for SOCP

Input: Data in Problems (P|SOCP) and (D|SOCP), $k = 0$,
$\left(\boldsymbol{x}^{(0)}; \boldsymbol{y}^{(0)}; \boldsymbol{s}^{(0)}\right) \in \mathcal{N}_{\mathrm{SOCP}}\left(\mu^{(0)}\right)$, $\epsilon > 0$, $\sigma^{(0)}, \theta \in (0, 1)$

Output: An ϵ-optimal solution to Problem (P|SOCP)

while ${\boldsymbol{x}^{(k)}}^{\mathsf{T}}\boldsymbol{s}^{(k)} \geq \epsilon$ **do**

  choose $\boldsymbol{p}^{(k)} \in C\left(\boldsymbol{x}^{(k)}, \boldsymbol{s}^{(k)}\right)$

  compute $\left(\overline{\boldsymbol{x}^{(k)}}; \boldsymbol{y}^{(k)}; \underline{\boldsymbol{s}^{(k)}}\right)$ by applying scaling to $\left(\boldsymbol{x}^{(k)}; \boldsymbol{y}^{(k)}; \boldsymbol{s}^{(k)}\right)$

  set $\mu^{(k)} \triangleq \frac{1}{r}\overline{\boldsymbol{x}^{(k)}}^{\mathsf{T}}\underline{\boldsymbol{s}^{(k)}}$, $\boldsymbol{h}^{(k)} \triangleq \sigma^{(k)}\mu^{(k)}\boldsymbol{e} - \overline{\boldsymbol{x}^{(k)}} \circ \underline{\boldsymbol{s}^{(k)}}$ and $\Psi^{(k)} \triangleq \frac{1}{\mu}\underline{A}\,\overline{\boldsymbol{x}^{(k)2}}\,\underline{A}^{\mathsf{T}}$

  compute $\left(\overline{\Delta\boldsymbol{x}^{(k)}}; \Delta\boldsymbol{y}^{(k)}; \underline{\Delta\boldsymbol{s}^{(k)}}\right)$ by solving the scaled system (11.15) to get
  $\left(\overline{\Delta\boldsymbol{x}^{(k)}}; \Delta\boldsymbol{y}^{(k)}; \underline{\Delta\boldsymbol{s}^{(k)}}\right) \triangleq \left(\left(\boldsymbol{h}^{(k)} - \overline{\boldsymbol{x}^{(k)}} \circ \underline{\Delta\boldsymbol{s}^{(k)}}\right) \circ \underline{\boldsymbol{s}^{(k)}}^{-1}; -{\Psi^{(k)}}^{-1}\underline{A}\left(\underline{\boldsymbol{s}^{(k)}}^{-1} \circ \boldsymbol{h}^{(k)}\right); -\underline{A}^{\mathsf{T}}\Delta\boldsymbol{y}\right)$

  compute $\left(\Delta\boldsymbol{x}^{(k)}; \Delta\boldsymbol{y}^{(k)}; \Delta\boldsymbol{s}^{(k)}\right)$ by applying inverse scaling to $\left(\overline{\Delta\boldsymbol{x}^{(k)}}; \Delta\boldsymbol{y}^{(k)}; \underline{\Delta\boldsymbol{s}^{(k)}}\right)$

  set the new iterate according to
  $\left(\boldsymbol{x}^{(k+1)};\ \boldsymbol{y}^{(k+1)};\ \boldsymbol{s}^{(k+1)}\right) \triangleq \left(\boldsymbol{x}^{(k)} + \alpha^{(k)}\Delta\boldsymbol{x}^{(k)};\ \boldsymbol{y}^{(k)} + \alpha^{(k)}\Delta\boldsymbol{y}^{(k)};\ \boldsymbol{s}^{(k)} + \alpha^{(k)}\Delta\boldsymbol{s}^{(k)}\right)$

  set $k = k + 1$

end

Algorithm 11.1 selects a sequence of displacement steps $\{\alpha^{(k)}\}$ and centrality parameters $\{\sigma^{(k)}\}$ according to the following rule: For all $k \geq 0$, we take $\sigma^{(k)} = 1 - \delta/\sqrt{r}$, where $\delta \in [0, \sqrt{r})$. The author in Alzalg [2018] discusses in Section 4 various selections for calculating the displacement step $\alpha^{(k)}$.

In the rest of this section, we prove that the complementary gap and the function f_μ decrease for a given displacement step. The proof of this result depends essentially on the following lemma.

Lemma 11.8 *Let $(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{s}) \in int\,\mathbb{R}^n_+ \times \mathbb{R}^m \times int\,\mathbb{E}^n_{r+}$, $(\overline{\boldsymbol{x}}; \boldsymbol{y}; \underline{\boldsymbol{s}})$ be obtained by applying scaling to $(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{s})$, and $\left(\overline{\Delta\boldsymbol{x}}; \Delta\boldsymbol{y}; \underline{\Delta\boldsymbol{s}}\right)$ be a solution of System (11.15). Then we have*

(a) $\overline{\Delta\boldsymbol{x}}^{\mathsf{T}}\underline{\Delta\boldsymbol{s}} = 0$.

(b) $\overline{\boldsymbol{x}}^{\mathsf{T}}\underline{\Delta\boldsymbol{s}} + \overline{\Delta\boldsymbol{x}}^{\mathsf{T}}\underline{\boldsymbol{s}} = trace(\boldsymbol{h})$, *where* $\boldsymbol{h} \triangleq \sigma\mu\boldsymbol{e} - \overline{\boldsymbol{x}} \circ \underline{\boldsymbol{s}}$ *such that* $\sigma \in (0, 1)$ *and* $\mu = \frac{1}{r}\overline{\boldsymbol{x}}^{\mathsf{T}}\underline{\boldsymbol{s}}$.

(c) $\overline{x}^{+\mathsf{T}}\underline{s}^{+} = (1 - \alpha(1 - \sigma/2))\, \overline{x}^{\mathsf{T}}\underline{s},\ \forall \alpha \in \mathbb{R}$, *where* $\overline{x}^{+} \triangleq \overline{x} + \alpha\overline{\Delta x}$ *and* $\underline{s}^{+} \triangleq \underline{s} + \alpha\underline{\Delta s}$.

(d) $x^{+\mathsf{T}}s^{+} = (1 - \alpha(1 - \sigma/2))\, x^{\mathsf{T}}s,\ \forall \alpha \in \mathbb{R}$, *where* $x^{+} \triangleq x + \alpha\Delta x$ *and* $s^{+} \triangleq s + \alpha\Delta s$.

Proof By the first two equations of System (11.15), we get

$$\overline{\Delta x}^{\mathsf{T}}\underline{\Delta s} = -\overline{\Delta x}^{\mathsf{T}}\underline{A}^{\mathsf{T}}\Delta y = -\left(\underline{A}\overline{\Delta x}\right)^{\mathsf{T}}\Delta y = 0.$$

This proves item (a). We prove item (b) by noting that

$$\begin{aligned}\operatorname{trace}(h) &= \operatorname{trace}\left(\sigma\mu e - \overline{x}\circ\underline{s}\right)\\ &= \operatorname{trace}\left(\overline{x}\circ\underline{\Delta s} + \overline{\Delta x}\circ\underline{s}\right)\\ &= \operatorname{trace}\left(\overline{x}\circ\underline{\Delta s}\right) + \operatorname{trace}\left(\overline{\Delta x}\circ\underline{s}\right) \;=\; \overline{x}^{\mathsf{T}}\underline{\Delta s} + \overline{\Delta x}^{\mathsf{T}}\underline{s},\end{aligned}$$

where we used the last equation of System (11.15) to obtain the first equality.

Item (c) is left as an exercise for the reader (see Exercise 11.4). Item (d) follows from item (c) and the fact that $\overline{x}^{\mathsf{T}}\underline{s} = x^{\mathsf{T}}s$ (see (11.16)), and similarly that $\overline{x}^{+\mathsf{T}}\underline{s}^{+} = x^{+\mathsf{T}}s^{+}$. The proof is complete. ■

The result of the following theorem is a special case of that in [Alzalg, 2018, Lemma 5.2].

Theorem 11.3 *Let* $(x; y; s)$ *and* $(x^{+}; y^{+}; s^{+})$ *be strictly feasible solutions of the pair of problems* (P|SOCP$_\mu$) *and* (D|SOCP$_\mu$) *with*

$$(x^{+}; y^{+}; s^{+}) = (x + \alpha\Delta x;\ y + \alpha\Delta y;\ s + \alpha\Delta s),$$

where α *is a displacement step and* $(\Delta x; \Delta y; \Delta s)$ *is the Newton direction. Then*

(a) $x^{+\mathsf{T}}s^{+} < \overline{x}^{\mathsf{T}}\underline{s}$. (b) $f_\mu(x^{+}) < f_\mu(x)$.

Proof Note that

$$x^{+\mathsf{T}}s^{+} = \left(1 - \alpha\left(1 - \frac{\sigma}{2}\right)\right)\overline{x}^{\mathsf{T}}\underline{s} < \overline{x}^{\mathsf{T}}\underline{s},$$

where the equality follows from item (d) of Lemma 11.8 and the strict inequality follows from $(1 - \alpha(1 - \sigma/2)) < 1$ (as $\alpha > 0$ and $\sigma \in (0, 1)$). This proves item (a).

To prove item (b), note that $f_\mu(x^{+}) \simeq f_\mu(x) + \nabla_x f_\mu(x)^{\mathsf{T}}(x^{+} - x)$, and therefore $f_\mu(x^{+}) - f_\mu(x) \simeq \alpha\nabla_x f_\mu(x)^{\mathsf{T}}\Delta x$. Since $\nabla_x f_\mu(x) = -\nabla^2_{xx} f_\mu(x)\Delta x$, we have that

$$f_\mu(x^{+}) - f_\mu(x) \simeq -\alpha\ \Delta x^{\mathsf{T}}\nabla^2_{xx} f_\mu(x)\Delta x < 0,$$

where the strict inequality follows from the positive definiteness of the Hessian matrix $\nabla^2_{xx} f_\mu(x)$ (as f_μ is strictly convex). Thus, $f_\mu(x^{+}) < f_\mu(x)$. The proof is complete. ■

Complexity estimates

In this part, we analyze the complexity of the proposed path-following algorithm for SOCP. More specifically, we prove that the iteration-complexity of Algorithm 11.1 is bounded by

$$O\left(\sqrt{r}\ln\left(\epsilon^{-1}\,\boldsymbol{x}^{(0)^{\mathsf{T}}}\boldsymbol{s}^{(0)}\right)\right).$$

Our proof depends essentially on the following two lemmas.

Lemma 11.9 *Let $(\boldsymbol{x};\boldsymbol{y};\boldsymbol{s}) \in \mathcal{F}^{\circ}_{P|SOCP}\times\mathcal{F}^{\circ}_{D|SOCP}$. Let also $(\overline{\boldsymbol{x}};\boldsymbol{y};\underline{\boldsymbol{s}})$ be obtained by applying scaling to $(\boldsymbol{x};\boldsymbol{y};\boldsymbol{s})$ with $\boldsymbol{h} = \sigma\mu\boldsymbol{e}-\overline{\boldsymbol{x}}\circ\underline{\boldsymbol{s}}$, and $(\overline{\Delta\boldsymbol{x}};\Delta\boldsymbol{y};\underline{\Delta\boldsymbol{s}})$ be a solution of System (11.15). For any $\alpha\in\mathbb{R}$, we set*

$$\begin{aligned}
&(\boldsymbol{x}(\alpha);\boldsymbol{y}(\alpha);\boldsymbol{s}(\alpha)) \triangleq (\overline{\boldsymbol{x}};\boldsymbol{y};\underline{\boldsymbol{s}}) + \alpha(\overline{\Delta\boldsymbol{x}};\Delta\boldsymbol{y};\underline{\Delta\boldsymbol{s}}),\\
&\mu(\alpha) \triangleq \tfrac{1}{r}\boldsymbol{x}(\alpha)^{\mathsf{T}}\boldsymbol{s}(\alpha),\\
&\boldsymbol{v}(\alpha) \triangleq \boldsymbol{x}(\alpha)\circ\boldsymbol{s}(\alpha) - \mu(\alpha)\boldsymbol{e}.
\end{aligned}$$

Then

$$\boldsymbol{v}(\alpha) = (1-\alpha)(\overline{\boldsymbol{x}}\circ\underline{\boldsymbol{s}}-\mu\boldsymbol{e}) + \alpha^2\,\overline{\Delta\boldsymbol{x}}\circ\underline{\Delta\boldsymbol{s}}. \tag{11.17}$$

Proof See Exercise 11.5. ■

Lemma 11.10 *Let $(\boldsymbol{x};\boldsymbol{y};\boldsymbol{s}) \in \mathcal{F}^{\circ}_{P|SOCP}\times\mathcal{F}^{\circ}_{D|SOCP}$, and $(\overline{\boldsymbol{x}};\boldsymbol{y};\underline{\boldsymbol{s}})$ be obtained by applying scaling to $(\boldsymbol{x};\boldsymbol{y};\boldsymbol{s})$ such that $\|\overline{\boldsymbol{x}}\circ\underline{\boldsymbol{s}}-\mu\boldsymbol{e}\|\le\theta\mu$, for some $\theta\in[0,1)$ and $\mu>0$. Let also $(\overline{\Delta\boldsymbol{x}};\Delta\boldsymbol{y};\underline{\Delta\boldsymbol{s}})$ be a solution of System (11.15), $\boldsymbol{h}=\sigma\mu\boldsymbol{e}-\overline{\boldsymbol{x}}\circ\underline{\boldsymbol{s}}$, $\delta_x \triangleq \mu\|\overline{\Delta\boldsymbol{x}}\circ\overline{\boldsymbol{x}}^{-1}\|_F$, $\delta_s \triangleq \|\overline{\boldsymbol{x}}\circ\underline{\Delta\boldsymbol{s}}\|_F$. Then, we have*

$$\delta_x\delta_s \le \frac{1}{2}\left(\delta_x^2+\delta_s^2\right) \le \frac{\|\boldsymbol{h}\|_F^2}{2(1-\theta)^2}. \tag{11.18}$$

Proof See Exercise 11.6. ■

The following theorem analyzes the behavior of one iteration of Algorithm 11.1. This theorem is a special case of [Alzalg, 2018, Theorem 6.1].

Theorem 11.4 *Let $\theta\in(0,1)$ and $\delta\in[0,\sqrt{r})$ be given such that*

$$\frac{\theta^2+\delta^2}{2(1-\theta)^2\left(1-\frac{\delta}{\sqrt{r}}\right)} \le \theta \le \frac{1}{2}. \tag{11.19}$$

Suppose that $(\overline{\boldsymbol{x}};\boldsymbol{y};\underline{\boldsymbol{s}}) \in \mathcal{N}_{SOCP}(\mu)$ and let $(\overline{\Delta\boldsymbol{x}};\Delta\boldsymbol{y};\underline{\Delta\boldsymbol{s}})$ denote the solution of system (11.15) with $\boldsymbol{h}=\sigma\mu\boldsymbol{e}-\overline{\boldsymbol{x}}\circ\underline{\boldsymbol{s}}$ and $\sigma = 1-\delta/\sqrt{r}$. Then, we have

(a) $\overline{\boldsymbol{x}}^{+^{\mathsf{T}}}\underline{\boldsymbol{s}}^{+} = (1-\delta/\sqrt{r})\overline{\boldsymbol{x}}^{\mathsf{T}}\underline{\boldsymbol{s}}$.

(b) $(\overline{\boldsymbol{x}}^{+};\boldsymbol{y}^{+};\underline{\boldsymbol{s}}^{+}) = (\overline{\boldsymbol{x}};\boldsymbol{y};\underline{\boldsymbol{s}}) + (\overline{\Delta\boldsymbol{x}};\Delta\boldsymbol{y};\underline{\Delta\boldsymbol{s}}) \in \mathcal{N}_{SOCP}(\mu)$.

(c) $(x^+; y^+; s^+) = (x; y; s) + (\Delta x; \Delta y; \Delta s) \in \mathcal{N}_{SOCP}(\mu)$.

Proof Item (a) follows directly from item (c) of Lemma 11.8 with $\alpha = 1$ and $\sigma = 1 - \delta/\sqrt{r}$. We now prove item (b). Define

$$\mu^+ \triangleq \frac{1}{r}\,\overline{x}^{+\mathsf{T}}\underline{s}^+ = \left(1 - \frac{\delta}{r}\right)\mu \tag{11.20}$$

and let $(\overline{x}; y; \underline{s}) \in \mathcal{N}_{\mathrm{SOCP}}(\mu)$, we then have

$$\begin{aligned} \left\|\sigma\mu e - \overline{x} \circ \underline{s}\right\|_F^2 &\le \left\|(\sigma - 1)\mu e\right\|_F^2 + \left\|\mu e - \overline{x} \circ \underline{s}\right\|_F^2 \\ &\le \left((\sigma - 1)^2 r + \theta^2\right)\mu^2 = \left(\delta^2 + \theta^2\right)\mu^2. \end{aligned} \tag{11.21}$$

Since $\|\overline{x} \circ \underline{s} - \mu e\| \le \theta\mu$ and $h = \sigma\mu e - \overline{x} \circ \underline{s}$, using Lemma 11.10 it follows that

$$\left\|\overline{\Delta x} \circ \overline{x}^{-1}\right\|_F \left\|\overline{x} \circ \underline{\Delta s}\right\|_F \le \frac{\left\|\sigma\mu e - \overline{x} \circ \underline{s}\right\|_F^2}{2(1 - \theta)^2\mu}. \tag{11.22}$$

Defining $v^+ \triangleq v(1) = \overline{x}^+ \circ \underline{s}^+ - \mu^+ e$ and using (11.17) with $\alpha = 1$, (11.22), (11.21), (11.19) and (11.20), we get

$$\begin{aligned} \|v^+\|_F &= \left\|\overline{\Delta x} \circ \underline{\Delta s}\right\|_F \\ &\le \left\|\overline{\Delta x} \circ \overline{x}^{-1}\right\|_F \left\|\overline{x} \circ \underline{\Delta s}\right\|_F \\ &\le \frac{\left\|\sigma\mu e - \overline{x} \circ \underline{s}\right\|_F^2}{2(1 - \theta)^2\mu} \\ &\le \frac{(\delta^2 + \theta^2)\mu}{2(1 - \theta)^2} \\ &\le \theta\left(1 - \tfrac{\delta}{\sqrt{r}}\right)\mu = \theta\mu^+. \end{aligned}$$

Consequently,

$$\left\|\overline{x}^+ \circ \underline{s}^+ - \mu^+ e\right\|_F \le \theta\mu^+. \tag{11.23}$$

By using the right-hand side inequality in (11.18), and using (11.21) and (11.19), we have

$$\left\|\overline{\Delta x} \circ \overline{x}^{-1}\right\|_F \le \frac{\left\|\sigma\mu e - \overline{x} \circ \underline{s}\right\|_F}{(1 - \theta)\mu} \le \frac{\sqrt{\delta^2 + \theta^2}}{(1 - \theta)} \le \sqrt{2\theta\left(1 - \frac{\delta}{\sqrt{r}}\right)} < 1,$$

where the strict inequality follows from $\theta \le \frac{1}{2}$ and $0 < 1 - \delta/\sqrt{r} < 1$.

One can easily see that $\|\overline{\Delta x} \circ \overline{x}^{-1}\|_F < 1$ implies that $e + \overline{\Delta x} \circ \overline{x}^{-1} \succ \mathbf{0}$, and therefore

$$\overline{x}^+ = \overline{\Delta x} + \overline{x} = \left(e + \overline{\Delta x} \circ \overline{x}^{-1}\right) \circ \overline{x} \succ \mathbf{0}.$$

Note that, from (11.23), we have $\lambda_{\min}(\overline{x}^+ \circ \underline{s}^+) \ge (1 - \theta)\mu^+ > 0$, and therefore $\overline{x}^+ \circ \underline{s}^+ \succ \mathbf{0}$. Since $\overline{x}^+ \succ \mathbf{0}$ and $\overline{x}^+$ and $\underline{s}^+$ operator commute, we conclude that $\underline{s}^+ \succ \mathbf{0}$. Using the first equation of system (11.15), we get

$$\underline{A}\overline{x}^+ = \underline{A}\left(\overline{x} + \overline{\Delta x}\right) = \underline{A}\overline{x} + \underline{A}\overline{\Delta x} = b, \quad \text{and hence } \overline{x}^+ \in \mathcal{F}^{\circ}_{\mathrm{PISOCP}}.$$

By using the second equation of system (11.15), we get

$$\underline{A}^{\mathsf{T}} y^{+} + \bar{s}^{+} = \underline{A}^{\mathsf{T}}(y + \Delta y) + \left(\bar{s} + \overline{\Delta s}\right) = \underline{A}^{\mathsf{T}} y + \bar{s} + \underline{A}^{\mathsf{T}} \Delta y + \overline{\Delta s} = c,$$

and hence $(y^{+}; \bar{s}^{+}) \in \mathcal{F}^{\circ}_{\text{DISOCP}}$.

Thus, in view of (11.23), we deduce that $(\bar{x}^{+}; y^{+}; \bar{s}^{+}) \in \mathcal{N}_{\text{SOCP}}(\mu)$. Item (b) is therefore established. Item (c) follows from item (b) and [Schmieta and Alizadeh, 2003, Proposition 29]. The proof is now complete. ■

We finally have the following corollary, which states that Algorithm 11.1 can find an iterate meeting a given tolerance within a limited number of iterations, determined by a logarithmic function.

Corollary 11.2 *Let θ and δ as given in Theorem 11.4 and $(x^0; y^0; s^{(0)}) \in \mathcal{N}_{SOCP}(\mu)$. Then Algorithm 11.1 generates a sequence of points $\{(x^{(k)}; y^{(k)}; s^{(k)})\} \subset \mathcal{N}_{SOCP}(\mu)$ such that*

$$x^{(k)^{\mathsf{T}}} s^{(k)} = \left(1 - \frac{\delta}{\sqrt{r}}\right)^{k} x^{(0)^{\mathsf{T}}} s^{(0)}, \quad \forall k \geq 0.$$

Moreover, given a tolerance $\epsilon > 0$, Algorithm 11.1 computes an iterate $\{(x^{(k)}; y^{(k)}; s^{(k)})\}$ satisfying $x^{(k)^{\mathsf{T}}} s^{(k)} \leq \epsilon$ in at most

$$O\left(\sqrt{r} \ln\left(\frac{x^{(0)^{\mathsf{T}}} s^{(0)}}{\epsilon}\right)\right)$$

iterations.

Proof Looking recursively at item (a) of Theorem 11.4, for each k we have that

$$x^{(k)^{\mathsf{T}}} s^{(k)} = \left(1 - \frac{\delta}{\sqrt{r}}\right)^{k} x^{(0)^{\mathsf{T}}} s^{(0)} \leq \epsilon.$$

By taking natural logarithm of both sides, we get

$$k \ln\left(1 - \frac{\delta}{\sqrt{r}}\right) \leq \ln\left(\frac{\epsilon}{x^{(0)^{\mathsf{T}}} s^{(0)}}\right),$$

which holds only if the inequality $k(-\delta/\sqrt{r}) \leq \ln(\epsilon / x^{(0)^{\mathsf{T}}} s^{(0)})$ holds, or equivalently, $k \geq \delta^{-1} \sqrt{r} \ln(x^{(0)^{\mathsf{T}}} s^{(0)} / \epsilon)$. The proof is complete. ■

11.6 A homogeneous self-dual algorithm

In this section, we present a homogeneous self-dual algorithm for SOCP. The material of this section has appeared in [Alzalg, 2014a, Section 3]. The algorithm presented in this section for SOCP generalizes the one proposed in Section 10.7 for linear programming.

The following primal-dual SOCP model provides sufficient conditions (but not always necessary) for an optimal solution of (P|SOCP) and (D|SOCP).

$$\begin{array}{rcl} A\boldsymbol{x} & = & \boldsymbol{b}, \\ A^{\mathsf{T}}\boldsymbol{y}+\boldsymbol{s} & = & \boldsymbol{c}, \\ \boldsymbol{x}^{\mathsf{T}}\boldsymbol{s} & = & 0, \\ \boldsymbol{x},\boldsymbol{s} & \succeq & \boldsymbol{0}. \end{array} \tag{11.24}$$

The homogeneous SOCP model for the pair (P|SOCP) and (D|SOCP) is as follows:

$$\begin{array}{rrrrrcl} A\boldsymbol{x} & & & -\boldsymbol{b}\tau & & = & \boldsymbol{0}, \\ & -A^{\mathsf{T}}\boldsymbol{y} & -\boldsymbol{s} & +\boldsymbol{c}\tau & & = & \boldsymbol{0}, \\ -\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & +\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & & -\kappa & = & 0, \\ \boldsymbol{x} & & & & & \succeq & \boldsymbol{0}, \\ & & \boldsymbol{s} & & & \succeq & \boldsymbol{0}, \\ & & & \tau & & \geq & 0, \\ & & & & \kappa & \geq & 0. \end{array} \tag{11.25}$$

The first two equations in (11.25), when $\tau = 1$, establish conditions for primal and dual feasibility (with $\boldsymbol{x},\boldsymbol{s} \succeq \boldsymbol{0}$) and exhibit reversed weak duality. As a result, these equations, when combined with the third equation after setting $\kappa = 0$, define optimal solutions for both the primal and dual problems. It is worth noting that when we homogenize τ (i.e., make it a variable), introduce the necessary dual variable into the third equation, incorporate the artificial variable κ to ensure feasibility, and include the third equation from (11.25), we achieve self-duality.

It can be easily shown that $\boldsymbol{x}^{\mathsf{T}}\boldsymbol{s} + \tau\kappa = 0$. The following theorem relates (11.24) to (11.25), and it is easily proved.

Theorem 11.5 *The primal-dual SOCP model (11.24) has a solution if and only if the homogeneous SOCP model (11.25) has a solution*

$$(\boldsymbol{x}^\star;\boldsymbol{y}^\star;\boldsymbol{s}^\star;\tau^\star;\kappa^\star) \in \mathbb{E}^n_{r+} \times \mathbb{R}^m \times \mathbb{E}^n_{r+} \times \mathbb{R}_+ \times \mathbb{R}_+$$

such that $\tau^\star > 0$ *and* $\kappa^\star = 0$.

At each iteration of the homogeneous interior-point algorithm designed for solving (P|SOCP) and (D|SOCP), a key step involves computing the search direction $(\Delta\boldsymbol{x};\Delta\boldsymbol{y};\Delta\boldsymbol{s})$. This direction is determined from the symmetrized Newton equations with respect to an invertible vector $\boldsymbol{p}$, which is selected as a function of $(\boldsymbol{x};\boldsymbol{y};\boldsymbol{s})$. These equations are defined by the system:

$$\begin{array}{ccccccccccc} A\,\Delta\boldsymbol{x} & & & & -\boldsymbol{b}\,\Delta\tau & & & = & \eta\boldsymbol{r}_p, \\ & - & A^{\mathsf{T}}\,\Delta\boldsymbol{y} & - \;\Delta\boldsymbol{s} & +\boldsymbol{c}\,\Delta\tau & & & = & \eta\boldsymbol{r}_d, \\ -\boldsymbol{c}^{\mathsf{T}}\,\Delta\boldsymbol{x} & + & \boldsymbol{b}^{\mathsf{T}}\,\Delta\boldsymbol{y} & & & - & \Delta\kappa & = & \eta r_g, \\ & & & & \kappa\,\Delta\tau & + & \tau\,\Delta\kappa & = & \gamma\mu-\tau\kappa, \\ (Q_{\boldsymbol{p}}\Delta\boldsymbol{x}) & \circ & (Q_{\boldsymbol{p}^{-1}}\boldsymbol{s}) & + & (Q_{\boldsymbol{p}}\boldsymbol{x}) & \circ & (Q_{\boldsymbol{p}^{-1}}\Delta\boldsymbol{s}) & = & \gamma\mu\boldsymbol{e}-(Q_{\boldsymbol{p}}\boldsymbol{x})\circ(Q_{\boldsymbol{p}^{-1}}\boldsymbol{s}), \end{array} \tag{11.26}$$

where

$$\begin{array}{rclrcl} \boldsymbol{r}_p & \triangleq & \boldsymbol{b}\tau - A\boldsymbol{x}, & \boldsymbol{r}_d & \triangleq & A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} - \tau\boldsymbol{c}, \\ r_g & \triangleq & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} + \kappa, & \mu & \triangleq & \frac{1}{2r+1}(\boldsymbol{x}^{\mathsf{T}}\boldsymbol{s} + \tau\kappa), \end{array}$$

and η and γ are two parameters.

The vectors $\boldsymbol{x}$ and $\boldsymbol{s}$ do not necessarily operator commute. However, as we discussed in the preceding section, it is often advantageous to analyze interior-point methods under the assumption that $\boldsymbol{x}$ and $\boldsymbol{s}$ operator commute. To facilitate this analysis, we will now introduce a scaling transformation to the primal-dual pair (P|SOCP) and (D|SOCP). This scaling will ensure that the scaled decision variables in the resulting pair of problems, which should remain equivalent to the original (P|SOCP) and (D|SOCP), indeed operator commute. To achieve this, we will employ an effective scaling method demonstrated in (11.14). With this change of variables, the transformed pair (P|SOCP) and (D|SOCP) becomes:

$$\begin{array}{llll} & \min & \underline{\boldsymbol{c}}^{\mathsf{T}}\overline{\boldsymbol{x}} \\ (\overline{\text{P|SOCP}}) & \text{s.t.} & \underline{A}\overline{\boldsymbol{x}} = \boldsymbol{b}, \\ & & \overline{\boldsymbol{x}} \succeq \boldsymbol{0}; \end{array} \qquad \begin{array}{llll} & \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} \\ (\underline{\text{D|SOCP}}) & \text{s.t.} & \underline{A}^{\mathsf{T}}\boldsymbol{y} + \underline{\boldsymbol{s}} = \underline{\boldsymbol{c}}, \\ & & \underline{\boldsymbol{s}} \succeq \boldsymbol{0}. \end{array}$$

Note that $\boldsymbol{x}^{\mathsf{T}}\boldsymbol{s} = \overline{\boldsymbol{x}}^{\mathsf{T}}\underline{\boldsymbol{s}}$ (see (11.15)). From [Alizadeh and Goldfarb, 2003, Lemma 31] it can be seen that the search direction $(\Delta\boldsymbol{x}; \Delta\boldsymbol{y}; \Delta\boldsymbol{s})$ solves System (11.26) if and only if $(\overline{\Delta\boldsymbol{x}}; \Delta\boldsymbol{y}; \underline{\Delta\boldsymbol{s}})$ solves the system:

$$\begin{array}{rcrcrcrcrcl} \underline{A}\,\overline{\Delta\boldsymbol{x}} & & & & & & -\boldsymbol{b}\,\Delta\tau & & & = & \eta\hat{\boldsymbol{r}}_p, \\ & - & \underline{A}^{\mathsf{T}}\,\Delta\boldsymbol{y} & - & \underline{\Delta\boldsymbol{s}} & & +\underline{\boldsymbol{c}}\,\Delta\tau & & & = & \eta\hat{\boldsymbol{r}}_d, \\ -\underline{\boldsymbol{c}}^{\mathsf{T}}\,\overline{\Delta\boldsymbol{x}} & + & \boldsymbol{b}^{\mathsf{T}}\,\Delta\boldsymbol{y} & & & & & - & \Delta\kappa & = & \eta\hat{r}_g, \\ & & & & & & \kappa\,\Delta\tau & + & \tau\,\Delta\kappa & = & \gamma\mu - \tau\kappa, \\ \overline{\Delta\boldsymbol{x}} & \circ & \underline{\boldsymbol{s}} & + & & & \overline{\boldsymbol{x}} & \circ & \underline{\Delta\boldsymbol{s}} & = & \gamma\mu\boldsymbol{e} - \overline{\boldsymbol{x}} \circ \underline{\boldsymbol{s}}, \end{array} \tag{11.27}$$

where

$$\begin{array}{rclrcl} \hat{\boldsymbol{r}}_p & \triangleq & \boldsymbol{b}\tau - \underline{A}\overline{\boldsymbol{x}}, & \hat{\boldsymbol{r}}_d & \triangleq & \underline{A}^{\mathsf{T}}\boldsymbol{y} + \underline{\boldsymbol{s}} - \tau\underline{\boldsymbol{c}}, \\ \hat{r}_g & \triangleq & \underline{\boldsymbol{c}}^{\mathsf{T}}\overline{\boldsymbol{x}} - \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} + \kappa, & \mu & \triangleq & \frac{1}{2r+1}(\overline{\boldsymbol{x}}^{\mathsf{T}}\underline{\boldsymbol{s}} + \tau\kappa). \end{array}$$

For each choice of $\boldsymbol{p}$, we get a different search direction. The three choices of $\boldsymbol{p}$ that we discussed in the preceding section are the most common in practice. We emphasize that $(\overline{\Delta\boldsymbol{x}}; \Delta\boldsymbol{y}; \underline{\Delta\boldsymbol{s}})$ is the result of applying Newton's method to the primal and dual feasibility and complementarity relations arising from the scaled problems $(\overline{\text{P|SOCP}})$ and $(\underline{\text{D|SOCP}})$. It depends on the choice of $\boldsymbol{p}$, while $(\Delta\boldsymbol{x}; \Delta\boldsymbol{y}; \Delta\boldsymbol{s})$ results as a special case when $\boldsymbol{p} = \boldsymbol{e}$.

We state the generic homogeneous algorithm for solving the pair (P|SOCP) and (D|SOCP) in Algorithm 11.2.

The following theorem, which is known to hold, gives the computational complexity (worst behavior) of Algorithm 11.2 in terms of the rank of the underlying second-order cone.

Theorem 11.6 *Let $\epsilon_0 > 0$ be the residual error at a starting point, and $\epsilon > 0$ be a given tolerance. Under Assumptions 11.2 and 11.1, if the pair (P|SOCP) and (D|SOCP) has a solution $(\boldsymbol{x}^\star; \boldsymbol{y}^\star; \boldsymbol{s}^\star)$, then Algorithm 11.2 finds an ϵ-approximate solution (i.e., a* *solution with residual error less than or equal to ϵ) in at most*

$$O\left(\sqrt{2r}\,\ln\left(trace\,(\boldsymbol{x}^\star + \boldsymbol{s}^\star)\left(\frac{\epsilon_0}{\epsilon}\right)\right)\right)$$

iterations.

We point out that the result in Theorem 11.6 for SOCP is the counterpart of that in Theorem 10.12 for linear programming.

Algorithm 11.2: Generic homogeneous self-dual algorithm for SOCP

Input: Data in Problems (P|SOCP) and (D|SOCP) $(\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{s}; \tau; \kappa) \triangleq (\boldsymbol{e}; \boldsymbol{0}; \boldsymbol{e}; 1; 1)$
Output: An approximate optimal solution to Problem (P|SOCP)

```
choose a scaling element p and compute (x̄, s̲)
while a stopping criterion is not satisfied do
    choose η, γ
    compute the solution (Δx̄; Δy; Δs̲; Δτ; Δκ) of the linear system (11.27)
    compute (Δx; Δs) by applying inverse scaling to (Δx̄; Δs̲)
    compute a step length θ so that
    x + θΔx ≻ 0
    s + θΔs ≻ 0
    τ + θΔτ > 0
    κ + θΔκ > 0
    set the new iterate according to
    (x; y; s; τ; κ) ≜ (x; y; s; τ; κ) + θ(Δx; Δy; Δs; Δτ; Δκ)
end
```

In summary, second-order cone programming stands as a powerful mathematical framework, adept at solving a wide array of optimization problems, particularly those involving complex constraints and non-linear objectives, offering an efficient tool in diverse fields such as engineering, finance, and machine learning.

Exercises

11.1 Prove Lemma 11.2.

11.2 The pth-order cone of dimension n is defined as

$$\mathcal{P}_p^n \triangleq \left\{\boldsymbol{x} \in \mathbb{E}^n : x_0 \geq \|\widetilde{\boldsymbol{x}}\|_p\right\},$$

where $\|\cdot\|_{p\geq 1}$, is the p-norm of $\boldsymbol{\xi}$:

$$\|\boldsymbol{\xi}\|_p := \left(\sum_{i=1}^{m} |\xi_i|^p\right)^{1/p}, \text{ for } \boldsymbol{\xi} \in \mathbb{R}^m.$$

When $p = 2$, the pth-order cone reduces to the second-order cone, i.e., $\mathbb{E}^n_+ = \mathcal{P}^n_2$. In the proof of Lemma 11.3, we used the Cauchy-Schwartz inequality to show that the second-order cone is self-dual (i.e., $\mathcal{P}^{n\star}_2 = \mathcal{P}^n_2$). More generally, use the Hölder's inequality to show that $\mathcal{P}^{n\star}_p = \mathcal{P}^n_q$ for $p \in [1, \infty]$, where q is the conjugate of p.

11.3 The nth-dimensional elliptic cone is defined as

$$\mathcal{K}^n_M \triangleq \left\{ \boldsymbol{x} = (x_0; \widetilde{\boldsymbol{x}}) \in \mathbb{R} \times \mathbb{R}^{n-1} : \ x_0 \geq \left\| M\widetilde{\boldsymbol{x}} \right\| \right\}, \tag{11.28}$$

where M be a nonsingular matrix of order $n-1$. Clearly, when $M = I_{n-1}$, the elliptic cone reduces to the second-order cone, i.e., $\mathbb{E}^n_+ = \mathcal{K}^n_{I_{n-1}}$. In the proof of Lemma 11.3, we showed that the second-order cone is self-dual (i.e., $(\mathcal{K}^n_{I_{n-1}})^\star = \mathcal{K}^n_{I_{n-1}}$). More generally, show that

$$(\mathcal{K}^n_M)^\star = \mathcal{K}^n_{(M^{-1})^\mathsf{T}}.$$

11.4 Prove item (c) in Lemma 11.8.

11.5 Prove Lemma 11.9.

11.6 Prove Lemma 11.10.

11.7 Implement Algorithm 11.1 and test it on the class of instances of the pair (P|SOCP) and (D|SOCP) with $n = 2m$ and

$$\begin{array}{rcl} \boldsymbol{c} & \triangleq & 10\boldsymbol{e} - 2\,\mathbf{1} + 4\,\textbf{rand}(n, 1) \in \mathbb{E}^n, \\ \boldsymbol{b} & \triangleq & 10\boldsymbol{e} - 2\,\mathbf{1} + 4\,\textbf{rand}(m, 1) \in \mathbb{R}^m, \\ A & \triangleq & \left[\hat{A} \,\vdots\, \text{Randn}(m, n-m) \right] \in \mathbb{R}^{m \times n}, \end{array}$$

where, for $1 \leq i, j \leq m$,

$$\hat{a}_{ij} \triangleq \begin{cases} 2, & \text{if } \ i = j - 1, \\ 100, & \text{if } \ i = j, \\ -2, & \text{if } \ i = j + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\mathbf{1}$ is a vector of ones with with an appropriate dimension, and $\textbf{rand}(\cdot, 1)$ (respectively, $\text{Randn}(\cdot, \cdot)$) is a random vector (respectively, matrix) with the indicated dimension (respectively, size). Take $\boldsymbol{x}^{(0)} = \boldsymbol{e} \in \mathbb{E}^n$ and $\boldsymbol{y}^{(0)} = \mathbf{0} \in \mathbb{R}^m$ as your initial strictly feasible points. You may also take $\epsilon = 10^{-6}, \sigma = 0.1$ and $\rho = 0.99$.

11.8 In this and the following exercises, we practice writing the homogeneous model of an SOCP problem and the corresponding search direction system. The underlying problems are SOCPs, in primal and dual standard forms, with block diagonal structures. Let $r \geq 1$ be an integer. For $i = 1, 2, \ldots, r$ and $k = 0, 1, \ldots, K$, let m, n_k, n_{ki} be positive integers such that $n_k = \sum_{i=1}^r n_{ik}$.

Let the data

$$\begin{array}{rcl} W_0 & \triangleq & (W_{01}, W_{02}, \dots, W_{0r}), \text{ where } W_{0i} \in \mathbb{R}^{m_0 \times n_{0i}}; \\ \boldsymbol{c}_0 & \triangleq & (\boldsymbol{c}_{01}; \boldsymbol{c}_{02}; \dots; \boldsymbol{c}_{0r}), \text{ where } \boldsymbol{c}_{0i} \in \mathbb{E}^{n_{0i}}; \\ B_k & \triangleq & (B_{k1}, B_{k2}, \dots, B_{kr}), \text{ where } B_{ki} \in \mathbb{R}^{m_1 \times n_{ki}}; \\ W_k & \triangleq & (W_{k1}, W_{k2}, \dots, W_{kr}), \text{ where } W_{ki} \in \mathbb{R}^{m_1 \times n_{ki}}; \\ \boldsymbol{c}_k & \triangleq & (\boldsymbol{c}_{k1}; \boldsymbol{c}_{k2}; \dots; \boldsymbol{c}_{kr}), \text{ where } \boldsymbol{c}_{ki} \in \mathbb{E}^{n_{ki}}, \end{array}$$

for $k = 1, 2, \dots, K$ be given. Consider the SOCP problem:

$$\begin{array}{lccccccccc} \min & \boldsymbol{c}_0^\mathsf{T}\boldsymbol{x}_0 & + & \boldsymbol{c}_1^\mathsf{T}\boldsymbol{x}_1 & + & \cdots & + & \boldsymbol{c}_K^\mathsf{T}\boldsymbol{x}_K & & \\ \text{s.t.} & W_0\boldsymbol{x}_0 & & & & & & & = & \boldsymbol{h}_0, \\ & B_1\boldsymbol{x}_0 & + & W_1\boldsymbol{x}_1 & & & & & = & \boldsymbol{h}_1, \\ & \vdots & & & & \ddots & & & & \vdots \\ & B_K\boldsymbol{x}_0 & + & & & & & W_K\boldsymbol{x}_K & = & \boldsymbol{h}_K, \\ & \boldsymbol{x}_0, & & \boldsymbol{x}_1, & & \dots, & & \boldsymbol{x}_K & \succeq & \boldsymbol{0}, \end{array} \tag{11.29}$$

where

$$\begin{array}{rcl} \boldsymbol{x}_0 & \triangleq & (\boldsymbol{x}_{01}; \boldsymbol{x}_{02}; \dots; \boldsymbol{x}_{0r}), \text{ where } \boldsymbol{x}_{0i} \in \mathbb{E}^{n_{0i}}; \\ \boldsymbol{x}_k & \triangleq & (\boldsymbol{x}_{k1}; \boldsymbol{x}_{k2}; \dots; \boldsymbol{x}_{kr}), \text{ where } \boldsymbol{x}_{ki} \in \mathbb{E}^{n_{ki}}, \end{array}$$

for $k = 1, 2, \dots, K$, are the primal decision variables, and $\boldsymbol{h}_0 \in \mathbb{R}^{m_0}, \boldsymbol{h}_k \in \mathbb{R}^{m_1}, k = 1, 2, \dots, K$, are right-hand side vectors. The dual of (11.29) is the problem:

$$\begin{array}{lccccccccccc} \max & \boldsymbol{h}_0^\mathsf{T}\boldsymbol{y}_0 & + & \boldsymbol{h}_1^\mathsf{T}\boldsymbol{y}_1 & + & \cdots & + & \boldsymbol{h}_K^\mathsf{T}\boldsymbol{y}_K & & & & \\ \text{s.t.} & W_0^\mathsf{T}\boldsymbol{y}_0 & + & B_1^\mathsf{T}\boldsymbol{y}_1 & + & \cdots & + & B_K^\mathsf{T}\boldsymbol{y}_K & + & \boldsymbol{s}_0 & = & \boldsymbol{c}_0, \\ & & & W_1^\mathsf{T}\boldsymbol{y}_1 & & & & & + & \boldsymbol{s}_1 & = & \boldsymbol{c}_1, \\ & & & & & \ddots & & & & & & \vdots \\ & & & & & & & W_K^\mathsf{T}\boldsymbol{y}_K & + & \boldsymbol{s}_K & = & \boldsymbol{c}_K, \\ & \boldsymbol{s}_0, & & \boldsymbol{s}_1, & & \dots, & & \boldsymbol{s}_K & & & \succeq & \boldsymbol{0}, \end{array} \tag{11.30}$$

where

$$\begin{array}{rcl} \boldsymbol{y} & \triangleq & (\boldsymbol{y}_0; \boldsymbol{y}_1; \dots; \boldsymbol{y}_K) \in \mathbb{R}^{m_0 + K m_1}; \\ \boldsymbol{s}_k & \triangleq & (\boldsymbol{s}_{k1}; \boldsymbol{s}_{k2}; \dots; \boldsymbol{s}_{kr}), \text{ where } \boldsymbol{s}_{ki} \in \mathbb{E}^{n_{ki}}, \end{array}$$

for $k = 0, 1, \dots, K$, are the dual decision variables. Write the homogeneous model for the pair (11.29) and (11.30).

11.9 Write the search direction system corresponding to the homogeneous model obtained in Exercise 11.8.

Notes and sources

The roots of conic optimization can be traced to the work of mathematicians like Michael Todd, Yinyu Ye, and Stephen Boyd in the 1980s and 1990s (refer to Nesterov and Todd [1998], Boyd et al. [2004]). In particular, second-order cone programming is a relatively recent development in the field of mathematical optimization, with its origins in the late 20th century. The foundations of second-order cone programming can be traced back to the early 1990s, with significant contributions from researchers such as Yurii Nesterov and Arkadi Nemirovski (refer to Nesterov and Nemirovskii [1994]). Second-order cone programming was introduced as an extension of linear and quadratic programming, offering a more versatile framework to solve complex optimization problems. It has found applications in various fields, including engineering, finance, and machine learning, due to its ability to handle a wide range of convex optimization problems efficiently.

In this chapter, we explored second-order cone programming problems and their associated interior-point methods, providing a comprehensive understanding of this important class of optimization problems. After studying the algebraic structure of the second-order cone, we delved into the theory and applications of second-order cone programming. Additionally, we illuminated the interior-point methods that were developed for solving second-order cone programming problems efficiently.

As we conclude this chapter, it is worth noting that the cited references and others, such as Ben-Tal and Nemirovski [2001], Renegar [2001], Davidsson [2013], Aliprantis and Tourky [2007], Jacobson [1968], Koecher et al. [1999], also serve as valuable sources of information pertaining to the subject matter covered in this chapter. The code that created the picture of the second-order cone in Section 11.2 was taken from the source file of Alzalg and Alioui [2022]. Exercise 11.2 is due to Alzalg [2011a]. Exercise 11.3 is due to Alzalg and Pirhaji [2017b]. Exercise 11.7 is due to Alzalg [2018]. Exercises 11.8 and 11.9 are due to Alzalg [2014a].

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CHAPTER 12

SEMIDEFINITE PROGRAMMING AND COMBINATORIAL OPTIMIZATION

Chapter overview: In semidefinite programming (SDP) problems, the variable is a symmetric matrix that is required to be positive semidefinite. Within this chapter, readers delve into the world of SDP, study some combinatorial applications of SDP, and explore the concept of SDP duality, shedding light on the intriguing interplay between primal and dual formulations. Furthermore, readers study efficient primal-dual methods developed for solving SDP problems, providing readers with valuable tools to address complex optimization challenges across different disciplines. The chapter concludes with a set of exercises to encourage readers to apply their knowledge and enrich their understanding.

Keywords: Semidefinite programming, SDP duality, Primal-dual methods

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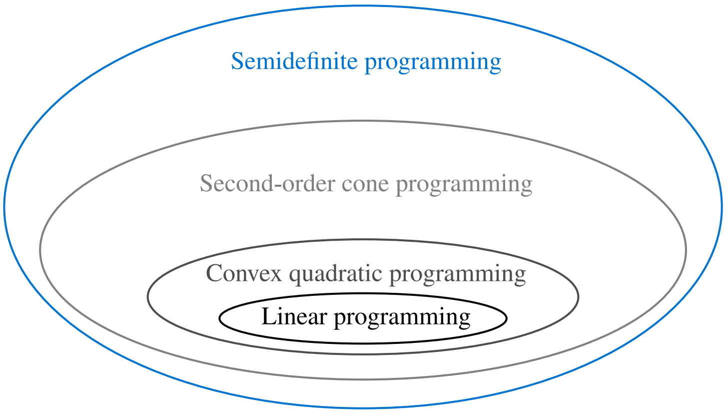

Figure 12.1: A Venn diagram of different classes of optimization problems.

In Chapters 10 and 11, we have studied linear programming and second-order cone programming, respectively. In modern convex optimization, the class of optimization that is an immediate enlargement of second-order cone optimization is the so-called semidefinite optimization. Figure 12.1 shows graphical relationships among different classes of optimization problems. Semidefinite programming (SDP for short) problems, which include linear programming problems and second-order cone programming problems as special cases, are a class of convex optimization problems in which the variable is not a vector that is required to be nonnegative, but rather a symmetric matrix that is required to be positive semidefinite (see Definition 3.3). This chapter delves into the SDP problems. We also refer to Todd [2001] for an excellent survey paper on this topic.

12.1 The cone of positive semidefinite matrices

Recall that a square matrix is positive semidefinite (respectively, positive definite) if it is symmetric and all its eigenvalues are nonnegative (respectively, positive). As an alternative to the above definition, a matrix $U \in \mathbb{R}^{n\times n}$ is positive semidefinite (respectively, positive definite) if it is symmetric and $\boldsymbol{x}^\mathsf{T} U\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$ (respectively, $\boldsymbol{x}^\mathsf{T} U\boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n - \{\boldsymbol{0}\}$). An immediate corollary here is that $\boldsymbol{x}\boldsymbol{x}^\mathsf{T}$ is a positive semidefinite matrix for all $\boldsymbol{x} \in \mathbb{R}^n$.

This section aims to introduce tools needed to study the SDP problems. We start this by introducing notations that will be used throughout this chapter. We use $\mathbb{S}^n$ to denote the space of real symmetric $n \times n$ matrices. An identity matrix of appropriate dimension is denoted by I. The set of the positive semidefinite matrices in $\mathbb{S}^n$ is a convex self-dual cone (see Lemma 12.1). See Figure 12.2 which shows a 3D plot of the boundary of the cone of the 2×2 positive semidefinite matrices.

For $U, V \in \mathbb{S}^n$, we write $U \succeq 0$ ($U \succ 0$) to mean that U is positive semidefinite (positive definite), and we use $U \succeq V$ or $V \preceq U$ to mean that $U - V \succeq 0$.

The bilinear map $\circ : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ is defined as

$$U \circ V \triangleq \frac{1}{2}(UV + VU). \tag{12.1}$$

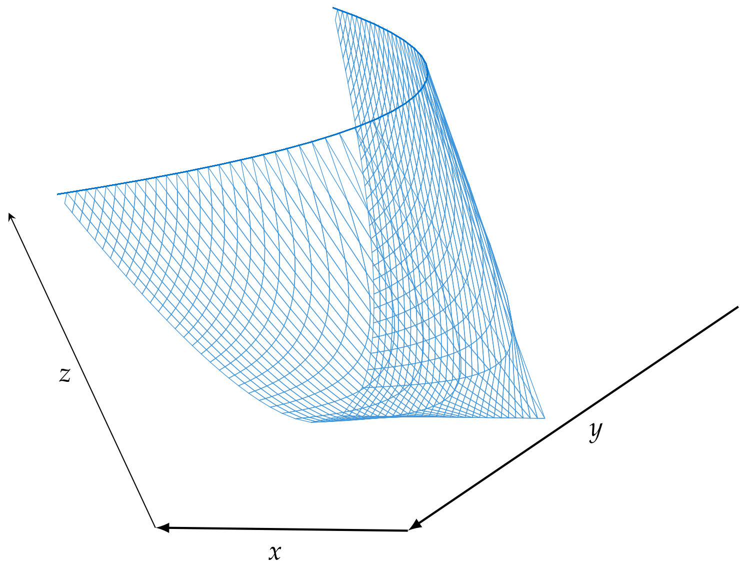

Figure 12.2: A 3D plot of the boundary of the cone of the positive semidefinite matrices in $\mathbb{S}^2$. The boundary is the continuous surface shaped by the above blue mesh.

The Frobenius inner product $\bullet : \mathbb{R}^{m\times n} \times \mathbb{R}^{mn\times n} \to \mathbb{R}$ between is defined as

$$U \bullet V \triangleq \text{trace}\,(U^{\mathsf{T}} V).$$

It is known that the space $\mathbb{S}^n$ under the bilinear map $\circ : \mathbb{S}^n \times \mathbb{S}^n \longrightarrow \mathbb{S}^n$ defined in (12.1) forms a Euclidean Jordan algebra (see Faraut [1994], Schmieta and Alizadeh [2003] for definitions) equipped with the standard inner product $\langle U, V\rangle \triangleq U \bullet V = \text{trace}\,(U \circ V) = \text{trace}\,(UV)$. The Frobenius norm of a matrix $U \in \mathbb{R}^{m\times n}$ is defined as

$$||U||_F \triangleq U \bullet U = \sum_i \lambda_i^2(U),$$

where $\lambda_i' s(U)$ are the eigenvalues of the matrix U.

It is known that, for any $U, V \in \mathbb{S}^n$, we have (see Horn and Johnson [1990])

$$\begin{array}{lcl} ||U+V||_F^2 & = & ||U||_F^2 + ||V||_F^2 + 2U \bullet V, \\ ||UV||_F & \leq & ||U||_2\, ||V||_F \;\leq\; ||U||_F\, ||V||_F, \\ ||UV||_F & \leq & ||U||_F\, ||V||_2 \;\leq\; ||U||_F\, ||V||_F, \end{array} \tag{12.2}$$

where $||U||_2 \triangleq \max\{||U\boldsymbol{x}||_2 : ||\boldsymbol{x}||_2 = 1\} = \max_i |\lambda_i(U)|$ is the operator norm (or 2-norm) of the matrix U.

The following properties of matrices are taken from [Todd, 2001, Section 3] and are given here without proofs. For proofs, see, for example, Horn and Johnson [1990], Watkins [1991], Todd [2001].

Property 12.1 (Trace commutativity) *If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$, then $\text{trace}(AB) = \text{trace}(BA)$.*

In particular, like any inner product, we have that $U \bullet V = V \bullet U$ for $U, V \in \mathbb{R}^{m\times n}$. The following property in $\mathbb{S}^n$ is the counterpart of Property 11.1 in $\mathbb{E}^n$.

Property 12.2 (Spectral decomposition in $\mathbb{S}^n$) *Any $A \in \mathbb{S}^n$ can be expressed in exactly one way as a product $A = Q(A)\Lambda(A)Q(A)^{\mathsf{T}}$, where $Q(A) \in \mathbb{R}^{n\times n}$ is an orthogonal matrix whose columns are the eigenvectors of A, and $\Lambda(A) \in \mathbb{R}^{n\times n}$ is a diagonal matrix whose diagonal entries are the eigenvalues of A.*

Recall that a square matrix is called lower triangular if all the entries above the main diagonal are zero. A unit lower triangular matrix A is a lower triangular matrix with $a_{ii} = 1$ for $1 \le i \le n$. Therefore, a unit lower triangular matrix has the form:

$$\begin{bmatrix} 1 & & & & \\ \times & 1 & & \mathbf{0} & \\ \times & \times & 1 & & \\ \times & \times & \times & 1 & \\ \times & \times & \times & \times & 1 \end{bmatrix}.$$

Unit lower triangular matrices arise in the following property about the LDL^{T} decomposition, which is a variant of the spectral decomposition.

Property 12.3 (LDL^{T} decomposition) *Any $A \succeq 0$ ($A \succ 0$) can be expressed in exactly one way as a product $A = L(A)D(A)L^{\mathsf{T}}(A)$, where $L(A) \in \mathbb{R}^{n\times n}$ is a unit lower triangular matrix, and $D(A) \in \mathbb{R}^{n\times n}$ is a diagonal matrix whose diagonal entries are nonnegative (positive).*

An immediate corollary of Property 12.3 is that every $A \succeq 0$ ($A \succ 0$) has a square root $A^{1/2} \succeq 0$ ($A^{1/2} \succ 0$). Simply, we take $A^{1/2} = L(A)D^{1/2}(A)L^{\mathsf{T}}(A)$. Another immediate corollary of this property is that every $A \succ 0$ has an inverse. Simply, we take $A^{-1} = L(A)D^{-1}(A)L^{\mathsf{T}}(A)$. The Cholesky decomposition is a variant of the LDL^{T} decomposition.

Property 12.4 (Cholesky decomposition) *Any $A \succ 0$ can be expressed in exactly one way as a product $A = LL^{\mathsf{T}}$, where $L \in \mathbb{R}^{n\times n}$ is lower triangular and has all main diagonal entries positive.*

The following property is used to represent the quadratic forms with the Frobenius inner product.

Property 12.5 (Representing quadratics) *If $A \in \mathbb{S}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$, then*

$$\boldsymbol{x}^{\mathsf{T}}A\boldsymbol{x} = A \bullet \boldsymbol{x}\boldsymbol{x}^{\mathsf{T}}.$$

In the following property and in what follows, we say that two matrices are simultaneously diagonalized if they have spectral decompositions with the same Q.

Property 12.6 (Commutativity and symmetry) *If $A, B \in \mathbb{S}^n$, then A and B commute if and only if AB is symmetric, if and only if A and B are simultaneously diagonalized.*

The Schur complement property presented in Property 12.7 has many applications in SDP.

Property 12.7 (Schur complement) *Let A and B be symmetric matrices and $A \succ 0$. Then*

$$\begin{bmatrix} A & B^\mathsf{T} \\ B^\mathsf{T} & C \end{bmatrix} \succeq 0 \ (\succ 0) \iff C - B^\mathsf{T} A^{-1} B \succeq 0 \ (\succ 0).$$

The matrix $C - B^\mathsf{T} A^{-1} B$ is called the Schur complement of the submatrix A.

Associated with $U_i \in \mathbb{S}^n (i = 1, 2, \ldots, m)$, we define the linear operator $\mathcal{U} : \mathbb{S}^n \to \mathbb{R}^m$ and its adjoint operator $\mathcal{U}^\star : \mathbb{R}^m \to \mathbb{S}^n$ as

$$\mathcal{U}X \triangleq \begin{bmatrix} U_1 \bullet X \\ U_2 \bullet X \\ \vdots \\ U_m \bullet X \end{bmatrix} \in \mathbb{R}^m, \text{ and } \mathcal{U}^\star z \triangleq \sum_{l=1}^{m} z_i U_i \in \mathbb{S}^n, \tag{12.3}$$

respectively. Note that

$$(\mathcal{U}X)^\mathsf{T} z = \sum_{l=l}^{m} (U_i \bullet X) z_i = (\sum_{l=1}^{m} z_i U_i \bullet X) = \mathcal{U}^\star z \bullet X, \tag{12.4}$$

for $X \in \mathbb{S}^n$ and $z \in \mathbb{R}^m$.

Associated with each nonsingular matrix $P \in \mathbb{R}^{n\times n}$, the symmetrization operator $\mathcal{H}_P : \mathbb{R}^{n\times n} \longrightarrow \mathbb{S}^n$ is defined as

$$\mathcal{H}_P(M) \triangleq \frac{1}{2}\left(PMP^{-1} + \left(PMP^{-1}\right)^\mathsf{T}\right) \tag{12.5}$$

for $M \in \mathbb{R}^{n\times n}$. We will use the operator $\mathcal{H}_P(\cdot)$ in order to symmetrize the optimality condition system, which is needed for applying Newton's method.

To end the notations, we point out that in the sequel $\mathrm{Diag}(x) \in \mathbb{S}^n$ is written for the diagonal matrix with the vector $x \in \mathbb{R}^n$ on its diagonal.

12.2 Semidefinite programming formulation

Within this section, we aim to familiarize readers with the SDP problem, providing an understanding of its form. Furthermore, we endeavor to delve deeper into this domain by demonstrating how various established categories of optimization problems can be approached through the lens of SDPs. This includes showcasing how certain well-recognized classes of optimization problems find their representation and resolution within the realm of SDP.

Problem formulation

We will be concentrating on SDP problems in primal standard form and in the corresponding dual form. The problem in primal form can be written as

$$\text{(P|SDP)} \qquad \begin{array}{rrcll} \min & C \bullet X & & & \\ & A_i \bullet X & = & b_i, & i = 1, \dots, m, \\ & X & \succeq & 0, & \end{array}$$

where the data $C, A_i, i = 1, \dots, m$, are real symmetric $n \times n$ matrices while $\boldsymbol{b}$ is a real m-vector and the variable X is a real symmetric $n \times n$ matrix.

It is convenient to introduce a *slack matrix* S and rewrite the problem as

$$\text{(D|SDP)} \qquad \begin{array}{rrcrcl} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & & & \\ & \sum_{i=1}^{m} y_i A_i & + & S & = & C, \\ & & & S & \succeq & 0. \end{array}$$

Here the variable $\boldsymbol{y}$ is a real m-vector while S is a real symmetric $n \times n$ matrix.

Using the notations introduced in (12.3), we can write our problems more compactly as

$$\text{(P|SDP)} \quad \begin{array}{rrcl} \min & C \bullet X & & \\ & \mathcal{A}X & = & \boldsymbol{b}, \\ & X & \succeq & 0; \end{array} \qquad \text{(D|SDP)} \quad \begin{array}{rrcrcl} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & & & \\ & \mathcal{A}^{\star}\boldsymbol{y} & + & S & = & C, \\ & & & S & \succeq & 0. \end{array}$$

Formulating problems as SDPs

In this part, we formulate four general classes of optimization problems as SDPs. We start with linear optimization.

Linear programming The linearity in linear programming can be transformed into a quadratic form that fits the criteria for SDP. Through this transformation, a linear program can be viewed as a specific instance of SDP, underscoring the relationship and demonstrating how the latter encapsulates a broader spectrum of optimization problems, including the more restrictive linear programming.

It is easy to see that the following pair of primal and dual linear programming problems:

$$\begin{array}{lrcl} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\ \text{s.t.} & A\boldsymbol{x} & = & \boldsymbol{b}, \\ & \boldsymbol{x} & \geq & \boldsymbol{0}; \end{array} \qquad \begin{array}{lrcl} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} & = & \boldsymbol{c}, \\ & \boldsymbol{s} & \geq & \boldsymbol{0}, \end{array}$$

is equivalent to the pair (P|SDP) and (D|SDP), respectively, with

$$C = \text{Diag}(\boldsymbol{c}), X = \text{Diag}(\boldsymbol{x}), S = \text{Diag}(\boldsymbol{s}), \text{ and } A_i = \text{Diag}(\boldsymbol{a}_i),$$

where $\boldsymbol{a}_i$ is the ith row of A.

Convex quadratic programming In convex quadratic optimization problems, we minimize a strictly convex quadratic function subject to affine constraint functions:

$$\begin{array}{ll} \min & q(\boldsymbol{x}) \triangleq \boldsymbol{x}^{\mathsf{T}} Q \boldsymbol{x} + \boldsymbol{c}^{\mathsf{T}} \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array} \tag{12.6}$$

Since Problem (12.6) is strictly convex, the matrix Q must be a symmetric positive definite matrix (i.e., $Q = Q^{\mathsf{T}}$ and $Q \succ O$). Let $Q = LL^{\mathsf{T}}$ (see Property 12.4). Then, using Schur complements (Property 12.7), the inequality $q(\boldsymbol{x}) \leq t$ can be written as

$$\begin{bmatrix} I & L^{\mathsf{T}}\boldsymbol{x} \\ \boldsymbol{x}^{\mathsf{T}}L & t - \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \end{bmatrix} \succeq 0.$$

Therefore, the quadratic optimization problem (12.6) can be formulated as the following SDP relaxation.

$$\begin{array}{ll} \min & t \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \begin{bmatrix} I & L^{\mathsf{T}}\boldsymbol{x} \\ \boldsymbol{x}^{\mathsf{T}}L & t - \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \end{bmatrix} \succeq 0, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

We also point out that, more generally, convex quadratically constrained quadratic optimization problems can also be formulated as SDP problems (see for example [Todd, 2001, Example 5]).

Second-order cone programming Consider the second-order cone programming problem (as introduced in Section 11.2):

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \succeq \boldsymbol{0}. \end{array}$$

The second-order cone constraint $\boldsymbol{x} \succeq \boldsymbol{0}$ is the inequality $x_0 \geq \|\widetilde{\boldsymbol{x}}\|$ (i.e., $\boldsymbol{x} \in\in \mathbb{E}^n_+$; see Definition 11.1). Using Schur complements (Property 12.7), we have that

$$\boldsymbol{x} = (x_0; \widetilde{\boldsymbol{x}}) \succeq \boldsymbol{0} \iff \begin{bmatrix} x_0 I & \widetilde{\boldsymbol{x}} \\ \widetilde{\boldsymbol{x}}^{\mathsf{T}} & x_0 \end{bmatrix} \succeq 0, \tag{12.7}$$

and equivalently, $\mathrm{Arw}(\boldsymbol{x}) \succeq 0$, where $\mathrm{Arw}(\boldsymbol{x})$ is the arrow-shaped matrix introduced in Definition 11.4.

Rotated quadratic cone programming Rotated quadratic cone programs were introduced in Section 11.2 in the context of second-order cone optimization. Let n be a positive integer and M be a nonsingular matrix of order $n-2$. Recall that the nth dimensional rotated quadratic cone is

$$\mathcal{K}^n = \left\{ \boldsymbol{x} = (x_0; x_1; \hat{\boldsymbol{x}}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} : 2x_0x_1 \geq \|\hat{\boldsymbol{x}}\|^2,\ x_0 \geq 0, x_1 \geq 0 \right\}.$$

Recall also that the constraint on $\boldsymbol{x}$ that satisfies the inequality $2x_0x_1 \geq \|\hat{x}\|^2$ is called the hyperbolic constraint. As mentioned in Section 11.2, a rotated quadratic cone optimization problems, a linear objective function is minimized subject to linear constraints and hyperbolic constraints. Note that

$$
\begin{aligned}
(x_0; x_1; \hat{x}) \in \mathcal{K}^n \quad &\iff \quad (2x_0 + x_1; 2x_0 - x_1; 2\hat{x}) \in \mathbb{E}^n_+ && \text{(By (11.6))}\\
&\iff \begin{bmatrix} (2x_0 + x_1)I & (2x_0 - x_1; 2\hat{x}) \\ (2x_0 - x_1; 2\hat{x})^\mathsf{T} & 2x_0 + x_1 \end{bmatrix} \in \mathbb{S}^n_+. && \text{(By (12.7))}
\end{aligned}
$$

This means that a rotated quadratic cone optimization problem can be expressed as an SDP problem because the hyperbolic constraint is equivalent to a linear matrix inequality.

12.3 Applications in combinatorial optimization

In this section, we describe three combinatorial applications of SDP. For non-combinatorial applications, we point out that the applications that we described in Section 11.3 in the context of second-order cone programming were first formulated as applications of SDP (see Vandenberghe and Boyd [1999]). The focus in this chapter is only on those applications in combinatorial optimization. The material of this section has appeared in some sections of Goemans [1998], Vandenberghe and Boyd [1999], Todd [2001]. For more applications of SDP, we refer the reader to Goemans [1998], Karger et al. [1998], Vandenberghe and Boyd [1999], Rendl [1999], Todd [2001], Lidický and Pfender [2021]. In particular, the work of Lidický and Pfender [2021] presents an interesting application in Ramsey theory.

Shannon capacity of graphs

In this problem, the objective is to bound the Shannon capacity, or the independence number of graphs. We need some definitions.

Definition 12.1 *Let $G = (V, E)$ be an undirected graph. Then:*

(a) *A clique set (or simply, a clique) of G is a set $S \subseteq V$ of mutually adjacent vertices.*

(b) *A clique cover of G is a collection $\mathcal{C}$ of cliques that together include all V. That is, $V \subseteq \cup_{S \in \mathcal{C}} S$.*

(c) *The clique cover number of G, denoted by $\bar{\chi}(G)$, is the minimum cardinality of a clique cover of G. That is, $\bar{\chi}(G)$ is the smallest number of cliques of G whose union covers the set V.*

Note that the clique cover number of a graph and the chromatic number (see Definition 4.7) of its complement are identical, i.e., $\bar{\chi}(G) = \chi(\bar{G})$.

Definition 12.2 *Let $G = (V, E)$ be an undirected graph. Then:*

(a) *An independent (or stable) set of G is a set $S \subseteq V$ of mutually nonadjacent vertices.*

(*b*) *The independence (or stability) number of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set G. In other words, $\alpha(G)$ is the largest number of vertices that can be colored with the same color in a graph coloring of G.*

Note that each node in a stable set must be in a different clique in a clique cover, hence we have

$$\alpha(G) \leq \bar{\chi}(G). \tag{12.8}$$

If the equality holds in (12.8), then we say that G is a perfect graph.

Computing both $\alpha(G)$ and $\bar{\chi}(G)$ is a well-known NP-hard problem. In the context of the Shannon capacity problem, the goal is to approximate a function that falls between these two values. This intermediary function is referred to as Lovász's theta function, denoted as $\theta(G)$, and it represents the optimal solution to the SDP problem:

$$\begin{array}{ll} \max & \mathbf{1}\mathbf{1}^{\mathsf{T}} \bullet X \\ \text{s.t.} & I \bullet X = 1, \\ & x_{ij} = 0, \ \ \text{for } (i,j) \in E, \\ & X \succeq 0, \end{array} \tag{12.9}$$

where $\mathbf{1}$ is a vector of ones with an appropriate dimension.

Note that the SDP problem (12.9) is in primal form, but in maximization form. The dual of Problem (12.9) is the SDP problem Todd [2001]:

$$\begin{array}{ll} \min & s \\ \text{s.t.} & sI + \displaystyle\sum_{(i,j)\in E} y_{ij} M_{ij} \succeq \mathbf{1}\mathbf{1}^{\mathsf{T}}, \end{array}$$

where M_{ij} is the symmetric matrix that is all zero except for ones in the ijth and jith positions. Clearly, the above SDP models calculate the numbers $\alpha(G)$ and $\bar{\chi}(G)$ exactly if G s perfect.

Max-cut of graphs

In a max-cut graph problem, the objective is to find the cut of maximum weight in undirected graphs. This problem arises in finding the ground state of a spin glass; see Poljak and Tuza [1993]. We need the following definition.

Definition 12.3 *Let $G = (V, E)$ be an undirected graph with a nonnegative weight vector $\boldsymbol{w} = (w_{ij})_{(i,j)\in E} \in \mathbb{R}_+^{|E|}$. The cut determined by a subset $S \subseteq V$ is the set $\delta(S) \triangleq \{(i,j) \in E : i \in S, j \notin S\}$, and its weight is defined as $w(\delta(S)) \triangleq \sum_{(i,j)\in\delta(S)} w_{ij}$.*

In light of Definition 12.3, we can assume that the underlying graph is complete by letting $w_{ij} = 0$ for all $(i,j) \notin E$ (we let also $w_{ii} = 0$ for all i). To represent the cut $\delta(S)$, we introduce the variable $\boldsymbol{x} \in \mathbb{R}^{|V|}$, which is defined as

$$x_i = \begin{cases} 1, & \text{if } i \in S, \\ -1, & \text{if } i \in V - S. \end{cases}$$

It follows that

$$1 - x_i x_j = \begin{cases} 2, & \text{if } (i,j) \in \delta(S), \\ 0, & \text{if } (i,j) \notin \delta(S). \end{cases}$$

We also define the matrix $C \in \mathbb{S}^{|V|}$ as

$$c_{ij} = \begin{cases} -w_{ij}/4, & \text{if } i \neq j, \\ \sum_j w_{ij}/4, & \text{if } i = j. \end{cases}$$

Note that

$$\begin{aligned} w(\delta(S)) &= \sum_{(i,j)\in\delta(S)} w_{ij} \\ &= \frac{1}{2}\sum_{i<j} w_{ij}(1 - x_i x_j) \\ &= \frac{1}{4}\sum_i \sum_j w_{ij}(1 - x_i x_j) \\ &= -\sum_i \left(x_i \left(\sum_j \frac{w_{ij}}{4} x_j \right) \right) \\ &= \sum_i \left(x_i \left(\sum_j c_{ij} x_j \right) \right) \\ &= \boldsymbol{x}^{\mathsf{T}} C \boldsymbol{x} \;=\; C \bullet \boldsymbol{x}\boldsymbol{x}^{\mathsf{T}}. \end{aligned}$$

Because every $(+1,-1)$-vector in $\mathbb{R}^{|V|}$ corresponds to a cut in the graph $G = (V, E)$, the max-cut problem can be formulated as the integer program:

$$\begin{array}{ll} \max & C \bullet \boldsymbol{x}\boldsymbol{x}^{\mathsf{T}} \\ \text{s.t.} & \boldsymbol{x} \in \{+1,-1\}^{|V|}, \end{array} \tag{12.10}$$

or equivalently, as the quadratic program:

$$\begin{array}{ll} \max & \boldsymbol{x}^{\mathsf{T}} C \boldsymbol{x} \\ \text{s.t.} & x_i^2 = 1,\ i \in V. \end{array} \tag{12.11}$$

Observe that Problem (12.11) is linear in the products $x_i x_j$, and these are the entries of the rank one matrix $X \triangleq \boldsymbol{x}\boldsymbol{x}^{\mathsf{T}}$. Observe also that $X \in \mathbb{S}_+^{|V|}$, with $x_{ii} = 1$ for all $i \in V$. It follows that Problem (12.10) can be written as

$$\begin{array}{ll} \max & C \bullet X \\ \text{s.t.} & x_{ii} = 1, i \in V, \\ & X \succeq 0, \\ & X \text{ is of rank one.} \end{array} \tag{12.12}$$

Relaxing the last constraint in Problem (12.12), we obtain the SDP problem Todd [2001]:

$$\begin{array}{ll} \max & C \bullet X \\ \text{s.t.} & x_{ii} = 1, i \in V, \\ & X \succeq 0. \end{array}$$

We also mention that Todd Todd [2001] discusses two other different ways to arrive at an SDP relaxation of the max-cut problem.

Combinatorial topology optimization

The Ben-Tal and Bendsøe's paper Ben-Tal and Bendsøe [1993] addresses a problem from combinatorial topology optimization of truss structures. Specifically, the authors investigate a scenario where a structure comprises k linear elastic bars connecting a set of p nodes. In this analysis, it is assumed that the geometry, including the topology and lengths of the bars, as well as the material, remain fixed. The primary objective of this study is to determine the optimal sizes of the bars, which involves selecting appropriate cross-sectional areas for them.

For $i = 1, 2, \ldots, k$, and $j = 1, 2, \ldots, p$, we define the following decision variables and parameters:

- $f_j \triangleq$ the external force applied on the j^{th} node,
- $d_j \triangleq$ the (small) displacement of the j^{th} node resulting from the load force f_j,
- $x_i \triangleq$ the cross-sectional area of the i^{th} bar,
- $\underline{x}_i \triangleq$ the lower bound on the cross-sectional area of the i^{th} bar,
- $\overline{x}_i \triangleq$ the upper bound on the cross-sectional area of the i^{th} bar,
- $l_i \triangleq$ the length of the i^{th} bar,
- $v \triangleq$ the maximum allowed volume of the bars of the structure,
- $G(\boldsymbol{x}) \triangleq \sum_{i=1}^{k} x_i G_i$ is the stiffness matrix, where the matrices $G_i \in \mathbb{S}^p$, $i = 1, 2, \ldots, k$, depend only on fixed parameters (such as length of bars and material).

In the simplest version of the problem, one considers one fixed set of externally applied nodal forces $f_j, j = 1, 2, \ldots, p$. Given this, the elastic stored energy within the structure is given by

$$\varepsilon = \boldsymbol{f}^{\mathsf{T}} \boldsymbol{d},$$

which serves as an indicator of the measure of the inverse of the structure's stiffness. In view of the definition of the stiffness matrix $G(\boldsymbol{x})$, we can also conclude that the following linear relationship between $\boldsymbol{f}$ and $\boldsymbol{d}$:

$$\boldsymbol{f} = G(\boldsymbol{x})\, \boldsymbol{d}.$$

The objective is to find the stiffest truss by minimizing ε subject to the inequality $\boldsymbol{l}^{\mathsf{T}} \boldsymbol{x} \le v$ as a constraint on the total volume (or equivalently, weight) and the constraint $\underline{\boldsymbol{x}} \le \boldsymbol{x} \le \bar{\boldsymbol{x}}$ as upper and lower bounds on the cross-sectional areas.

For simplicity, we assume that $\underline{x} > \mathbf{0}$ and $G(x) \succ 0$, for all $x > \mathbf{0}$. In this case, we can express the elastic stored energy in terms of the inverse of the stiffness matrix and the external applied nodal force as follows:

$$\varepsilon = f^{\mathsf{T}} G(x)^{-1} f.$$

In summary, they consider the problem (see also Vandenberghe and Boyd [1999]):

$$\begin{array}{ll} \min & f^{\mathsf{T}}\, G(x)^{-1} f \\ \text{s.t.} & \underline{x} \leq x \leq \bar{x}, \\ & l^{\mathsf{T}}\, x \leq v, \end{array}$$

which is equivalent to:

$$\begin{array}{ll} \min & s \\ \text{s.t.} & f^{\mathsf{T}}\, G(x)^{-1} f \leq s, \\ & \underline{x} \leq x \leq \bar{x}, \\ & l^{\mathsf{T}}\, x \leq v. \end{array} \tag{12.13}$$

The first inequality constraint in (12.13) is just fractional quadratic function inequality constraint and it can be formulated as a positive semidefinite constraint. This problem can be cast as an SDP problem as follows:

$$\begin{array}{ll} \min & s \\ \text{s.t.} & \begin{bmatrix} G(x) & f \\ f^{\mathsf{T}} & s \end{bmatrix} \succeq 0, \\ & \underline{x} \leq x \leq \bar{x}, \\ & l^{\mathsf{T}}\, x \leq v. \end{array}$$

12.4 Duality in semidefinite programming

In this section, we offer a brief overview of the duality theory pertaining to SDP and introduce the complementarity condition as one of the optimality criteria for SDP. While much of the duality theory for SDP shares similarities with that of linear programming (as well as second-order cone programming), it is important to note that there are certain distinctions.

Recall that a regular cone is self-dual if it equals its dual cone (see Definition 3.22). Now we need the following lemma.

Lemma 12.1 *The symmetric positive semidefinite cone $\mathbb{S}^n_+$ is self-dual under the Frobenius inner product.*

Proof We verify that the symmetric positive semidefinite cone $\mathbb{S}^n_+$ equals its dual cone, which is defined as

$$(\mathbb{S}^n_+)^\star \triangleq \{X \in \mathbb{S}^n : X \bullet Y \geq 0 \text{ for all } Y \in \mathbb{S}^n_+\}.$$

We first prove that $(\mathbb{S}^n_+)^\star \subseteq \mathbb{S}^n_+$. Assume that $X \notin \mathbb{S}^n_+$, then there exists a nonzero vector $y \in \mathbb{R}^n$ such that $X \bullet yy^{\mathsf{T}} = y^{\mathsf{T}} X y < 0$, which shows that $X \notin (\mathbb{S}^n_+)^\star$.

Now, we prove that $\mathbb{S}^n_+ \subseteq (\mathbb{S}^n_+)^\star$. Letting $X \in \mathbb{S}^n_+$, then for any $Y \in \mathbb{S}^n_+$, we have that

$$X \bullet Y = \mathrm{trace}(XY) = \mathrm{trace}\left(X^{\frac{1}{2}} Y X^{\frac{1}{2}}\right) \geq 0,$$

where we used the fact that $X^{\frac{1}{2}} Y X^{\frac{1}{2}} \in \mathbb{S}^n$ to obtain the inequality. In fact, if $U \in \mathbb{S}^n_+$ and Q and Λ are respectively its corresponding orthogonal and diagonal matrices obtained from the spectral decomposition (see Property 12.2), then we have

$$\mathrm{trace}(U) = \mathrm{trace}(Q\Lambda Q^\mathsf{T}) = \mathrm{trace}(\Lambda QQ^\mathsf{T}) = \mathrm{trace}(\Lambda) = \sum_{i=1}^{n} \lambda_i \geq 0.$$

We have shown that $X \in (\mathbb{S}^n_+)^\star$ whenever $X \in \mathbb{S}^n_+$. The proof is complete. ■

The weak duality property in SDP is encapsulated by Theorem 12.1, which asserts that the objective function values of primal feasible solutions consistently surpass those of dual feasible solutions.

Theorem 12.1 (Weak duality in SDP) *Let X and (y, S) be feasible for (P|SDP) and (D|SDP) respectively, then*

$$C \bullet X - b^\mathsf{T} y = S \bullet X \geq 0.$$

Proof Using (12.4), we have

$$\begin{aligned} C \bullet X - b^\mathsf{T} y &= (\mathcal{A}^\star y + S) \bullet X - (\mathcal{A}X)^\mathsf{T} y \\ &= \mathcal{A}^\star y \bullet X + S \bullet X - (\mathcal{A}X)^\mathsf{T} y \\ &= \mathcal{A}^\star y \bullet X + S \bullet X - \mathcal{A}^\star y \bullet X \\ &= S \bullet X \ \geq \ 0, \end{aligned}$$

where the last inequality follows from the self-duality of the cone of the symmetric positive semidefinite matrices. ■

As we mentioned in Chapter 11, it is important to note that the strong duality property may not hold in general conic optimization, as highlighted in Nesterov and Nemirovskii [1994]. Nevertheless, even in cases where strong duality does not apply, we can establish a somewhat weaker property that consistently holds in conic optimization. In the context of SDP, we will now outline the conditions under which this slightly weaker property remains valid. We say that the primal problem is strictly feasible if there exists a primal feasible point $\hat{X}$ such that $\hat{X} \succ 0$. We make the following assumption for convenience.

Assumption 12.1 *The m matrices $A_1, A_2, \ldots, A_m$ are linearly independent in $\mathbb{S}^n$.*

Now we state and prove the following semi-strong duality result.

Lemma 12.2 (Semi-strong duality in SDP) *Consider the primal–dual pair (P|SDP) and (D|SDP). If the primal problem is strictly feasible and solvable, then the dual problem is solvable and their optimal values are equal.*

Proof By the assumption of the lemma, the primal problem is strictly feasible and solvable. So, let X be an optimal solution of the primal problem where we can apply the KKT conditions (see Theorem 11.1). This implies that there are Lagrange multipliers $\boldsymbol{y}$ and S such that $(X, \boldsymbol{y}, S)$ satisfies:

$$
\begin{aligned}
&\mathcal{A}X = \boldsymbol{b},\\
&\mathcal{A}^\star \boldsymbol{y} + S = C,\\
&X \bullet S = 0,\\
&X, S \succeq 0.
\end{aligned}
$$

This means that $(\boldsymbol{y}, S)$ is feasible for the dual problem. Let $(\boldsymbol{\nu}, Z)$ be any feasible solution of the dual problem, then we have that $\boldsymbol{b}^\top \boldsymbol{\nu} \le C \bullet X = S \bullet X + \boldsymbol{b}^\top \boldsymbol{y} = \boldsymbol{b}^\top \boldsymbol{y}$, where we used the weak duality to obtain the inequality and the complementary slackness to obtain the last equality. Thus, $(\boldsymbol{y}, S)$ is an optimal solution of the dual problem and $C \bullet X = \boldsymbol{b}^\top \boldsymbol{y}$ as desired. ∎

The following strong duality result can be obtained by applying the duality relations to our problem formulation (see also [Todd, 2001, Theorem 4.1]).

Theorem 12.2 (Strong duality in SDP) *Consider the primal–dual pair (P|SDP) and (D|SDP). If both the primal and dual problems have strictly feasible solutions, then they both have optimal solutions $X^\star$ and $(\boldsymbol{y}^\star, S^\star)$, respectively, and*

$$
p^\star \triangleq C \bullet X^\star = d^\star \triangleq \boldsymbol{b}^\top \boldsymbol{y}^\star (\textit{i.e.}, X^\star \bullet S^\star = 0).
$$

The following lemma describes the complementarity condition as one of the optimality conditions of SDP. This lemma is not hard to show using the spectral decomposition of X, and considering its positive and zero eigenvalues separately.

Lemma 12.3 (Complementarity condition in SDP) *If $X, S \succeq 0$, then*

$$
X \bullet S = 0 \iff XS = 0.
$$

As a result of Lemma 12.3, the complementary slackness condition for the primal and dual SDP problems (P|SDP) and (D|SDP) can be equivalently represented by the equation $XS = 0$. From the above results, we get the following corollary.

Corollary 12.1 (Optimality conditions in SDP) *Consider the primal–dual pair (P|SDP) and (D|SDP). Assume that both the primal and dual problems are strictly feasible, then $(X, (\boldsymbol{y}, S)) \in \mathbb{S}^n \times \mathbb{R}^m \times \mathbb{S}^n$ is a pair of optimal solutions to the SDP (P|SDP) and (D|SDP) if and only if*

$$
\begin{aligned}
&\mathcal{A}X = \boldsymbol{b},\\
&\mathcal{A}^\star \boldsymbol{y} + S = C,\\
&XS = 0,\\
&X, S \succeq 0.
\end{aligned} \tag{12.14}
$$

We have established the duality relations in SDP. The focus in the remaining part of this chapter is to solve SDP algorithmically.

12.5 A primal-dual path-following algorithm

In Section 11.5, we presented a primal-dual path-following algorithm for solving SOCP problems. In this section, we present a primal-dual path-following algorithm for solving SDP problems. The material presented in this section is based on, and similar to, any primal-dual path-following algorithm proposed for SDP (see for instance Touil et al. [2017]). The general scheme of the path-following algorithms for SDP is as follows. We associate the perturbed problems to semidefinite programming problems (P|SDP) and (D|SDP), then we draw a path of the centers defined by the perturbed KKT optimality conditions. After that, Newton's method is applied to treat the corresponding perturbed equations in order to obtain a descent search direction.

Let $\mu > 0$ be a barrier parameter. The perturbed primal problem corresponding to the primal problem (P|SDP) is

$$
\begin{array}{lll}
 & \min & f_\mu(X) \triangleq C \bullet X - \mu \ln\det(X) + n\mu \ln \mu \\
(\text{P|SDP}_\mu) & \text{s.t.} & \mathcal{A}X = \boldsymbol{b}, \\
 & & X \succ 0,
\end{array}
$$

and the perturbed dual problem corresponding to the dual problem (D|SDP) is

$$
\begin{array}{lll}
 & \max & g_\mu(\boldsymbol{y}, S) \triangleq \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} + \mu \ln\det(S) - n\mu \ln \mu \\
(\text{D|SDP}_\mu) & \text{s.t.} & \mathcal{A}^\star \boldsymbol{y} + S = C, \\
 & & S \succ 0.
\end{array}
$$

Now, we define the following feasibility sets:

$$
\begin{array}{rcl}
\mathcal{F}_{\text{P|SDP}} & \triangleq & \{X \in \mathbb{S}^n : \mathcal{A}X = \boldsymbol{b},\ X \succeq 0\}, \\
\mathcal{F}_{\text{D|SDP}} & \triangleq & \{(\boldsymbol{y}, S) \in \mathbb{R}^m \times \mathbb{S}^n : \mathcal{A}^\star \boldsymbol{y} + S = C,\ S \succeq 0\}, \\
\mathcal{F}^\circ_{\text{P|SDP}} & \triangleq & \{X \in \mathbb{S}^n : \mathcal{A}X = \boldsymbol{b},\ X \succ 0\}, \\
\mathcal{F}^\circ_{\text{D|SDP}} & \triangleq & \{(\boldsymbol{y}, S) \in \mathbb{R}^m \times \mathbb{S}^n : \mathcal{A}^\star \boldsymbol{y} + S = C,\ S \succ 0\}, \\
\mathcal{F}^\circ_{\text{SDP}} & \triangleq & \mathcal{F}^\circ_{\text{P|SDP}} \times \mathcal{F}^\circ_{\text{D|SDP}}.
\end{array}
$$

We also make the following assumption about the primal-dual pair (P|SDP) and (D|SDP).

Assumption 12.2 *The set $\mathcal{F}^\circ_{SDP}$ is nonempty.*

Assumption 12.2 requires that Problem (P|SDP$_\mu$) and its dual (D|SDP$_\mu$) have strictly feasible solutions, which guarantees strong duality for the semidefinite programming problem. Note that the feasible region for Problems (P|SDP$_\mu$) and (D|SDP$_\mu$) is described implicitly by $\mathcal{F}^\circ_{\text{SDP}}$. Due to the coercivity of the function f_μ on the feasible set of (P|SDP$_\mu$), Problem (P|SDP$_\mu$) has an optimal solution.

The following lemma proves the convergence of the optimal solution of Problem (P|SDP$_\mu$) to the optimal solution of Problem (P|SDP) when μ approaches zero.

Lemma 12.4 *Let $\bar{X}_\mu$ be an optimal primal solution of (P|SDP$_\mu$), then $\bar{X} = \lim_{\mu\to 0} \bar{X}_\mu$ is an optimal solution of Problem (P|SDP).*

Proof Let $f_\mu(X) \triangleq f(X,\mu)$ and $f(X) \triangleq f(X,0)$. Due to the coercivity of the function f_μ on the feasible set of (P|SDP$_\mu$), Problem (P|SDP$_\mu$) has an optimal solution, say $\bar{X}_\mu$, such that

$$\nabla_X f_\mu(\bar{X}_\mu) = \nabla_X f(\bar{X}_\mu, \mu) = 0.$$

Then, for all $X \in \mathcal{F}^\circ_{\text{P|SDP}}$, we have that

$$\begin{array}{rcl} f(X) & \geq & f(\bar{X}_\mu,\mu) + (X - \bar{X}_\mu) \bullet \nabla_X f(\bar{X}_\mu,\mu) + (0-\mu)\frac{\partial}{\partial\mu} f(\bar{X}_\mu,\mu) \\ & \geq & f(\bar{X}_\mu,\mu) + \mu \ln\det \bar{X}_\mu - n\mu\ln\mu - n\mu \\ & \geq & C \bullet \bar{X}_\mu - \mu\ln\det\bar{X}_\mu + n\mu\ln\mu + \mu\ln\det\bar{X}_\mu - n\mu\ln\mu - n\mu \\ & \geq & C \bullet \bar{X}_\mu - n\mu. \end{array}$$

Since X was arbitrary in $\mathcal{F}^\circ_{\text{P|SDP}}$, this implies that

$$\min_{X\in\mathcal{F}^\circ_{\text{P|SDP}}} f(X) \geq C \bullet \bar{X}_\mu - n\mu \geq C \bullet \bar{X}_\mu = f(\bar{X}_\mu).$$

On the other side, we have $f(\bar{X}_\mu) \geq \min_{X\in\mathcal{F}^\circ_{\text{P|SDP}}} f(X)$. As μ goes to 0, it immediately follows that $f(\bar{X}) = \min_{X\in\mathcal{F}^\circ_{\text{P|SDP}}} f(X)$. Thus, $\bar{X}$ is an optimal solution of Problem (P|SDP). The proof is complete. ■

Newton's method and commutative directions

As we mentioned, the objective function of Problem (P|SDP$_\mu$) is strictly convex, hence the KKT conditions are necessary and sufficient to characterize an optimal solution of Problem (P|SDP$_\mu$). Consequently, and in light of Lemma 12.3, the points $\bar{X}_\mu$ and $(\bar{\boldsymbol{y}}_\mu, \bar{S}_\mu)$ are optimal solutions of (P|SDP$_\mu$) and (D|SDP$_\mu$) respectively if and only if they satisfy the perturbed nonlinear system

$$\begin{array}{rclcrcl} \mathcal{A}X & = & \boldsymbol{b}, & & X & \succ & 0, \\ \mathcal{A}^\star\boldsymbol{y} + S & = & C, & & S & \succ & 0, \\ XS & = & \mu I, & & \mu & > & 0, \end{array} \tag{12.15}$$

where I is the identity matrix of $\mathbb{S}^n$.

We call the set of all solutions of system (12.15), denoted by $(X_\mu, \boldsymbol{y}_\mu, S_\mu)$ with $\mu > 0$, the central path. We say that a point $(X, \boldsymbol{y}, S)$ is near to the central path if it belongs to the set $\mathcal{N}_{\text{SDP}}(\mu)$, which is defined as

$$\mathcal{N}_{\text{SDP}}(\mu) \triangleq \left\{(X,\boldsymbol{y},S) \in \mathcal{F}^\circ_{\text{P|SDP}} \times \mathcal{F}^\circ_{\text{D|SDP}} : d_{\text{SDP}}(X,S) \leq \theta\mu, \theta \in (0,1)\right\},$$

where

$$d_{\text{SDP}}(X,S) \triangleq \left\|X \circ S - \mu I\right\|_F.$$

Since for $X, S \in \mathbb{S}^n$, the product XS is generally not in $\mathbb{S}^n$, so the left-hand side of System (12.15) is a map from $\mathbb{R}^{n\times n} \times \mathbb{R}^m \times \mathbb{S}^n$ to $\mathbb{S}^n \times \mathbb{R}^m \times \mathbb{S}^n$. Thus, System (12.15) is not a square system when X and S are restricted to $\mathbb{S}^n$, which is needed for applying Newton's method. A remedy for this is to make System (12.15) square by modifying the left-hand side to a map from $\mathbb{S}^n \times \mathbb{R}^m \times \mathbb{S}^n$ to itself. To this end, we use the symmetrization operator $\mathcal{H}_P(\cdot)$ defined in (12.5). The following lemma is due to Zhang [1998b].

Lemma 12.5 *For $M \in \mathbb{S}^n$, nonsingular $P \in \mathbb{R}^{n\times n}$, and a scalar τ, we have*

$$\mathcal{H}_P(M) = \tau I \iff M = \tau I.$$

In light of Lemma 12.5, for any nonsingular matrix P, System (12.15) is equivalent to

$$\begin{array}{rcllcl} \mathcal{A}X & = & \boldsymbol{b}, & X & \succ & 0, \\ \mathcal{A}^\star \boldsymbol{y} + S & = & C, & S & \succ & 0, \\ \mathcal{H}_P(XS) & = & \mu I, & \mu & > & 0. \end{array} \tag{12.16}$$

Recall that $\mathcal{H}_P(XS) = \frac{1}{2}(PXSP^{-1} + P^{-1}SXP^{\mathsf{T}})$. So, an alternative way to view the above development is to scale (P|SDP) so that the variable X is replaced by $\overline{X} \triangleq PXP^{\mathsf{T}}$ and to scale (D|SDP) so that the variable S is replaced by $\underline{S} \triangleq {P^{-1}}^{\mathsf{T}} SP^{-1}$.

The need for the above symmetrization occurs because X and S do not commute (see Property 12.6). So, we choose P so that the scaled matrices commute. Denote by $C(X, S)$ the set of all matrices so that the scaled matrices commute. That is,

$$C(X, S) \triangleq \left\{ P \in \mathbb{S}^n : P^{-1} \text{ exists, and } PXP^{\mathsf{T}} \,\&\, {P^{-1}}^{\mathsf{T}} SP^{-1} \text{ commute} \right\}. \tag{12.17}$$

Now, we can apply Newton's method to System (12.16) and obtain the following linear system

$$\begin{array}{rcl} \mathcal{A}\Delta X & = & 0, \\ \mathcal{A}^\star \Delta \boldsymbol{y} + \Delta S & = & 0, \\ \mathcal{H}_P(X\Delta S + \Delta XS) & = & \sigma\mu I - \mathcal{H}_P(XS). \end{array} \tag{12.18}$$

where $(\Delta X, \Delta \boldsymbol{y}, \Delta S) \in \mathbb{S}^n \times \mathbb{R}^m \times \mathbb{S}^n$ is the search direction, $\sigma \in (0, 1)$ is the centering parameter, and $\mu = \frac{1}{n} X \bullet S = \frac{1}{n} \overline{X} \bullet \underline{S}$ is the normalized duality gap corresponding to $(X, \boldsymbol{y}, S)$. In fact,

$$\frac{1}{n}\overline{X} \bullet \underline{S} = \frac{1}{n}\left(PXP^{\mathsf{T}}\right) \bullet \left({P^{-1}}^{\mathsf{T}} SP^{-1}\right) = \frac{1}{n} X \bullet S.$$

Solving the scaled Newton system (12.18) yields the search direction $(\Delta X, \Delta \boldsymbol{y}, \Delta S)$. Note that the search direction $(\Delta X, \Delta \boldsymbol{y}, \Delta S)$ belongs to the so-called the *MZ family of directions* (due to Monteiro [1997], Zhang [1998a]). In fact, such a way of scaling originally proposed for semidefinite programming by Monteiro [1997] and Zhang [1998a], and after that it was used for second-order cone programming in Schmieta and Alizadeh's paper Schmieta and Alizadeh [2003]. Clearly, the set $C(X, S)$ defined in (12.17) is a subclass of the MZ family of search directions. Our focus is in matrices $P \in C(X, S)$. We discuss the following three choices of P (see [Todd, 2001, Section 6]):

- The first one is to choose $P = X^{-1/2}$, which gives $\overline{X} = I$.
- The second one is to choose $P = S^{1/2}$, which gives $\underline{S} = I$.
- The third choice of P is given by $P = (X^{1/2}(X^{1/2}SX^{1/2})^{-1/2}X^{1/2})^{-1/2}$, which yields $P^2XP^2 = S$, and therefore $\overline{X} = \underline{S}$.

As we mentioned in Chapter 11 for the case of second-order cone programming, the first two choices of directions are respectively called the HRVW/KSH/M direction and dual HRVW/K-

SH/M direction (due to Helmberg et al. [1996], Monteiro [1997], Kojima et al. [1997]). The third choice of direction is called the NT direction (due to Nesterov and Todd [1998]).

Path-following algorithm

We formally state the path-following algorithm for solving SDP problem in Algorithm 12.1.

Algorithm 12.1: Path-following algorithm for SDP

Input: Data in Problems (P|SDP) and (D|SDP), $k = 0$, $\left(X^{(0)}, \boldsymbol{y}^{(0)}, S^{(0)}\right) \in \mathcal{N}_{\text{SDP}}\left(\mu^{(0)}\right)$, $\epsilon > 0$, $\sigma^{(0)}, \theta \in (0, 1)$

Output: An ϵ-optimal solution to Problem (P|SDP)

while $X^{(k)} \bullet S^{(k)} \geq \epsilon$ **do**

  choose $P^{(k)} \in \mathcal{C}\left(X^{(k)}, S^{(k)}\right)$

  set $\mu^{(k)} \triangleq \frac{1}{n} X^{(k)} \bullet S^{(k)}$
  $H^{(k)} \triangleq \sigma^{(k)} \mu^{(k)} I - X^{(k)} \circ S^{(k)}$

  compute $\left(\Delta X^{(k)}; \Delta \boldsymbol{y}^{(k)}; \Delta S^{(k)}\right)$ by solving the scaled system (12.18) to get

$$
\begin{aligned}
\Delta \boldsymbol{y}^{(k)} &= -\left(\mathcal{A}\left(S^{(k)^{-1}} \circ X^{(k)}\right)\mathcal{A}^{\star}\right)^{-1} \mathcal{A}\left(S^{(k)^{-1}} \circ H^{(k)}\right)\\
\Delta S^{(k)} &= -\mathcal{A}^{\star} \Delta \boldsymbol{y}^{(k)}\\
\Delta X^{(k)} &= S^{(k)^{-1}} \circ \left(H^{(k)} - X^{(k)} \circ \Delta S^{(k)}\right)
\end{aligned}
$$

  set the new iterate according to

$$
\begin{aligned}
X^{(k+1)} &\triangleq X^{(k)} + \alpha^{(k)} \Delta X^{(k)}\\
\boldsymbol{y}^{(k+1)} &\triangleq \boldsymbol{y}^{(k)} + \alpha^{(k)} \Delta \boldsymbol{y}^{(k)}\\
S^{(k+1)} &\triangleq S^{(k)} + \alpha^{(k)} \Delta S^{(k)}
\end{aligned}
$$

  set $k = k + 1$

end

Algorithm 12.1 selects a sequence of displacement steps $\{\alpha^{(k)}\}$ and centrality parameters $\{\sigma^{(k)}\}$ according to the following rule: For all $k \geq 0$, we take $\sigma^{(k)} = 1 - \delta/\sqrt{n}$, where $\delta \in [0, \sqrt{n})$. The paper Touil et al. [2017] discusses in Section 3 various selections for calculating the displacement step $\alpha^{(k)}$.

In the rest of this section, we prove that the complementary gap and the function f_μ decrease for a given displacement step. The proof of this result depends essentially on the following lemma.

Lemma 12.6 *Let* $(X, \boldsymbol{y}, S) \in \textit{int}\, \mathbb{S}^n_+ \times \mathbb{R}^m \times \textit{int}\, \mathbb{S}^n_+$, $(X, \boldsymbol{y}, S)$ *be obtained by applying scaling to* $(X, \boldsymbol{y}, S)$, *and* $(\Delta X, \Delta \boldsymbol{y}, \Delta S)$ *be a solution of System* (12.18). *Then we have*

(a) $\Delta X \bullet \Delta S = 0$.

(b) $X \bullet \Delta S + \Delta X \bullet S = \textit{trace}(H)$, *where* $H \triangleq \sigma\mu I - X \circ S$ *such that* $\sigma \in (0, 1)$ *and* $\mu = \frac{1}{n} X \bullet S$.

(c) $X^+ \bullet S^+ = \left(1 - \alpha\left(1 - \frac{\sigma}{2}\right)\right) X \bullet S$, $\forall \alpha \in \mathbb{R}$, *where* $X^+ \triangleq X + \alpha \Delta X$ *and* $S^+ \triangleq S + \alpha \Delta S$.

Proof By the first two equations of System (12.18), we get

$$\Delta X \bullet \Delta S = -\Delta X \bullet \mathcal{A}^\star \Delta \boldsymbol{y} = -(\mathcal{A}\Delta X)^\mathsf{T} \Delta \boldsymbol{y} = 0.$$

This proves item (a).

We prove item (b) by noting that

$$\begin{aligned} \operatorname{trace}(H) &= \operatorname{trace}(\sigma\mu I - X \circ S) \\ &= \operatorname{trace}(X \circ \Delta S + \Delta X \circ S) \\ &= \operatorname{trace}(X \circ \Delta S) + \operatorname{trace}(\Delta X \circ S) \;=\; X \bullet \Delta S + \Delta X \bullet S, \end{aligned}$$

where we used the last equation of System (12.18) to obtain the first equality. Item (c) is left as an exercise for the reader (see Exercise 12.1). The proof is complete. ■

The result in the following theorem is essentially those in [Touil et al., 2017, Lemmas 4.2 and 4.3]. The result given here in SDP is the counterpart of that in Theorem 11.3 for second-order cone programming, and its proof is also the same as the proof of its counterpart.

Theorem 12.3 *Let $(X, \boldsymbol{y}, S)$ and $(X^+, \boldsymbol{y}^+, S^+)$ be strictly feasible solutions of the pair of problems $(P|SDP_\mu)$ and $(D|SDP_\mu)$ with*

$$(X^+, \boldsymbol{y}^+, S^+) = (X + \alpha\Delta X,\ \boldsymbol{y} + \alpha\Delta\boldsymbol{y},\ S + \alpha\Delta S),$$

where α is a displacement step and $(\Delta X, \Delta\boldsymbol{y}, \Delta S)$ is the Newton direction. Then

(a) $X^+ \bullet S^+ < X \bullet S$. *(b)* $f_\mu(X^+) < f_\mu(X)$.

Proof Note that

$$X^+ \bullet S^+ = \left(1 - \alpha\left(1 - \frac{\sigma}{2}\right)\right) X \bullet S < X \bullet S,$$

where the equality follows from item (d) of Lemma 12.6 and the strict inequality follows from $(1 - \alpha(1 - \sigma/2)) < 1$ (as $\alpha > 0$ and $\sigma \in (0, 1)$). This proves item (a).

To prove item (b), note that

$$f_\mu(X^+) \simeq f_\mu(X) + \nabla_X f_\mu(X) \bullet (X^+ - X),$$

and hence

$$f_\mu(X^+) - f_\mu(X) \simeq \alpha \nabla_X f_\mu(X) \bullet \Delta X.$$

Since

$$\nabla_X f_\mu(X) = -\nabla^2_{XX} f_\mu(X)\Delta X,$$

we have

$$f_\mu(X^+) - f_\mu(X) \simeq -\alpha\, \Delta X \bullet \nabla^2_{XX} f_\mu(X)\Delta X < 0,$$

where the strict inequality follows from the positive definiteness of the Hessian matrix $\nabla^2_{XX} f_\mu(X)$ (as f_μ is strictly convex). Thus, $f_\mu(X^+) < f_\mu(X)$. The proof is complete. ■

Complexity estimates

In this part, we analyze the complexity of the proposed path-following algorithm for SDP. More specifically, we prove that the iteration-complexity of Algorithm 12.1 is bounded by

$$O\left(\sqrt{n}\ln\left[\epsilon^{-1} X^{(0)} \bullet S^{(0)}\right]\right).$$

Our proof depends essentially on the following two lemmas.

Lemma 12.7 *Let* $(X, \boldsymbol{y}, S) \in \mathcal{F}^{\circ}_{P|SDP} \times \mathcal{F}^{\circ}_{D|SDP}$, *and* $(X, \boldsymbol{y}, S)$ *be obtained by applying scaling to* $(X, \boldsymbol{y}, S)$ *with* $H = \sigma\mu I - XS$, *and* $(\Delta X, \Delta \boldsymbol{y}, \Delta S)$ *be a solution of System (12.18). For any* $\alpha \in \mathbb{R}$, *we set*

$$\begin{aligned}
&(X(\alpha), \boldsymbol{y}(\alpha), S(\alpha)) \triangleq (X, \boldsymbol{y}, S) + \alpha(\Delta X, \Delta \boldsymbol{y}, \Delta S),\\
&\mu(\alpha) \triangleq \tfrac{1}{n} X(\alpha) \bullet S(\alpha),\\
&V(\alpha) \triangleq X(\alpha) \circ S(\alpha) - \mu(\alpha) I.
\end{aligned}$$

Then

$$V(\alpha) = (1-\alpha)(X \circ S - \mu I) + \alpha^2\, \Delta X \circ \Delta S. \tag{12.19}$$

Proof See Exercise 12.2. ■

Lemma 12.8 *Let* $(X, \boldsymbol{y}, S) \in \mathcal{F}^{\circ}_{P|SDP} \times \mathcal{F}^{\circ}_{D|SDP}$, *and* $(X, \boldsymbol{y}, S)$ *be obtained by applying scaling to* $(X, \boldsymbol{y}, S)$ *such that* $\|XS - \mu I\| \leq \theta\mu$, *for some* $\theta \in [0, 1)$ *and* $\mu > 0$. *Let also* $(\Delta X, \Delta \boldsymbol{y}, \Delta S)$ *be a solution of System* (12.18), $H = \sigma\mu I - XS$, $\delta_X \triangleq \mu\|\Delta X X^{-1}\|_F, \delta_S \triangleq \|X\Delta S\|_F$. *Then, we have*

$$\delta_X \delta_S \leq \frac{1}{2}\left(\delta_X^2 + \delta_S^2\right) \leq \frac{\|H\|_F^2}{2(1-\theta)^2}. \tag{12.20}$$

Proof See Exercise 12.3. ■

The following theorem analyzes the behavior of one iteration of Algorithm 12.1. This theorem is due to [Touil et al., 2017, Theorem 4.6]. The result here in SDP is the counterpart of that in Theorem 11.4 for second-order cone programming, and its proof is also the same as the proof of its counterpart.

Theorem 12.4 *Let* $\theta \in (0, 1)$ *and* $\delta \in \left[0, \sqrt{n}\right)$ *be given such that*

$$\frac{\theta^2 + \delta^2}{2(1-\theta)^2\left(1 - \frac{\delta}{\sqrt{n}}\right)} \leq \theta \leq \frac{1}{2}. \tag{12.21}$$

Suppose that $(X, \boldsymbol{y}, S) \in \mathcal{N}_{SDP}(\mu)$ *and let* $(\Delta X, \Delta \boldsymbol{y}, \Delta S)$ *denote the solution of system* (12.18) *with*

$$H = \sigma\mu I - X \circ S \text{ and } \sigma = 1 - \delta/\sqrt{n}.$$

Then, we have

(a) $X^+ \bullet S^+ = (1 - \delta/\sqrt{n})\, X \bullet S$.

(b) $(X^+, \boldsymbol{y}^+, S^+) = (X, \boldsymbol{y}, S) + (\Delta X, \Delta \boldsymbol{y}, \Delta S) \in \mathcal{N}_{SDP}(\mu)$.

Proof Item (a) follows directly from item (c) of Lemma 12.6 with $\alpha = 1$ and $\sigma = 1 - \delta/\sqrt{n}$. We now prove item (b). Define

$$\mu^+ \triangleq \frac{1}{n} X^+ \bullet S^+ = (1 - \delta/n)\, \mu, \tag{12.22}$$

and let $(X, \boldsymbol{y}, S) \in \mathcal{N}_{\mathrm{SDP}}(\mu)$, we then have

$$\begin{aligned} \left\|\sigma\mu I - X \circ S\right\|_F^2 &\leq \left\|(\sigma - 1)\mu I\right\|_F^2 + \left\|\mu I - X \circ S\right\|_F^2 \\ &\leq \left((\sigma - 1)^2 r + \theta^2\right)\mu^2 = \left(\delta^2 + \theta^2\right)\mu^2. \end{aligned} \tag{12.23}$$

Since $\|X \circ S - \mu I\| \leq \theta\mu$ and $H = \sigma\mu I - X \circ S$, using Lemma 12.8 it follows that

$$\left\|\Delta \boldsymbol{x} X^{-1}\right\|_F \|X\Delta S\|_F \leq \frac{\left\|\sigma\mu I - X \circ S\right\|_F^2}{2(1 - \theta)^2\mu}. \tag{12.24}$$

Defining $V^+ \triangleq V(1) = X^+ \circ S^+ - \mu^+ I$ and using (12.19) with $\alpha = 1$, (12.24), (12.23), (12.21) and (12.22), we get

$$\|V^+\|_F = \|\Delta X \Delta S\|_F \leq \left\|\Delta X X^{-1}\right\|_F \|X\Delta S\|_F \leq \frac{\left\|\sigma\mu I - X \circ S\right\|_F^2}{2(1 - \theta)^2\mu} \leq \frac{(\delta^2 + \theta^2)\mu}{2(1 - \theta)^2} \leq \theta\left(1 - \frac{\delta}{\sqrt{n}}\right)\mu.$$

Consequently,

$$\left\|X^+ \circ S^+ - \mu^+ I\right\|_F \leq \theta\mu^+. \tag{12.25}$$

By using the right-hand side inequality in (12.20), and using (12.23) and (12.21), we have

$$\left\|\Delta X X^{-1}\right\|_F \leq \frac{\left\|\sigma\mu I - X \circ S\right\|_F}{(1 - \theta)\mu} \leq \frac{\sqrt{\delta^2 + \theta^2}}{(1 - \theta)} \leq \sqrt{2\theta\left(1 - \frac{\delta}{\sqrt{n}}\right)} < 1,$$

where the strict inequality follows from $\theta \leq \frac{1}{2}$ and $0 < 1 - \frac{\delta}{\sqrt{n}} < 1$.

One can easily see that $\left\|\Delta X X^{-1}\right\|_F < 1$ implies that $I + \Delta X X^{-1} \succ 0$, and therefore

$$X^+ = \Delta X + X = \left(I + \Delta X X^{-1}\right) X \succ 0.$$

Note that, from (12.25), we have $\lambda_{\min}(X^+ \circ S^+) = \lambda_{\min}(X^+ S^+) \geq (1 - \theta)\mu^+ > 0$, and therefore $X^+ S^+ \succ 0$. Since $X^+ \succ 0$ and X^+ and S^+ commute, we conclude that $S^+ \succ 0$. Using the first equation of system (12.18), we get

$$\mathcal{A}X^+ = \mathcal{A}(X + \Delta X) = \mathcal{A}X + A\Delta X = \boldsymbol{b}, \quad \text{and hence } X^+ \in \mathcal{F}^{\circ}_{\mathrm{P|SDP}}.$$

By using the second equation of system (12.18), we get

$$\mathcal{A}^\star \boldsymbol{y}^+ + S^+ = \mathcal{A}^\star(\boldsymbol{y} + \Delta\boldsymbol{y}) + (S + \Delta S) = \mathcal{A}^\star \boldsymbol{y} + S + \mathcal{A}^\star \Delta\boldsymbol{y} + \Delta S = C,$$

and hence $(\boldsymbol{y}^+, S^+) \in \mathcal{F}^\circ_{\text{D|SDP}}$. Thus, in view of (12.25), we deduce that $(X^+, \boldsymbol{y}^+, S^+) \in \mathcal{N}_{\text{SDP}}(\mu)$. Item (b) is therefore established. The proof is now complete. ∎

Now we present Corollary 12.2 in SDP to be the counterpart of Corollary 11.2 for second-order cone programming. The proof of Corollary 12.2 is also the same as that of its counterpart.

Corollary 12.2 *Let θ and δ as given in Theorem 12.4 and $(X^0, \boldsymbol{y}^0, S^{(0)}) \in \mathcal{N}_{SDP}(\mu)$. Then Algorithm 12.1 generates a sequence of points $\{(X^{(k)}, \boldsymbol{y}^{(k)}, S^{(k)})\} \subset \mathcal{N}_{SDP}(\mu)$ such that*

$$X^{(k)} \bullet S^{(k)} = \left(1 - \frac{\delta}{\sqrt{n}}\right)^k X^{(0)} \bullet S^{(0)}, \quad \forall k \geq 0.$$

Moreover, given a tolerance $\epsilon > 0$, Algorithm 12.1 computes an iterate $\{(X^{(k)}, \boldsymbol{y}^{(k)}, S^{(k)})\}$ satisfying $X^{(k)} \bullet S^{(k)} \leq \epsilon$ in at most

$$O\left(\sqrt{n} \ln\left(\frac{X^{(0)} \bullet S^{(0)}}{\epsilon}\right)\right)$$

iterations.

Proof Looking recursively at item (a) of Theorem 12.4, for each k we have that

$$X^{(k)} \bullet S^{(k)} = \left(1 - \frac{\delta}{\sqrt{n}}\right)^k X^{(0)} \bullet S^{(0)} \leq \epsilon.$$

By taking natural algorithm of both sides, we get $k \ln(1 - \delta/\sqrt{n}) \leq \ln(\epsilon/(X^{(0)} \bullet S^{(0)}))$, which holds only if

$$k\left(-\frac{\delta}{\sqrt{n}}\right) \leq \ln\left(\frac{\epsilon}{X^{(0)} \bullet S^{(0)}}\right),$$

or equivalently,

$$k \geq \frac{\sqrt{n}}{\delta} \ln\left(\frac{X^{(0)} \bullet S^{(0)}}{\epsilon}\right).$$

The proof is complete. ∎

In this section, we have presented and analyzed a path-following algorithm for SDP by extending the path-following algorithm that was presented in Section 11.5 for second-order cone programming. Exercises 12.5-12.8 aim to derive a homogeneous self-dual algorithm for SDP by extending the homogeneous self-dual algorithms that were presented in Section 10.7 for linear programming, and in Section 11.6 for second-order cone programming. In fact, there are several interior-point methods that can be extended from linear programming and second-order cone programming to SDP. In this context, we end this chapter by presenting Table 12.1, which shows gradient and Hessian derivatives for the logarithmic barriers applied on the most three well-known examples of conic constraints: the polyhedral, second-order cone and semidefinite constraints.

	Polyhedral cone [a]	Second-order cone [b]	Semidefinite cone [c]
Space	$\mathbb{R}^n$	$\mathbb{E}^n = \left\{ \boldsymbol{s} = \begin{pmatrix} s_0 \\ \bar{\boldsymbol{s}} \end{pmatrix} : \boldsymbol{s} \in \mathbb{R} \times \mathbb{R}^{n-1} \right\}$	$\mathbb{S}^n = \left\{ S \in \mathcal{M}^n : S = S^\mathsf{T} \right\}$
Cone	$\mathbb{R}^n_+ = \{ \boldsymbol{s} \in \mathbb{R}^n : \boldsymbol{s} \geq \mathbf{0} \}$	$\mathbb{E}^n_+ = \{ \boldsymbol{s} \in \mathbb{E}^n : s_0 \geq \|\bar{\boldsymbol{s}}\|_2 \}$	$\mathbb{S}^n_+ = \{ S \in \mathbb{S}^n : S \succeq 0 \}$
Feasibility constraint	$\boldsymbol{s}(\boldsymbol{x}) = A\boldsymbol{x} - \boldsymbol{b} \in \mathbb{R}^n_+$	$\boldsymbol{s}(\boldsymbol{x}) = A\boldsymbol{x} - \boldsymbol{b} \in \mathbb{E}^n_+$	$S(\boldsymbol{x}) = \sum_i x_i A_i - B \in \mathbb{S}^n_+$
Logarithmic barrier $\ell(\boldsymbol{x})$	$-\mathbf{1}^\mathsf{T} \ln \boldsymbol{s}(\boldsymbol{x})$	$-\ln \det(\boldsymbol{s}(\boldsymbol{x}))$	$-\ln \det(S(\boldsymbol{x}))$
Gradient $\nabla_x \ell(\boldsymbol{x})$	$-A^\mathsf{T} \boldsymbol{s}^{-1}$	$-A^\mathsf{T} \boldsymbol{s}^{-1}$	$-\mathcal{A}^\mathsf{T} \mathbf{vec}\left(S^{-1}\right)$
Hessian $\nabla^2_{xx} \ell(\boldsymbol{x})$	$A^\mathsf{T} S^{-2} A$	$A^\mathsf{T} Q_{\boldsymbol{s}^{-1}} A$	$\mathcal{A}^\mathsf{T} \left(S^{-1} \otimes S^{-1}\right) \mathcal{A}$

Table 12.1: Comparison of gradient and Hessian derivatives of the logarithmic barrier for polyhedral, second-order cone, and semidefinite constraints.

[a]**[Notations for specifying derivatives in $\mathbb{R}^n_+$]:** We use $\mathbf{1}$ to denote the vector of all ones of appropriate dimension. For any strictly positive vector $\boldsymbol{s} \in \mathbb{R}^n$, we define $\ln \boldsymbol{s} \triangleq (\ln s_1, \ldots, \ln s_n)^\mathsf{T}$ and $\boldsymbol{s}^{-1} \triangleq \left(s_1^{-1}, \ldots, s_n^{-1}\right)^\mathsf{T}$. We use $S \triangleq \operatorname{diag}(\boldsymbol{s})$ to denote the $n \times n$ diagonal matrix whose diagonal entries are $s_1, \ldots, s_n$. We define $P \triangleq P(\boldsymbol{s}) \triangleq S^{-1} A^\mathsf{T} (A S^{-2} A^\mathsf{T})^{-1} A S^{-1}$ to act as the orthogonal projection onto the range of AS^{-1}. We define $\sigma_i \triangleq P_{ii}$ for $i = 1, 2, \ldots, n$, and $\Sigma \triangleq \operatorname{diag}(\sigma)$.

[b]**[Notations for specifying derivatives in $\mathbb{E}^n_+$]:** All these notations have been introduced in Table 11.1 including the arrow-shaped matrix and the quadratic representation. Having the quadratic representation specified for $\mathbb{E}^n$, we can now introduce the vector $\boldsymbol{\sigma}$ in exactly the same way as we defined for any Euclidean Jordan algebra $\mathcal{J}$ (described early in this section) taking under consideration that the inner product "$\bullet$" is the standard dot product.

[c]**[Notations for specifying derivatives in $\mathbb{S}^n_+$]:** For two matrices $S, T \in \mathbb{S}^n$, the Kronecker product $S \otimes T$ is the $n^2 \times n^2$ block matrix whose i, j block is $s_{ij} T, i, j = 1, \ldots, n$. The vectorization of a matrix S, denoted $\mathbf{vec}(S)$, is the vector obtained by stacking the columns of S on top of one another. We use $\mathcal{A}$ to denote the matrix whose i^{th} column is $\mathbf{vec}(A_i)$. We now define the matrix Σ used for $\mathbb{S}^n$. By applying a Gram-Schmidt procedure to $\{S^{-1/2} A_i S^{-1/2}, i = 1, \ldots, n\}$, we obtain symmetric matrices $U_i, i = 1, \ldots, n$ having $\|U_i\| = 1$ for all i and $\operatorname{trace}(U_i U_j) = 0, i \neq j$, such that the linear span of $\{U_i, i = 1, \ldots, n\}$ is equal to the span of $\{S^{-1/2} A_i S^{-1/2}, i = 1, \ldots, n\}$. We then define $\Sigma \triangleq \sum_{i=1}^n U_i^2$.

Exercises

12.1 Prove item (c) in Lemma 12.6.

12.2 Prove Lemma 12.7.

12.3 Prove Lemma 12.8.

12.4 Implement Algorithm 12.1 and test it on the pair (P|SDP) and (D|SDP) with

$$C \triangleq \mathrm{Diag}(5;8;8;5) \in \mathbb{S}^4, \;\; \boldsymbol{b} \triangleq (1;1;1;2) \in \mathbb{R}^4, \;\; A_4 \triangleq I \in \mathbb{S}^4,$$

and A_i's, $i = 1, 2, 3$, are so that

$$(a_i)_{jk} \quad \triangleq \quad \begin{cases} 1, & \text{if } \; j = k = i, \text{ or } j = k = i + 1, \\ -1, & \text{if } \; j = i, k = j + 1, \text{ or } j = i + 1, k = i, \\ 0, & \text{otherwise.} \end{cases}$$

Take $X^{(0)} = \frac{1}{2} I \in \mathbb{S}^4$, $\boldsymbol{y}^{(0)} = (1.5; 1.5; 1.5; 1.5) \in \mathbb{R}^4$, and

$$S^{(0)} = \begin{bmatrix} 2 & 1.5 & 0 & 0 \\ 1.5 & 3.5 & 1.5 & 0 \\ 0 & 1.5 & 3.5 & 1.5 \\ 0 & 0 & 1.5 & 2 \end{bmatrix}$$

as your initial strictly feasible points. You may also take $\epsilon = 10^{-6}, \sigma = 0.1$ and $\rho = 0.99$.

12.5 In this and the following exercises, we practice extending interior-point methods from linear programming and second-order cone programming to SDP. In particular, this and the following exercises aim to derive a homogeneous self-dual algorithm for SDP by extending the homogeneous self-dual algorithms that were presented in Section 10.7 for linear programming, and in Section 11.6 for second-order cone programming. In this exercise, we ask the reader to write the homogeneous model for the pair (P|SDP) and (D|SDP).

12.6 Write the search direction system corresponding to the homogeneous model obtained in Exercise 12.5.

12.7 State the generic homogeneous algorithm for solving the pair (P|SDP) and (D|SDP) based on the search direction system obtained in Exercise 12.6.

12.8 Let $\epsilon_0 > 0$ be the residual error at a starting point, and $\epsilon > 0$ be a given tolerance. Estimate (without proof) the computational complexity (worst behavior) of the algorithm obtained in Exercise 12.7. This computational complexity must be in terms of the rank of the underlying positive semidefinite cone (n).

Notes and sources

Semidefinite programming is a branch of mathematical optimization that emerged in the late 20th century. It was initially developed by mathematicians and computer scientists as an extension of linear and convex programming. The concept of semidefinite programming can be traced back to the works of Leonid Khachiyan, who introduced a polynomial-time algorithm for solving linear matrix inequalities in 1979; see Khachiyan [1979]. The pivotal moment for the formalization of semidefinite programming was the development of the interior-point method for semidefinite programming problems by Yinyu Ye in the 1990s (refer to Ye [1997]). Since then, semidefinite programming has found numerous applications in optimization, control theory, combinatorial optimization, and other fields due to its ability to handle a wide range of complex optimization problems.

In this chapter, readers delved into semidefinite optimization problems, studied some of their combinatorial applications, and explored the concept of semidefinite programming duality, shedding light on the intriguing interplay between primal and dual formulations. Furthermore, readers studied efficient primal-dual methods that had been developed for solving semidefinite programming problems, providing readers with valuable tools to address complex optimization challenges across different disciplines.

As we conclude this chapter, it is worth noting that the cited references and others, such as Boyd et al. [2004], Ben-Tal and Nemirovski [2001], Renegar [2001], Davidsson [2013], Blekherman et al. [2013], Wolkowicz et al. [2000], Aliprantis and Tourky [2007], also serve as valuable sources of information pertaining to the subject matter covered in this chapter. The code that created Figure 12.2 is due to StackExchange [2018]. Exercises 12.5-12.7 are due to Potra and Sheng [1998].

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APPENDIX A

SOLUTIONS TO CHAPTER EXERCISES

1.1 (a) (i).
(b) (iii).
(c) (i).
(d) (i).
(e) (i).
(f) (ii).
(g) (ii).
(h) (iv).
(i) (iv).
(j) (iii).
(k) (iii).
(l) (iv).
(m) (i).
(n) (i).
(o) (iv).
(p) (ii).

1.2 (a) A true proposition.
(b) Not a proposition.
(c) Not a proposition.
(d) A true proposition.
(e) Not a proposition.
(f) A true proposition.
(g) Not a proposition.
(h) A true proposition.
(i) Not a proposition.
(j) Not a Proposition.
(k) Not a proposition.
(l) A true proposition.
(m) Not a Proposition.
(n) Not a proposition.

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1.3 (a) Today is not Thursday.

(b) There is pollution in New Jersey.

(c) $2 + 1 \neq 3$.

(d) Sara's first answer to item (l) in Exercise 1.1 was correct.

(e) The Summer in Santiago is hot.

(f) The summer in Santiago is not hot or not bearable.

(g) The summer in Santiago is hot and humid.

(h) The sun is shining in Santiago's sky and I am not going to the nearest beach.

(i) The sun is shining in Santiago's sky and I am not going to the nearest beach or not doing a little physical exercise.

1.4 (a) I did not buy a lottery ticket this week.

(b) I bought a lottery ticket this week or I won the million dollar jackpot on Friday.

(c) I bought a lottery ticket this week and I won the million dollar jackpot on Friday.

(d) I did not buy a lottery ticket this week and I did not win the million dollar jackpot on Friday.

1.5 (a) $P \wedge Q$. (b) $P \wedge \neg Q$. (c) $\neg P \wedge \neg Q$. (d) $P \vee Q$.

1.6
- If you have a passing score on the final exam, then you will receive a passing grade for the course.
- You will receive a passing grade for the course if you get a passing score on the final exam.
- Receiving a passing grade for the course is necessary for getting a passing score on the final exam.
- Getting a passing score on the final exam is sufficient to receive a passing score for the course.
- You will get a passing score on the final exam only if you receive a passing grade for the course.

1.7 (a) We are on the line TF, so it is false.

(b) We cannot be on the line TF, so it is true.

(c) We cannot be on the line TF, so it is true.

(d) We are on the line FF, so it is true.

(e) We are on the line TT or the line FF only, so it is true.

(f) We are on the line TF, so it is false.

(g) We are on the line TT, so it is true.

1.8 Restating the theorem gives us "If x and y are odd integers, then their sum $x + y$ is even." From here, all odd integers can be represented as $2k + 1$ for some integer k. That is,

$$x = 2n + 1 \text{ and } y = 2m + 1 \text{ for some integers } n \text{ and } m.$$

It follows that

$$x + y = (2n + 1) + (2m + 1) = 2(m + n + 1).$$

Given that m and n are integers, $(m + n + 1)$ is also an integer. Thus, x is an integer multiplied by 2, hence it must be an even integer.

1.9 The implication statement is “If the Sun is shrunk to the size of your head, then the Earth will be the size of the pupil of your eye”. So the contrapositive is “If the Earth will not be the size of the pupil of your eye, then the Sun is not shrunk to the size of your head”, the converse is “If the Earth will be the size of the pupil of your eye, then the Sun is shrunk to the size of your head”, and the is inverse “If the Sun is not shrunk to the size of your head, then the Earth will not be the size of the pupil of your eye”.

1.10 (a) Any statement "If P then Q" for which P and Q are both true or both false. Take, for instance, the statement "If 1+1=2 then 2-1=1".

(b) Impossible.

1.11
- The contrapositive of this theorem is “If x is odd, then x^2 is odd”. An odd integer can be represented as $2n + 1$, where n is some integer.

$$\text{Now } x = 2n + 1, \text{ which implies that } x^2 = (2n + 1)^2 = 2(2n^2 + 2n) + 1.$$

Given that n is an integer, $(2n^2 + 2n)$ must also be an integer. This means that x^2 must also be an odd integer, proving that the theorem is true.

- The theorem says that “x^2 is odd $\leftrightarrow$ x is odd”.

($\rightarrow$) The part that “If x is odd, then x^2 is odd" was already proved in item (a).

($\leftarrow$) To prove the part that “If x^2 is odd, then x is odd", note that the contrapositive of this statement is "If x is even, then x^2 is even". Given that n is an integer, an even number can be represented as $2n$ where n is some integer.

$$\text{Now } x = 2n, \text{ which implies that } x^2 = (2n)^2 = 4n^2 = 2(2n^2).$$

Given that n is an integer, $2n^2$ must also be an integer. This means that x^2 must also be an even integer, proving that the theorem is true.

1.12 (a)

$$\begin{aligned}
\neg[P \leftrightarrow Q] &\equiv \neg[(P \rightarrow Q) \wedge (Q \rightarrow P)] && \text{(The hint)}\\
&\equiv \neg(P \rightarrow Q) \vee \neg(Q \rightarrow P) && \text{(DeMorgan's law)}\\
&\equiv \neg(\neg P \vee Q) \vee \neg(\neg Q \vee P) && \text{(Implication law)}\\
&\equiv (\neg\neg P \wedge \neg Q) \vee (\neg\neg Q \wedge \neg P) && \text{(DeMorgan's law)}\\
&\equiv (P \wedge \neg Q) \vee (Q \wedge \neg P). && \text{(Double negation law)}
\end{aligned}$$

(b)

$$\begin{aligned}
\neg[P \oplus Q] &\equiv \neg[(P \vee Q) \wedge \neg(P \wedge Q)] && \text{(The hint)}\\
&\equiv \neg(P \vee Q) \vee \neg\neg(P \wedge Q) && \text{(DeMorgan's law)}\\
&\equiv \neg(P \vee Q) \vee (P \wedge Q) && \text{(Double negation law)}\\
&\equiv (\neg P \wedge \neg Q) \vee (P \wedge Q). && \text{(Double negation law)}
\end{aligned}$$

1.13 We construct the following truth table.

P	Q	$P \leftrightarrow Q$	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$
F	F	T	T	T	T
F	T	F	T	F	F
T	F	F	F	T	F
T	T	T	T	T	T

Because the truth values for $P \leftrightarrow Q$ and $(P \to Q) \wedge (Q \to P)$ are identical, this imply that they are logically equivalent.

1.14 (a) Below is "$P \wedge \neg P$" truth table.

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

(b) Below is "$(P \vee \neg Q) \to Q$" truth table.

P	Q	$\neg Q$	$(P \vee \neg Q)$	$(P \vee \neg Q) \to Q$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

(c) Below is "$(P \vee Q) \to (P \wedge Q)$" truth table.

P	Q	$P \vee Q$	$P \wedge Q$	$(P \vee Q) \to (P \wedge Q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(d) Below is "$(P \to Q) \leftrightarrow (\neg Q \to \neg P)$" truth table.

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$	$(P \to Q) \leftrightarrow (\neg Q \to \neg P)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

(e) Below is "$P \oplus P$" truth table.

P	$P \oplus P$
T	F
F	F

(f) Below is "$P \oplus \neg Q$" truth table.

P	Q	$\neg Q$	$P \oplus \neg Q$
T	T	F	T
T	F	T	F
F	T	F	F
F	F	T	T

(g) Below is "$\neg P \oplus \neg Q$" truth table.

P	Q	$\neg P$	$\neg Q$	$\neg P \oplus \neg Q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	F

(h) Below is "$(P \oplus Q) \wedge (P \oplus \neg Q)$" truth table.

P	Q	$\neg Q$	$P \oplus Q$	$P \oplus \neg Q$	$(P \oplus Q) \wedge (P \oplus \neg Q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	F

(i) Below is "$P \rightarrow \neg Q$" truth table.

P	Q	$\neg Q$	$P \rightarrow \neg Q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

(j) Below is "$\neg P \leftrightarrow Q$" truth table.

P	Q	$\neg P$	$\neg P \leftrightarrow Q$
T	T	F	F
T	F	F	T
F	T	T	T
F	F	T	F

(k) Below is "$(P \rightarrow Q) \vee (\neg P \rightarrow Q)$" truth table.

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$\neg P \rightarrow Q$	$(P \rightarrow Q) \vee (\neg P \rightarrow Q)$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	T	T
F	F	T	T	T	F	T

(l) Below is "$(P \leftrightarrow Q) \vee (\neg P \leftrightarrow Q)$" truth table.

P	Q	$\neg P$	$P \leftrightarrow Q$	$\neg P \leftrightarrow Q$	$(P \leftrightarrow Q) \vee (\neg P \leftrightarrow Q)$
T	T	F	T	F	T
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	T	F	T

(m) Below is "$(P \wedge Q) \vee R$" truth table.

P	Q	R	$P \wedge Q$	$(P \wedge Q) \vee R$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

(n) "$(P \wedge Q) \wedge R$" truth table.

P	Q	R	$(P \wedge Q) \wedge R$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

(o) Below is "$(P \vee Q) \vee R)$" truth table.

P	Q	R	$P \vee Q$	$(P \vee Q) \vee R$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

(p) Below is "$(P \wedge Q) \vee \neg R$" truth table.

P	Q	R	$\neg R$	$P \wedge Q$	$(P \wedge Q) \vee \neg R$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	T	F	T

(q) Below is "$P \to (\neg Q \vee R)$" truth table.

P	Q	R	$\neg Q$	$(\neg Q \vee R)$	$P \to (\neg Q \vee R)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

(r) Below is "$(P \to Q) \vee (\neg P \to R)$" truth table.

P	Q	R	$\neg P$	$P \to Q$	$\neg P \to R$	$(P \to Q) \vee (\neg P \to R)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

(s) Below is "$(P \to Q) \wedge (\neg P \to R)$" truth table.

P	Q	R	$\neg P$	$P \to Q$	$\neg P \to R$	$(P \to Q) \wedge (\neg P \to R)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	F	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	F

(t) Below is "$(P \leftrightarrow Q) \vee (\neg Q \leftrightarrow R)$" truth table.

P	Q	R	$\neg Q$	$P \leftrightarrow Q$	$\neg Q \leftrightarrow R$	$(P \leftrightarrow Q) \vee (\neg Q \leftrightarrow R)$
T	T	T	F	T	F	T
T	T	F	F	T	T	T
T	F	T	T	F	T	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

1.15 Let F be "Lillian is forceful", G be "Lillian will be a good executive", E be "Lillian is efficient", and C be "Lillian is creative". Then the problem statement can be formulated in the following propositional logical model.

$$\text{Problem model: } \begin{cases} F \vee C, \\ F \to G, \\ \neg(E \wedge C), \\ \neg E \to (F \vee G). \end{cases}$$

The desired conclusion is to conclude that the following propositional formula

$$[(F \vee C) \wedge (F \to G) \wedge (\neg(E \wedge C)) \wedge (\neg E \to (F \vee G))] \longrightarrow G \tag{A.1}$$

is a tautology. Otherwise, we cannot conclude that Lillian will be a good executive. Using the DeMorgan's law and double negation law, the conditional statement (the fourth proposition) in (A.1) can be written as

$$[(F \vee C) \wedge (F \to G) \wedge \neg(E \wedge C) \wedge (E \vee F \vee G)] \longrightarrow G. \tag{A.2}$$

Now, our objective is to establish whether the conditional statement presented in equation (A.1) possesses the property of being a tautology or not. The most straightforward approach to ascertain this characteristic is by constructing a truth table. Nevertheless, it is essential to acknowledge that many rows in this truth table can be omitted from our analysis due to the fact that the only circumstance under which the formula can yield a "false" result is when the variable G is assigned a "false" value, and concurrently, all the components on the left-hand side of the implication are assigned "true" values. Therefore, if we can successfully demonstrate that for each and every conceivable combination of values where G is "false", the entirety of the expressions on the left side of the implication is also "false", we can conclude that the implication in question will invariably hold true.

F	E	C	G	(1) $F \vee C$	(2) $F \rightarrow G$	(3) $\neg(E \wedge C)$	(4) $(E \vee F \vee G)$	(1) $\wedge$ (2) $\wedge$ (3) $\wedge$ (4)
F	F	F	F	F	T	T	F	F
F	F	T	F	T	T	T	F	F
F	T	F	F	F	T	T	T	F
F	T	T	F	T	T	F	T	F
T	F	F	F	T	F	T	T	F
T	F	T	F	T	F	T	T	F
T	T	F	F	T	F	T	T	F
T	T	T	F	T	F	F	T	F

As you can see from the table, for every combination of values where G is false, everything to the left of the implication is also false. This means the original implication in (A.1) can never be false, so G must be true. Thus, Lillian will be a good executive.

1.16 Implication law, associative law, commutative law, distributive law, contradiction law, domination law and idempotent law.

1.17 All the given implications are tautologies as shown below.

(a)

$$\begin{aligned}
[\neg P \wedge (P \vee Q)] \rightarrow Q &\equiv [(\neg P \wedge P) \vee (\neg P \wedge Q)] \rightarrow Q \\
&\equiv [F \vee (\neg P \wedge Q)] \rightarrow Q \\
&\equiv (\neg P \wedge Q) \rightarrow Q \\
&\equiv \neg(\neg P \wedge Q) \vee Q \\
&\equiv (P \vee \neg Q) \vee Q \\
&\equiv P \vee (\neg Q \vee Q) \\
&\equiv P \vee T \equiv T.
\end{aligned}$$

(b)

$$\begin{aligned}
[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R) &\equiv \neg[(\neg P \vee Q) \wedge (\neg Q \vee R)] \vee (\neg P \vee R) \\
&\equiv [\neg(\neg P \vee Q) \vee \neg(\neg Q \vee R)] \vee (\neg P \vee R) \\
&\equiv [(P \wedge \neg Q) \vee (Q \wedge \neg R)] \vee (\neg P \vee R) \\
&\equiv (P \wedge \neg Q) \vee (Q \wedge \neg R) \vee \neg P \vee R \\
&\equiv [(P \wedge \neg Q) \vee \neg P] \vee [(Q \wedge \neg R) \vee R] \\
&\equiv [(P \vee \neg P) \wedge (\neg Q \vee \neg P)] \vee [(Q \vee R) \wedge (\neg R \vee R)] \\
&\equiv [T \wedge (\neg Q \vee \neg P)] \vee [(Q \vee R) \wedge T] \\
&\equiv (\neg Q \vee \neg P) \vee (Q \vee R) \\
&\equiv (\neg Q \vee Q) \vee (\neg P \vee R) \\
&\equiv T \vee (\neg P \vee R) \equiv T.
\end{aligned}$$

(c)

$$\begin{aligned}
[P \wedge (P \to Q)] \to Q &\equiv [P \wedge (\neg P \vee Q)] \to Q \\
&\equiv [(P \wedge \neg P) \vee (P \wedge Q)] \to Q \\
&\equiv [F \vee (P \wedge Q)] \to Q \\
&\equiv (P \wedge Q) \to Q \\
&\equiv \neg(P \wedge Q) \vee Q \\
&\equiv (\neg P \vee \neg Q) \vee Q \\
&\equiv \neg P \vee (\neg Q \vee Q) \\
&\equiv \neg P \vee T \equiv T.
\end{aligned}$$

(d)

$$\begin{aligned}
[(P \vee Q) \wedge (P \to R) \wedge (Q \to R)] \to R &\equiv \neg[(P \vee Q) \wedge (\neg P \vee R) \wedge (\neg Q \vee R)] \vee R \\
&\equiv [\neg(P \vee Q) \vee \neg(\neg P \vee R) \vee \neg(\neg Q \vee R)] \vee R \\
&\equiv (\neg P \wedge \neg Q) \vee (P \wedge \neg R) \vee (Q \wedge \neg R) \vee R \\
&\equiv (\neg P \wedge \neg Q) \vee (P \wedge \neg R) \vee [(Q \vee R) \wedge (\neg R \vee R)] \\
&\equiv (\neg P \wedge \neg Q) \vee (P \wedge \neg R) \vee [(Q \vee R) \wedge T] \\
&\equiv (\neg P \wedge \neg Q) \vee (P \wedge \neg R) \vee Q \vee R \\
&\equiv [(\neg P \wedge \neg Q) \vee Q] \vee [(P \wedge \neg R) \vee R] \\
&\equiv [(\neg P \vee Q) \wedge (\neg Q \vee Q)] \vee [(P \vee R) \wedge (\neg R \vee R)] \\
&\equiv [(\neg P \vee Q) \wedge T] \vee [(P \vee R) \wedge T] \\
&\equiv (\neg P \vee Q) \vee (P \vee R) \\
&\equiv (\neg P \vee P) \vee (Q \vee R) \\
&\equiv T \vee (Q \vee R) \equiv T.
\end{aligned}$$

1.18 Starting with item (a), we have

$$\begin{aligned}
P \leftrightarrow Q &\equiv (P \to Q) \wedge (Q \to P) \\
&\equiv (\neg P \vee Q) \wedge (\neg Q \vee P) \\
&\equiv [(\neg P \vee Q) \wedge \neg Q] \vee [(\neg P \vee Q) \wedge P] \\
&\equiv [(\neg P \wedge \neg Q) \vee (Q \wedge \neg Q)] \vee [(\neg P \wedge P) \vee (Q \wedge P)] \\
&\equiv [(\neg P \wedge \neg Q) \vee F] \vee [F \vee (Q \wedge P)] \\
&\equiv (\neg P \wedge \neg Q) \vee (Q \wedge P) \quad \text{(This is item (b))} \\
&\equiv \neg(P \vee Q) \vee (Q \wedge P) \\
&\equiv \neg\neg[\neg(P \vee Q) \vee (Q \wedge P)] \\
&\equiv \neg[(P \vee Q) \wedge \neg(Q \wedge P)] \equiv \neg[P \oplus Q]. \quad \text{(This is item (c)).}
\end{aligned}$$

1.19 To construct a DNF having the given truth table, note that

P	Q	R	Statement	Equivalent To
T	T	F	T	$P \wedge Q \wedge \neg R$
T	F	T	T	$P \wedge \neg Q \wedge R$
F	T	F	T	$\neg P \wedge Q \wedge \neg R$

A DNF would be $(P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$.

1.20 (a) $\neg(A \lor B) \equiv \neg A \land \neg B$ (CNF).

(b) $\neg(A \land B) \equiv \neg A \lor \neg B \equiv (\neg A \lor \neg B) \land (A \lor \neg A)$ (CNF).

(c) $A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$ (CNF).

1.21 $(P \land \neg P \land Q) \lor (Q \land \neg Q) \lor (R \land \neg R) \lor (P \land \neg Q \land \neg R \land \neg P)$.

1.22 Note that $P \land Q \land (\neg R \lor S \lor \neg T) \equiv (P \land Q \land \neg R) \lor (P \land Q \land S) \lor (P \land Q \land \neg T)$, which is in DNF. According to Remark 1.5, the given proposition is satisfiable.

1.23 (a) (iii). (b) (iv). (c) (ii). (d) (iv). (e) (ii). (f) (iii).

1.24 (a) False. The value $x = -1$ can make this proposition false.

(b) False. It is true only if we take $x = 0$, but this is not in the domain.

(c) True. The value $x = 2$ can make this proposition true.

(d) False. The value $x = -1$ can make this proposition false.

(e) True. The value $x = 1$ can make this proposition true.

1.25 Note that $\mathbb{N}$ is defined as $\mathbb{N} \triangleq \{1, 2, 3, \ldots\}$.

(a) Take $P(x, y) =$ "$x + y = x + y$".

(b) Take $P(x, y) =$ "$xy = \frac{1}{2}$".

(c) Take $P(x, y) =$ "$xy = y$".

(d) Take $P(x, y) =$ "$xy = x$".

1.26 The negation of the given proposition is $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y \neq 0$ as it is seen below.

$$\begin{aligned}
\neg(\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 0) &\equiv \forall x \in \mathbb{R}, \neg(\forall y \in \mathbb{R}, x + y = 0) \\
&\equiv \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \neg(x + y = 0) \\
&\equiv \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y \neq 0.
\end{aligned}$$

1.27 Denote by D the set of all people.

(a) $\exists x \in D, \neg L(x)$.

(b) $\exists x \in D, (I(x) \land \neg C(x) \land \neg A(x))$.

(c) $\forall x \in D, ([A(x) \lor C(x)] \land I(x) \to L(x))$.

1.28 The negation of the sentence "Some children do not like mimes" is the sentence "Every child has some mime that he/she likes", which can be symbolized as $\forall x \in D, [\text{Child}(x) \to (\exists y \in D, [\text{Mime}(y) \land \text{Like}(x, y)])]$. As another approach, the quantified statement "Some children do not like mimes" is symbolized as $\exists x \in D, [\text{Child}(x) \land (\forall y \in D, [\text{Mime}(y) \to \neg\text{Like}(x, y)])]$, which can be negated as $\forall x \in D, [\text{Child}(x) \to \neg(\forall y \in D, [\text{Mime}(y) \to \neg\text{Like}(x, y)])]$.

1.29 We let D_1 be the set of all houses and D_2 be the set of all owners. We also define the following two predicates: $\text{OWN}(v, x) =$ "The house v has owner x", and $\text{ADJ}(v, u) =$ "The house v is adjacent to house u".

(a) $\forall x \in D_2, \exists v \in D_1, \text{OWN}(v, x)$.

(b) $\forall v \in D_1, \forall u \in D_1, \forall x \in D_2, (\mathrm{ADJ}(v,u) \ \wedge \ \mathrm{OWN}(v,x) \longrightarrow \neg\mathrm{OWN}(u,x))$.

(c) $\forall v \in D_1, \forall x \in D_2, (\mathrm{OWN}(v,x)) \longrightarrow [\neg\exists y \in D_2, ((y \neq x) \wedge \mathrm{OWN}(v,y))])$.

1.30 Denote by D the set of all people. We also define the following three predicates: $\mathrm{Fr}(x,y)$ = "The two persons x and y are friends", $\mathrm{Sm}(x)$ = "The person x smokes", and $\mathrm{Ca}(x)$ = "The person x has a cancer".

(a) $\forall x \in D, \forall y \in D, \forall z \in D, (\mathrm{Fr}(x,y) \ \wedge \ \mathrm{Fr}(y,z) \longrightarrow \mathrm{Fr}(x,z))$.

(b) $\forall x \in D, (\neg[\exists y \in D, \mathrm{Fr}(x,y)] \longrightarrow \mathrm{Sm}(x))$.

(c) $\forall x \in D, (\mathrm{Sm}(x) \longrightarrow \mathrm{Ca}(x))$.

(d) $\forall x \in D, \forall y \in D, (\mathrm{Fr}(x,y) \longrightarrow [\mathrm{Sm}(x) \longleftrightarrow \mathrm{Sm}(y)])$.

2.1 (a) (iii). (b) (iv). (c) (ii). (d) (ii). (e) (iv). (f) (iv). (g) (ii). (h) (iii). (i) (ii). (j) (iv).

2.2 Suppose the principle is true when $n = k$, for some $k \in \mathbb{N}$. Then there is some natural number s such that $m < sk$. Choosing the same s, we have $m < sk < s(k+1)$, so the principle is true when $n = k+1$. The proof is complete.

2.3 By induction, let $P(n)$ be

$$\text{“}1 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}\text{”}. \tag{A.3}$$

First, we look at the base case for $n = 1$ to see whether or not $P(1)$ is true. Substituting $n = 1$ into both sides of the equation, we see

$$\sum_{i=1}^{1} i^2 = 1^2 = \frac{(1)(1+1)(2*1+1)}{6} = 1$$

is true. Now, we look at the inductive step for $n = k$ to see whether or not $P(k+1)$ is true. By substituting all n's with $(n+1)$ on the right side of (A.3), we get

$$\begin{aligned}\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n^2+3n+2)(2n+3)}{6} \\ &= \frac{2n^3+9n^2+13n+6}{6}.\end{aligned} \tag{A.4}$$

If the inductive hypothesis $P(k)$, which is

$$\text{“}1 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}\text{”}$$

is true, then $P(k+1)$, which is

$$\begin{aligned}
1 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k^2+k)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6},
\end{aligned}$$

is also true because this matches in (A.4) that we found by substituting $(n+1)$ into the right-hand side of (A.3). We conclude if $P(k)$ is true, then $P(k+1)$ is true for any $k \geq 1$ which proves the inductive conclusion.

2.4 Define the following predicate:

$P(n)$ = "The cardinality of the powerset of a finite set A is equal to 2^n if the cardinality of A is n."

Base case (for $n = 0$): The set A with cardinality 0 is the empty set $\emptyset$. Its powerset (set of all subsets) is $\emptyset$. Since $\mathcal{P}(A) = \{\emptyset\}$, we have $|\mathcal{P}(A)| = 1 = 2^0$. Therefore $P(0)$ is true.

Inductive step: Assume that $P(k)$ is true. To prove that $P(k+1)$ is also true, let B be any set of cardinality $k+1$. Enumerating the elements of B as $a_1, a_2, a_3, \ldots, a_k, a_{k+1}$. That is, $B = \{a_1, a_2, a_3, \ldots, a_k, a_{k+1}\}$ and $|B| = k+1$. Define

$$A = \{a_1, a_2, a_3, \ldots, a_k\},$$

then $|A| = k$ and $|\mathcal{P}(A)| = 2^k$ (by the inductive hypothesis).

Note that $B = A \cup \{a_{k+1}\}$ and that every subset of A is also a subset of B.

Now any subset of B either contains a_{k+1} or it does not. Every subset C (of B) that does not contain a_{k+1} is also a subset of A, of which there are 2^k (by the inductive hypothesis of $P(k)$). For every subset C not containing a_{k+1}, there is another subset of the form $C \cup \{a_{k+1}\}$ containing it. Since there are 2^k such possible subsets C, there are 2^k subsets of B containing a_{k+1}. We now have that B has 2^k subsets not containing a_{k+1} and 2^k containing it. Therefore, $|\mathcal{P}(B)| = 2^k + 2^k = 2^{k+1}$, and hence $P(k+1)$ is true.

Thus, by induction on n, $P(n)$ is true for all $n \in \mathbb{N}$.

2.5 The base case for $n = 1$ is seen to be true as $T(1) = 1 = 3^1 - 2 = 1$, where the first equality follows from the piecewise function. Now, assume that the inductive hypothesis holds for $n = k$, i.e., $T(k) = 3^k - 2$. Then

$$T(k+1) = 3T(k) + 4 = 3(3^k - 2) + 4 = 3(3^k) - 2 = 3^{k+1} - 2. \tag{A.5}$$

Therefore, if $T(k) = 3^k - 2$, then $T(k+1) = 3^{k+1} - 2$ for an integer $k \geq 0$. This proves the inductive conclusion. Thus, by induction on n, the proof is complete.

2.6 (a) False. To see this, let a be any real number, and consider $A = \{a\}$, $B = \{\{a\}\}$, and $C = \{\{\{a\}\}\}$. Then $A \in B$ and $B \in C$, but $A \notin C$ because C only contains one element which is not A.

Note that the statement becomes true if the symbol membership symbol "$\in$" is replaced with the inclusion symbol "$\subseteq$". That is, the following statement is true.

$$(A \subseteq B) \wedge (B \subseteq C) \longrightarrow (A \subseteq C).$$

(b) False. Take, for example, $A = \{1, 2\}, B = \{1, 3\}$ and $C = \{2, 3, 4\}$.

2.7 (a) Since $|S| = 5$, the number of subsets of S is $2^{|S|} = 2^5 = 32$.

(b) Note that $T = \{22, 24, 26, \ldots, 38, 40\}$ and that $|T| = 10$. So, the number of subsets of T is $2^{|T|} = 2^{10} = 1024$.

2.8 $(A \cup C') \cap B' = \{\text{u, w, r, t}\}$.

2.9 We have to show that the relation $\mathcal{R}$ is reflexive, symmetric, and transitive. For reflexivity, each person x has the same birthday of him/herself. For symmetry, if x and y have the same birthday, then y and x have the same birthday. For transitivity, if x and y have the same birthday and y and z have the same birthday, then x and z have the same birthday. Thus, the relation $\mathcal{R}$ is an equivalence relation. Since there are 365 ways two people can have the same birthday, $\mathcal{R}$ has 365 equivalence classes.

2.10 The exponential function $f : \mathbb{R} \to [0, \infty)$ defined by $f(x) = e^x$ is an injection. To see this, let $x, y \in \mathbb{R}$ such that $e^x = e^y$, then $x = \ln e^x = \ln e^y = y$. Now, we show that the function f is a surjection. To see this, note that any $y \in [0, \infty)$, we can choose $x = \ln y$ and have $f(x) = e^x = e^{\ln y} = y$. Therefore f is both an injective and surjective function. Thus, f is a bijection.

The exponential function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = e^x$ is not a bijection because it is not a surjection. For instance, because $e^x > 0$ for any $x \in \mathbb{R}$, there is no $x \in \mathbb{R}$ such that $g(x) = -1$.

3.1 (a) (iii). (c) (ii). (e) (i). (g) (ii).
(b) (iv). (d) (i). (f) (i). (h) (i).

3.2 Let S be the set of positive integers for which the statement is true. Since $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, $1 \in S$. Assume that $k \in S$, i.e., $\lim_{n\to\infty} \frac{(\ln n)^k}{n}$. Using Hospital's rule, it follows that

$$\lim_{n\to\infty} \frac{(\ln n)^{k+1}}{n} = \lim_{n\to\infty} \frac{(k+1)(\ln n)^k}{n} = (k+1) \lim_{n\to\infty} \frac{(\ln n)^k}{n} = 0.$$

Thus, $k + 1 \in S$. This proves the result by induction.

3.3 Using Table 3.1 (see also the result in Exercise 2.3), we have

$$\begin{aligned}\sum_{k=30}^{60} k^2 &= \sum_{k=1}^{60} k^2 - \sum_{k=1}^{29} k^2\\ &= \frac{60\cdot 61\cdot 62}{6} - \frac{29\cdot 30\cdot 31}{6}\\ &= 10\cdot 61\cdot 62 - 29\cdot 5\cdot 31\\ &= 37{,}820 - 4{,}495 = 33{,}325.\end{aligned}$$

3.4 Note that

$$\frac{1}{(2k+1)(2k-1)} = \frac{(2k+1)/2 - (2k-1)/2}{(2k+1)(2k-1)} = \frac{1/2}{2k-1} - \frac{1/2}{2k+1}.$$

The sequence of partial sums is

$$\begin{aligned}S_n &= \sum_{k=1}^{n} \frac{1}{(2k+1)(2k-1)}\\ &= \sum_{k=1}^{n}\left(\frac{1/2}{2k-1} - \frac{1/2}{2k+1}\right)\\ &= \left(\frac{1/2}{1} - \frac{1/2}{3}\right) + \left(\frac{1/2}{3} - \frac{1/2}{5}\right) + \cdots + \left(\frac{1/2}{2n-1} - \frac{1/2}{2n+1}\right)\\ &= \frac{1}{2} - \frac{1/2}{2n+1} \longrightarrow \frac{1}{2}.\end{aligned}$$

Thus, the series converges to 1/2.

3.5 (i) Note that $\lim_{k\to\infty} 3^{-k} = \lim_{k\to\infty} 1/3^k = 0$. It follows that

$$\lim_{k\to\infty} \frac{1}{2+3^{-k}} = \frac{1}{2+0} = \frac{1}{2}.$$

Using the divergence test, the series $\sum_{k=1}^{\infty} \frac{1}{2+3^{-k}}$ diverges.

(ii) Using Theorem 3.3 (iv), we have $\lim_{k\to\infty}(0.3)^k = 0$. It follows that

$$\lim_{k\to\infty} \frac{1}{2+(0.3)^k} = \frac{1}{2+0} = \frac{1}{2}.$$

Using the divergence test, the series $\sum_{k=1}^{\infty} \frac{1}{2+(0.3)^k}$ diverges.

3.6 (i) Since $2/3 < 1$, using the geometric series we have

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2.$$

The series converges to 2.

(*ii*) Note that

$$\sum_{k=1}^{\infty}\left(\frac{4}{5}\right)^{3-k} = \left(\frac{4}{5}\right)^{3}\sum_{k=1}^{\infty}\left(\frac{5}{4}\right)^{k}.$$

Since $5/4 > 1$, using the geometric series, we conclude that the series diverges.

(*iii*) Using the geometric series, we have

$$\sum_{k=2}^{\infty} x^k = \frac{x^2}{1-x}, \quad \text{if } |x| < 1.$$

Replacing x with $-x$, we get

$$\sum_{k=2}^{\infty}(-1)^k x^k = \sum_{k=2}^{\infty}(-x)^k = \frac{(-x)^2}{1-(-x)} = \frac{x^2}{1+x}.$$

When $x = 2/5$, we have

$$\sum_{k=2}^{\infty}(-1)^k\left(\frac{2}{5}\right)^{k-2} = \left(\frac{5}{2}\right)^2\sum_{k=2}^{\infty}(-1)^k\left(\frac{2}{5}\right)^k = \left(\frac{5}{2}\right)^2\frac{(\frac{2}{5})^2}{1+\frac{2}{5}} = \frac{1}{1+\frac{2}{5}} = \frac{5}{7}.$$

The series converges to 5/7.

(*iv*) Using the geometric series, we have

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}, \quad \text{if } |x| < 1.$$

Differentiating both sides with respect to x, we get

$$\sum_{k=1}^{\infty} kx^k = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2}, \quad \text{if } |x| < 1.$$

When $x = 2/3$, we have

$$\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k = \frac{1}{\left(1-\frac{2}{3}\right)^2} = \frac{1}{\left(\frac{1}{3}\right)^2} = 9.$$

The series converges to 9.

3.7 If $M \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, and $\boldsymbol{x} \in \mathbb{R}^n$ is a nonzero vector, then

$$\boldsymbol{x}^\top M^{-1}\boldsymbol{x} = \boldsymbol{x}^\top(MM^{-1}M)^{-1}\boldsymbol{x} = (M^{-1}\boldsymbol{x})^\top MM^{-1}\boldsymbol{x} = \boldsymbol{y}^\top M\boldsymbol{y} > 0,$$

where $\boldsymbol{y} = M^{-1}\boldsymbol{x}$. Thus, M^{-1} is also positive definite.

3.8 Using Proposition 3.1, we have

$$
\begin{aligned}
H^{\perp} &= \left\{x \in \mathbb{R}^2 : x_1 - 2x_2 = 0\right\}^{\perp} \\
&= \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : \begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0\right\}^{\perp} \\
&= \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} u, u \in \mathbb{R}\right\} = \left\{x \in \mathbb{R}^2 : 2x_1 = -x_2\right\}.
\end{aligned}
$$

3.9 Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\lambda \in [0,1]$. We have

$$
\begin{aligned}
f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &= \max_{i=1,\dots,m} f_i(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \\
&\leq \max_{i=1,\dots,m} (\lambda f_i(\boldsymbol{x}) + (1-\lambda) f_i(\boldsymbol{y})) \\
&\leq \lambda \max_{i=1,\dots,m} f_i(\boldsymbol{x}) + (1-\lambda) \max_{i=1,\dots,m} f_i(\boldsymbol{y}) \\
&= \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}),
\end{aligned}
$$

where the first inequality follows from the convexity of $f_i, i = 1, 2, \dots, m$.

3.10 Assume that $S_1, S_2, \dots, S_m$ are convex sets. Let $\boldsymbol{x}, \boldsymbol{y} \in \cap_{i=1}^m S_i$ and $\lambda \in [0,1]$. By the convexity of $S_i, i = 1, 2, \dots, m$, we have $\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in S_i, i = 1, 2, \dots, m$. It follows that $\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in \cap_{i=1}^m S_i$. Therefore, $\cap_{i=1}^m S_i$ is convex. If an infinite number of convex sets intersect, the result is also correct and a proof similar to the above one can be used. In addition, if the number of convex sets is infinite but countable, a proof by induction can be also thought of.

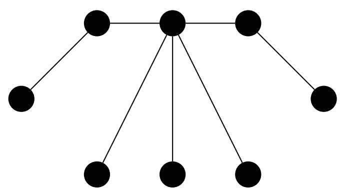
3.11 Under the assumption in Farkas' lemma (Version II). It is clear that if (2′) holds, then so does (2). Now, assume that (2) is true, i.e., $A\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} < 0$ and $\boldsymbol{x} \geq \boldsymbol{0}$, and let $\bar{\boldsymbol{x}} = -\boldsymbol{x}/(\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x})$. Then, the nonnegativity of $\bar{\boldsymbol{x}}$ follows from the nonnegativity of $\boldsymbol{x}$ and the negativity of $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$. Moreover,

$$
\boldsymbol{c}^{\mathsf{T}}\bar{\boldsymbol{x}} = \boldsymbol{c}^{\mathsf{T}}\left(\frac{-1}{\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}}\boldsymbol{x}\right) = -\frac{1}{\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}}\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} = -1, \quad \text{and} \quad A\bar{\boldsymbol{x}} = A\left(\frac{-1}{\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}}\boldsymbol{x}\right) = -\frac{1}{\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}}A\boldsymbol{x} = \boldsymbol{0}.
$$

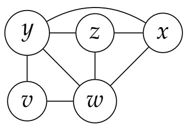
This proves that if (2) holds, then so does (2′).

4.1 (a) (iv). (b) (ii). (c) (i). (d) (iv). (e) (ii). (f) (iii). (g) (i). (h) (iii). (i) (iv). (j) (iii).

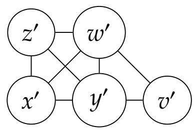
4.2 A spanning tree so that no vertex has a degree of 4 is shown to the right. (Other answers might be possible).



4.3 Yes. To see this, labeling the vertices of G_1 and that of G_2 creates a matching between them in a way that preserves adjacency as shown below.



G_1

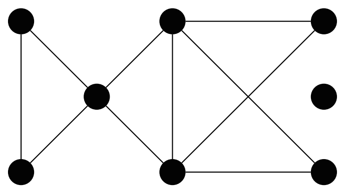


G_2

4.4 (*a*) The answer is no, because we have an odd number of odd degree vertices.

(*b*) No. The justification for this answer is that there are two 4-degree vertices. The minimum number of vertices (outside of themselves) that those two vertices must be connected to is 6. Therefore, all 8 vertices are connected. However, the degree sequence has 2 vertices of degree 0, contradicting this.

(*c*) Yes. To justify this answer, a graph that represents the given degree sequence is given in shown to the right.



4.5 Assume that we have an acyclic graph denoted as G, and the addition of the edge (x, y) to G results in the creation of a cycle for every pair of vertices x and y in the vertex set V where $(x, y) \notin E$. In order to establish that the graph G is a tree, we need to demonstrate its connectedness. Let us take two arbitrary vertices, say u and v, within G. If u and v are not already adjacent, the introduction of the edge (u, v) leads to the formation of a cycle. However, it is important to note that all edges in this cycle, except for (u, v), are part of the original graph G. Consequently, there exists a path between u and v within G. Given that our choice of vertices u and v was entirely arbitrary, we can conclude that G is indeed a connected graph.

4.6 Only one graph is possible, which is the complete graph K_5.

4.7 The graph $K_{5,7}$ has no a Hamiltonian cycle because the sizes of its partition subsets are not identical. The graph $K_{5,7}$ has no a Hamiltonian path because the sizes of its partition subsets differ by more than one.

4.8 (*a*) The tight upper bound is $k + 1$. We may need to add a single color, i.e., change the color of one endpoint to a new color.

(*b*) While k is still an upper bound, the tight upper bound would be $k - 1$. Let us consider worst case which is the complete graph K_n. The graph K_n is not $(n - 1)$-colorable (Theorem 4.12). Now, removing any edge from K_n reduces the number of colors required to properly color it to $n - 1$.

4.9 The statement is false. From Theorem 4.12, the complete graph K_{d+2} is not $(d + 1)$-colorable. Let $v_1, v_2, \ldots, v_d, v_{d+1}, v_{d+2}$ be the vertices of K_{d+2}. Let also G be the graph formed by adding a new vertex u of degree d and adjacent to each of $v_1, v_2, \ldots, v_d$. Then G has a vertex of degree d but it is not $(d + 1)$-colorable.

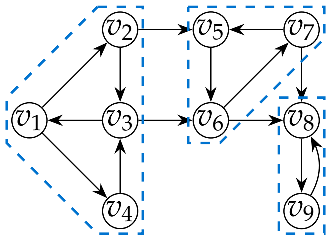


Figure A.1: The strongly connected components of the digraph of Exercise 4.10.

4.10 The strongly connected components of the given digraph are surrounded by blue dashed polygons in Figure A.1.

5.1 (a) (iv). (b) (ii). (c) (iii). (d) (iii). (e) (iv).

5.2 (a) The recurrence relation $T(n) = 2T(n/2) + n$ is of the merge sort algorithm. In Example 5.5, we used the iteration method to solve this recurrence. Now, we use the guess-and-confirm method to obtain the same answer. In this recurrence, each n should be a power of 2 (i.e., $n = 1, 2, 4, 8, \ldots$), otherwise we have either non-integer inputs or inputs with undefined references. It is given that $T(1) = 1$. Note that

$$\begin{array}{lcl} T(2) & = & 2T(1) + 2 = 4 = 2\log 2 + 2, \\ T(4) & = & 2T(2) + 4 = 12 = 4\log 4 + 4, \\ T(8) & = & 2T(4) + 8 = 32 = 8\log 8 + 8, \\ T(16) & = & 2T(8) + 16 = 80 = 16\log 16 + 16, \\ & \vdots & \\ T(k) & = & k\log k + k. \end{array}$$

It seems that $T(n) = n\log n + n$ for every $n = 2^i$ and $i = 0, 1, 2, 3, \ldots$. We prove this by induction on n. The base case is trivial: $T(1) = 1 = 1\log 1 + 1$. Assume that the statement is true for all $m < k$, i.e., $T(m) = m\log m + m$ for any $m < k$. Now, we prove that $T(k) = k\log k + k$. Since k is a power of 2, we have $k = 2^i$ (and hence $i = \log k$) for some $i \in \mathbb{N}$. Since $2^{i-1} < k$, we particularly have $T(2^{i-1}) = 2^{i-1}\log 2^{i-1} + 2^{i-1} = 2^{i-1}(i-1) + 2^{i-1} = i(2^{i-1})$ by the inductive step. It follows that $T(k) = 2T(k/2) + k = 2T(2^{i-1}) + k = 2i(2^{i-1}) + k = i(2^i) + k = k\log k + k$. This confirms that $T(n) = n\log n + n$.

(b) The recurrence relation $T(n) = 3T(n-1) + 4$ is the one given in (5.1). We use the guess-and-confirm method to solve this recurrence. It is given that $T(1) = 1$. Note that

$$\begin{aligned} T(2) &= 3T(1) + 4 = 7 = 3^2 - 2, \\ T(3) &= 3T(2) + 4 = 25 = 3^3 - 2, \\ T(4) &= 3T(3) + 4 = 79 = 3^4 - 2, \\ &\vdots \\ T(k) &= 3^k - 2. \end{aligned}$$

It seems that $T(n) = 3^n - 2$ for $n = 1, 2, 3, \ldots$. We prove this by induction on n. The base case is trivial: $T(1) = 1 = 3^1 - 2$. Assume that the statement is true for $n = k$, i.e., $T(k) = 3^k - 2$ for some $k \in \mathbb{N}$. Now, we prove that $T(k+1) = 3^{(k+1)} - 2$. Note that

$$T(k+1) = 3T(k) + 4 = 3\left(3^k - 2\right) + 4 = 3\left(3^k\right) - 6 + 4 = 3^{k+1} - 2.$$

Thus, $T(n) = 3^n - 2$ for any $n = 1, 2, 3, \ldots$. This confirms the solution given in (7.10).

5.3 Using repeated substitutions, we have

$$\begin{aligned} T(1) &= 1 = (0-3)2^0 + 4, \\ T(2) &= 0 = (1-3)2^1 + 4, \\ T(3) &= 0 = (2-3)2^2 + 4, \\ T(3) &= 5T(2) - 8T(1) + 4T(0) = 4 = (3-3)2^3 + 4, \\ &\vdots \\ T(n) &= 5T(n-1) - 8T(n-2) + 4T(n-3) = (n-3)2^n + 4, \text{ for all } n \geq 1. \end{aligned}$$

We prove the nth-term guess by mathematical induction. The base case is trivial: $T(0) = 1 = (0-3)2^0 + 4$, $T(1) = 0 = (1-3)2^1 + 4$, and $T(2) = 0 = (2-3)2^2 + 4$. Assume that the statement is true for all $m < k$, i.e., $T(m) = (m-3)2^m + 4$ for any $m < k$. Now, we prove that $T(k) = (k-3)2^k + 4$. This can be seen from the following:

$$\begin{aligned} T(k) &= 5T(k-1) - 8T(k-2) + 4T(k-3) \\ &= 5((k-4)2^{(k-1)} + 4) - 8((k-5)2^{(k-2)} + 4) + 4((k-6)2^{(k-3)} + 4) \\ &= (5k)2^{(k-1)} - (20)2^{(k-1)} + 20 - (8k)2^{(k-2)} + (40)2^{(k-2)} - 32 \\ &\quad + (4k)2^{(k-3)} - (24)2^{(k-3)} + 16 \\ &= (2.5)(k)2^k - (10)2^k + 20 - (2k)2^k + (10)2^k - 32 \\ &\quad + (0.5)(k)2^k - (3)2^k + 16 = (k-3)2^k + 4. \end{aligned}$$

5.4 (a) Note that

$$T(n) = 5T(n-1) = 5(5T(n-2)) = 5(5(5T(n-3))) = \cdots = 5^k T(n-k).$$

Let $k = n$, then $T(n) = 5^n T(0) = 3(5^n)$.

(b) Note that

$$\begin{aligned} T(n) &= 2T(\tfrac{n}{2}) + n\log n \\ &= 2(2T(\tfrac{n}{2^2}) + \tfrac{n}{2}\log(\tfrac{n}{2})) + n\log n \\ &= 2^2T(\tfrac{n}{2^2}) + n(\log n - \log 2) + n\log n \\ &= 2^2T(\tfrac{n}{2^2}) + 2n\log n - n \\ &= 2^3T(\tfrac{n}{2^3}) + 3n\log n - n - 2n \\ &\vdots \\ &= 2^kT(\tfrac{n}{2^k}) + kn\log n - n\textstyle\sum_{i=0}^{k-1} i \\ &= 2^kT(\tfrac{n}{2^k}) + kn\log n - \tfrac{n}{2}(k-1)k. \end{aligned}$$

Let $n = 2^k$, then

$$T(n) = nT(1) + n\log^2 n - \frac{n}{2}(\log n - 1)\log n = cn + \frac{1}{2}n\log^2 n + \frac{1}{2}n\log n.$$

5.5 Let $g(x)$ be the generating function for the sequence $\{a_n\} = \{T(n)\}$. Then

$$g(x) = \sum_{k=0}^{\infty} T(k)\,x^k.$$

Using the recurrence relation, we have

$$\begin{aligned} g(x) &= T(0) + T(1)x + \sum_{k=2}^{\infty} T(k)x^k \\ &= 1 + 3x + \sum_{k=2}^{\infty}(-T(k-1) + 6T(k-2))x^k \\ &= 1 + 3x - x\sum_{k=2}^{\infty} T(k-1)x^{k-1} + 6x^2\sum_{k=2}^{\infty} T(k-2)x^{k-2} \\ &= 1 + 3x + xT(0) - x\sum_{k=1}^{\infty} T(k-1)x^{k-1} + 6x^2\sum_{k=2}^{\infty} T(k-2)x^{k-2} \\ &= 1 + 4x - x\sum_{k=0}^{\infty} T(k)x^k + 6x^2\sum_{k=0}^{\infty} T(k)x^k = 1 + 4x - xg(x) + 6x^2g(x). \end{aligned}$$

It follows that

$$g(x) = 1 + 4x - xg(x) + 6x^2g(x).$$

Solving for $g(x)$, we get

$$\begin{aligned} g(x) &= \frac{1+4x}{1+x-6x^2} \\ &= \frac{1+4x}{(1-2x)(1+3x)} \\ &= \frac{\frac{-1}{5}(1-2x) + \frac{6}{5}(1+3x)}{(1-2x)(1+3x)} = \frac{6}{5}\frac{1}{1-2x} - \frac{1}{5}\frac{1}{1+3x}. \end{aligned}$$

Using the geometric series (5.7), we get

$$g(x) = \frac{6}{5}\sum_{k=0}^{\infty} 2^k x^k - \frac{1}{5}\sum_{k=0}^{\infty}(-3)^k x^k = \sum_{k=0}^{\infty}\left(\underbrace{\frac{6}{5}\,2^k - \frac{1}{5}\,(-3)^k}_{T(k)}\right)x^k.$$

Therefore, $T(n) = (1/5)\left((6)\,2^n + (-1)^{n+1}3^n\right)$ for any $n \geq 0$.

5.6 Let $g(x)$ be the generating function for the sequence $\{a_n\} = \{T(n)\}$. Then

$$g(x) = \sum_{k=0}^{\infty} T(k)\,x^k.$$

Using the recurrence relation, we have

$$\begin{aligned}
g(x) &= T(0) + T(1)x + \sum_{k=2}^{\infty} T(k)x^k \\
&= T(0) + T(1)x + \sum_{k=2}^{\infty}(T(k-1) + T(k-2))x^k \\
&= T(0) + T(1)x + \sum_{k=2}^{\infty} T(k-1)x^k + \sum_{k=2}^{\infty} T(k-2)x^k \\
&= T(0) + T(1)x + x\sum_{k=2}^{\infty} T(k-1)x^{k-1} + x^2\sum_{k=2}^{\infty} T(k-2)x^{k-2} \\
&= T(0) + T(1)x - xT(0) + x\sum_{k=1}^{\infty} T(k-1)x^{k-1} + x^2\sum_{k=2}^{\infty} T(k-2)x^{k-2} \\
&= x + x\sum_{k=0}^{\infty} T(k)x^k + x^2\sum_{k=0}^{\infty} T(k)x^k = x + xg(x) + x^2 g(x).
\end{aligned}$$

It follows that

$$g(x) = x + xg(x) + x^2 g(x).$$

Solving for $g(x)$, we get $g(x) = x/(x^2 + x - 1)$.

It can be shown that

$$1 - x - x^2 = -(x+\alpha)(x+\beta), \quad \text{and hence} \quad g(x) = \frac{1}{\sqrt{5}}\left(\frac{\beta}{\beta + x} - \frac{\alpha}{\alpha + x}\right),$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.

Note that $\alpha = -1/\beta$. It follows that

$$\frac{\beta}{\beta + x} = \frac{1}{1 + x/\beta} = \frac{1}{1 - \alpha x} = \sum_{k=0}^{\infty} \alpha^k x^k,$$

where the last equality was obtained by the geometric series (5.7). Similarly, we also have

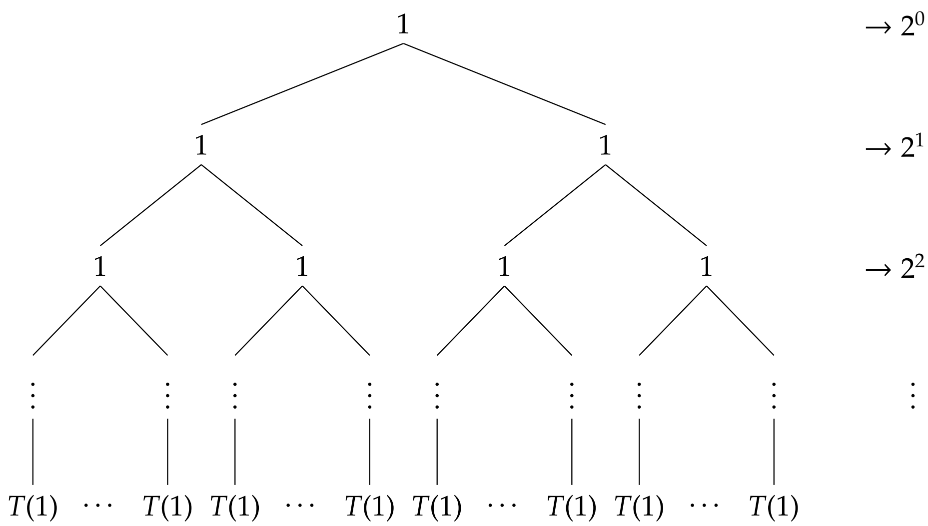
$$\frac{\alpha}{\alpha + x} = \sum_{k=0}^{\infty} \beta^k x^k.$$

Thus, we have

$$g(x) = \frac{1}{\sqrt{5}} \left(\sum_{k=0}^{\infty} \alpha^k x^k - \sum_{k=0}^{\infty} \beta^k x^k \right) = \sum_{k=0}^{\infty} \left(\underbrace{\frac{1}{\sqrt{5}} \left(\alpha^k - \beta^k \right)}_{T(k)} x^k \right).$$

Therefore, $T(n) = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n)$ for any $n \geq 0$.

5.7 The recursion tree for the recurrence $T(n) = 2T(n-1) + 1$ is shown below.



From the tree,the total cost at level i is $(3/2)^i n$. The height of the tree, h, is obtained when $n - h = 1$. So, we have $h = n - 1$. It follows that

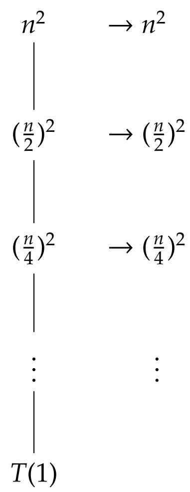
$$T(n) = \sum_{i=0}^{n-1} 2^i = \frac{1 - 2^n}{1 - 2} = 2^n - 1 = \Theta(2^n).$$

5.8 (a) The recursion tree for the recurrence $T(n) = T(n/2) + n^2$ is shown below.

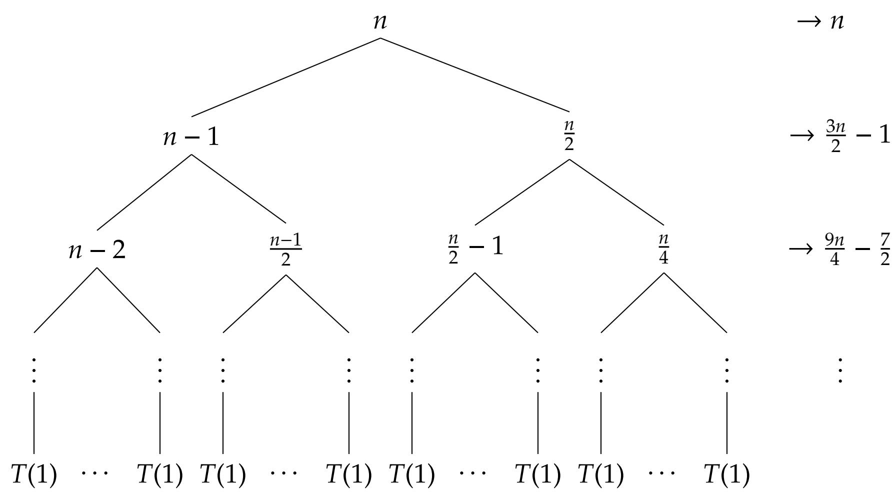
From the recursion tree shown to the right, the total cost at level i is $n/2^{2i}$. The height of the tree, h, is obtained when $n/2^h = 1$. So, we have $h = \log n$. It follows that

$$\begin{aligned} T(n) &= \sum_{i=0}^{\log n} \left(\frac{1}{2}\right)^{2i} n^2 \\ &\le n^2 \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^{i} \\ &= \frac{1}{1-\frac{1}{4}} n^2 = \frac{4}{3} n^2. \end{aligned}$$

Thus, using the asymptotic notation introduced in Chapter 7, we have $T(n) = O(n^2)$.



(b) The recursion tree for the recurrence $T(n) = T(n-1) + T(n/2) + n$ is shown below.



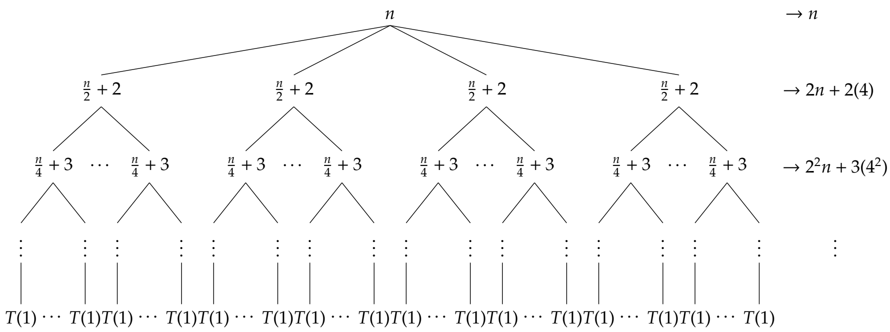
Since we are finding an upper bound, the height of the tree is decided by the *longest* path, which is the leftmost path. So, the height of the tree, h, is obtained when $n - h = 1$. Hence, $h = n - 1$.

Also, because we are finding an upper bound, we can drop the subtracted constant at each level. Then the total cost at level i can be taken to be $(3/2)^i n$. It follows that

$$T(n) \le \sum_{i=1}^{n-1} \left(\frac{3}{2}\right)^i n = \frac{1-\left(\frac{3}{2}\right)^n}{1-\frac{3}{2}} n = 2n\left(\left(\frac{3}{2}\right)^n - 1\right),$$

where we used the finite geometric series formula (3.3) to obtain the first equality. Thus, using the asymptotic notation introduced in Chapter 7, we have $T(n) = O(n(3/2)^n)$.

5.9 (a) The recursion tree for the recurrence $T(n) = 4T(\frac{n}{2} + 2) + n$ is shown below.



Although growing, the growth rate of the constant in front of 4^i is much smaller, so we can ignore it. So, we can take $2^i n + 4^i$ to be the total cost at level i. Also, the height of the tree, h, can obtained when $n/2^h = 1$. So, we have $h = \log n$. It follows that

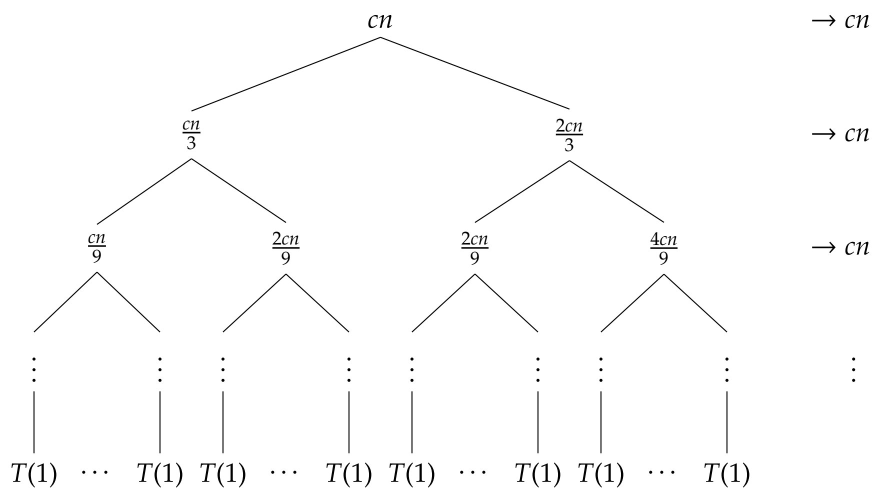
$$T(n) \geq \sum_{i=0}^{logn} \left(2^i n + 4^i\right) = n \sum_{i=0}^{\log n} 2^i + \sum_{i=0}^{\log n} 4^i.$$

Using the finite geometric series formula, we have

$$T(n) \geq \frac{1 - 2^{(1+\log n)}}{1-2} n + \frac{1 - 4^{(1+\log n)}}{1-4} = (2n-1)n + \frac{4n^2 - 1}{3} = \frac{10}{3}n^2 - n - \frac{1}{3}.$$

Thus, using the asymptotic notation introduced in Chapter 7, we have $T(n) = \Omega(n^2)$.

(b) The recursion tree for the recurrence $T(n) = T(n/3) + T(2n/3) + cn$ is shown in what follows.



Since we are finding a lower bound, the height of the tree is decided by the *shortest* path, which is the leftmost path. So, the height of the tree, h, is obtained when $n/3^h = 1$. Hence, $h = \log_3 n$. From the tree, the total cost at level i is cn. It follows that

$$T(n) \geq \sum_{i=0}^{log_3 n} cn = cn(1 + \log_3 n) = cn\left(1 + \frac{\log n}{\log 3}\right).$$

Thus, using the asymptotic notation introduced in Chapter 7, we have $T(n) = \Omega(n \log n)$.

6.1 (a) (ii). (b) (i). (c) (iv). (d) (ii). (e) (iii). (f) (iv). (g) (iii). (h) (ii). (i) (i). (j) (i). (k) (i). (l) (iv). (m) (i). (n) (ii). (o) (iii). (p) (iv).

6.2 From the binomial theorem it follows that

$$\begin{aligned}
(x+y)^5 &= \sum_{k=0}^{5} \binom{5}{k} x^{5-k} y^k \\
&= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5 \\
&= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.
\end{aligned}$$

6.3 A set consisting of n elements possesses a grand total of 2^n distinct subsets. Each of these subsets can contain zero elements, one element, two elements, and so forth, up to n elements. We find that there are $\binom{n}{0}$ subsets that contain precisely zero elements, $\binom{n}{1}$ subsets with one element, $\binom{n}{2}$ subsets with two elements, and so on, until $\binom{n}{n}$ subsets with a full complement of n elements. Hence, the summation $\sum_{k=0}^{n} \binom{n}{k}$ accounts for the overall count of subsets within a set that has n elements. By equating the two formulas we have developed to represent the number of subsets within a set of n elements, we find that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

6.4 Using the binomial theorem with $x = 2$ and $y = 1$, we have

$$3^n = (2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

6.5 Consider a set T consisting of n elements. Let us select an element, say a, from this set, and define another set S as the complement of a within T. It is important to note that within the set T, there are $\binom{n}{k}$ subsets containing exactly k elements. However, we can observe that a subset of T containing k elements can be in one of two forms: either it includes a along with $k-1$ elements from S, or it comprises k elements from S and does not include a. Given that there are $\binom{n-1}{k-1}$ subsets comprising $k-1$ elements from S, it follows that there are $\binom{n-1}{k-1}$ subsets consisting of k elements from T that include the element a. Moreover, there are $\binom{n-1}{k}$ subsets comprising k elements from T that do not include the element a, as $\binom{n-1}{k}$ is the count of subsets of k elements from S. Consequently, we arrive at the relationship: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

6.6 Consider a scenario where there are m items in one set and n items in a second set. The total count of ways to select r elements from the combination of these sets is denoted as $\binom{m+n}{r}$. Alternatively, we can approach the selection of r elements from the union by choosing k elements from the second set and $r-k$ elements from the first set, where k is an integer satisfying $0 \leq k \leq r$. Since there are $\binom{n}{k}$ ways to pick k elements from the second set and $\binom{m}{r-k}$ ways to pick $r-k$ elements from the first set, we can apply the product principle of counting to determine that this can be accomplished in $\binom{m}{r-k}\binom{n}{k}$ ways. Consequently, the total count of ways to choose r elements from the union is also equivalent to $\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$. We have now established two distinct expressions for the count of ways to select r elements from the union of a set containing m items and another set with n items. By equating these expressions, we arrive at Vandermonde's identity.

6.7 Define the following predicate:

$P(m)$ = "If there are n_k ways to perform task T_k for $k = 1, \ldots, m$, then there are $n_1 \times n_2 \times \cdots \times n_m$ ways to perform all m tasks."

Base case (for $m = 1$): $P(1)$ is trivially true.

Inductive step: Assume that $P(k)$ is true for all $k \leq m$. To prove that $P(m+1)$ is also true, assume that we have $m+1$ tasks, and that task T_k can be performed in n_k ways, for $k = 1, 2, \ldots, m+1$. We would like to show that the entire set of $m+1$ tasks can be performed in $n_1 \times n_2 \times \cdots \times n_m \times n_{m+1}$ ways.

Let T be a (large) task that comprises of the combination of tasks T_1 through T_m. Now, to do the tasks T_1 through T_{m+1}, we must perform the pair of tasks T and T_{m+1}. From the inductive hypothesis, we conclude that the task T can be done in n ways where $n = n_1 \times n_2 \times \cdots \times n_m$ ways. Now, the inductive hypothesis can be also used to conclude that the number of ways to perform the pair of tasks T and $Tm+1$ is $n \times n_{m+1}$ ways. Thus, the entire set of $m+1$ tasks can be performed in $n \times n_{m+1} = n_1 \times n_2 \times \cdots \times n_m \times n_{m+1}$ ways.

6.8 Define the following predicate:

$P(m)$ = "If a procedure can be done in one of n_1 ways, in one of n_2 ways, …, or in one of n_m ways, then the total number of ways to do the procedure is $n_1 + n_2 + \cdots + n_m$ ways, provided that none of the set of n_i ways of doing the procedure is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m$."

Base case (for $m = 1$): $P(1)$ is trivially true.

Inductive step: Assume that $P(k)$ is true for all $k \leq m$. To prove that $P(m+1)$ is also true, assume that a procedure can be done in one of n_1 ways, in one of n_2 ways, …, in one of n_m ways, or in one of n_{m+1} ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m+1$. We would like to show that the total number of ways to do the procedure is $n_1 + n_2 + \cdots + n_{m+1}$ ways.

Note that it is assumed that the procedure can be done in one of n_1 ways, in one of n_2 ways, …, in one of n_m ways. So, by the inductive hypothesis, we conclude that the procedure can also be done in one of n ways where $n = n_1 + n_2 + \cdots + n_m$ ways.

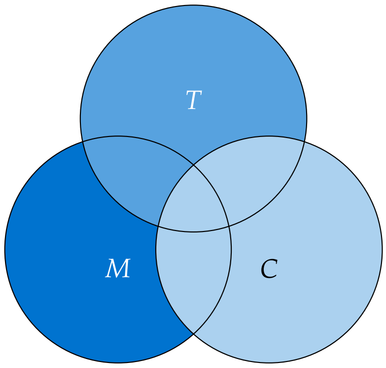


Figure A.2: A Venn diagram for Exercise 6.11.

Note that it is also assumed that the procedure can be done in one of n_{m+1} ways. Because none of the set of n_i ways of doing the procedure is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m+1$, it follows that none of the set of n ways of doing the procedure is the same as any of the set of n_{m+1} ways. Now, the inductive hypothesis can be also used to conclude that the procedure can be performed in $n + n_{m+1} = n_1 + n_2 + \cdots + n_m + n_{m+1}$ ways.

6.9 Suppose, in the contrary, that none of the k boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total number of objects would be at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where we have used the inequality $\lceil N/k \rceil < (N/k) + 1$. This contradicts the fact that there are a total of N objects.

6.10 Note that $A' \cap B \cap C = B \cap C \cap A' = B \cap C - A$. Then $|A' \cap B \cap C| = |B \cap C| - |A \cap B \cap C| = 19 - 11 = 8$.

6.11 Let S be the set of all people who took the survey, T be the set of all people who like tea, C be the set of all people who like coffee, and M be the set of all people who like milk. Then $|S| = 240, |T| = 91, |C| = 70, |T \cap C| = 31, |T' \cap C' \cap M'| = 91$, and $|T \cap C \cap M| = 7$. We are looking for $|T' \cap C' \cap M|$.

Note that $|T \cup C \cup M| = |S| - |(T \cup C \cup M)'| = |S| - |T' \cap C' \cap M'| = 240 - 91 = 149$. Note also that the subset $M \cap T' \cap C'$ is the dark-colored area in the Venn diagram in Figure A.2. It follows that

$$|M \cap T' \cap C'| = |M \cup T \cup C| - |T| - |C| + |T \cap C| = 149 - 91 - 70 + 31 = 19.$$

6.12 Note that the "for" loops in lines (5) and (8) are independent, but this pair of loops and "for" loops in lines (2) and (3) are nested. Let T_s be the number of times of executing the statement given in line (s) for each $s = 1, 2, 3, 4, 5, 6, 8, 9$. The initial value of "sum" is one. Each time the "sum" statement in each of lines (4), (6) and (9) is executed, 1 is added to "sum". Therefore, by the sum principle of counting, the value of "sum" after the fragment given in Algorithm 6.5 has been executed is equal to the number of ways to carry out the task T_1, plus the number of ways to carry out the task T_4, plus the number of ways to carry out the task T_6, plus the number of ways to carry out the task T_9. By using the product principle of counting, we can find the number of ways to carry out the task $T_s, s = 1, 2, 3, 4, 5, 6, 8, 9$. We added a column for these numbers in Algorithm A.1 (see the comments in gray in Algorithm A.1). Thus, the value of "sum" after the fragment given in Algorithm 6.5 has been executed is equal to

$$1 + n\,m + n\,m\,p + n\,m\,q = 1 + n\,m\,(1 + p + q).$$

6.13 Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain of f such that $f(x) = y$. Because the domain contains $k + 1$ or more elements and the codomain contains only k elements, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain. This means that f cannot be an injection.

Algorithm A.1: Algorithm 6.5 revisited

```
sum = 1                               // Task T1 is done in 1 way
for (i = 1; i ≤ n; i++) do            // T2 is done in n + 1 ways
    for (j = 0; j < m; j++) do        // T3 is done in n × (m + 1) ways
        sum = sum + 1                 // T4 is done in n × m ways
        for (k = p; k ≥ 1; k--) do    // T5 is done in n × m × (p + 1) ways
            sum = sum + 1             // T6 is done in n × m × p ways
        end
        for (r = q; r > 0; r--) do    // T8 is done in n × m × (q + 1) ways
            sum = sum + 1             // T9 is done in n × m × q ways
        end
    end
end
```

6.14 Note that the order in which the shirts can be selected does not matter, and the shirts can be repeated. Total number of different colored shirts is $n = 6$. The number of shirts to be selected is $r = 4$. Here, the desired number is equal to the number of 4-combinations with repetition allowed from a set with six elements. From Theorem 6.13, the shirts can be displayed in

$$C(6 + 4 - 1, 4) = C(9, 4) = C(9, 5) = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 126$$

different ways.

6.15 (*a*) We count permutations of 2 O's, 1 N, 1 E, 1 W, 1 R, and 1 D, a total of 7 symbols. By Theorem 6.14, the number of these is

$$\frac{7!}{2!\,1!\,1!\,1!\,1!\,1!} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2,520.$$

(*b*) Under the given condition, we should treat the O-block as a single unit "OO". We then have to count permutations of 1 N, 1 E, 1 W, 1 R, 1 D, and 1 "OO", a total of 6 symbols. The number of these is

$$\frac{6!}{1!\,1!\,1!\,1!\,1!\,1!} = 6! = 720.$$

6.16 (*a*) We count permutations of 3 P's, 2 E's, 1 I, 1 N, 1 A, and 1 L, a total of 9 symbols. By Theorem 6.14, the number of these is

$$\frac{9!}{3!\,2!\,1!\,1!\,1!\,1!} = 30,240.$$

(*b*) Under the given condition, we should treat the P-block as a single unit "PPP". We then have to count permutations of 2 E's, 1 I, 1 N, 1 A, 1 L, and 1 "PPP", a total of 8 symbols. The number of these is

$$\frac{8!}{2!\,1!\,1!\,1!\,1!\,1!} = 20,160.$$

7.1 (a) (iii). (b) (i). (c) (i). (d) (i). (e) (i). (f) (i).
(g) (ii). (h) (iv). (i) (iii). (j) (iv). (k) (iv). (l) (iv).
(m) (ii). (n) (iii). (o) (iv). (p) (iii). (q) (iii). (r) (iv).
(s) (ii). (t) (ii). (u) (iv). (v) (iv).

7.2 An algorithmic code that solves the given problem is stated in Algorithm A.2.

Algorithm A.2: The algorithm of Exercise 7.2

Input: An array $A[0 : n-1]$ of positive integers and length n
Output: The largest number of the array

```
find-maxi(A, n)
if (n = 1) then
    return A[0]
end
v1 = A[0]
v2 = find-maxi(A, n - 1)
if (v1 > v2) then
    return v_1
end
else
    return v_2
end
```

7.3 (a) Yes. The input for Algorithm A.2 is a finite array of positive numbers whose length is n.

(b) Yes. The output for Algorithm A.2 is the maximum in the array.

(c) Yes. Algorithm A.2 terminates when n is equal to 1.

(d) Yes. One can prove the following equality by mathematical induction.

$$\text{find-maxi}(A[i : n-1], n-i) = \max\{A[i], \text{find-maxi}(A[i+1 : n-1], n-i-1)\}.$$

7.4 Let $T(n)$ be the running time of Algorithm 7.12. Carrying out a line-by-line analysis for Algorithm 7.12, we have

Line	One-time Cost	Number of Times
2	c_1	1
3	c_2	$n+1$
4	c_3	n
5	c_4	n [worst-case]
8	c_5	1

The total runtime is

$$T(n) = c_1 + c_2(n+1) + c_3 n + c_4 n + c_5 = (c_2 + c_3 + c_4)n + (c_1 + c_2 + c_5).$$

7.5 Let $T(n)$ be the running time of Algorithm 7.33. Carrying out a line-by-line analysis for Algorithm 7.33, we have

Line	One-time Cost	Number of Times
1	c_1	1
2	c_2	$1 + \log n$
3	c_3	$\log n(1 + \log n)$
4	c_4	$\log^2 n$

The total runtime is

$$\begin{aligned} T(n) &= c_1 + c_2(1 + \log n) + c_3 \log n(1 + \log n) + c_4 \log^2 n \\ &= (c_3 + c_4)\log^2 n + (c_2 + c_3)\log n + c_1 + c_2. \end{aligned}$$

7.6 (a) Since the loop goes from p to q in increments of 1, and $p \le q$, the solution would be $q - p$. Since this loop includes q as part of the end condition ($\le q$), add one to the solution, making it $q - p + 1$.

(b) Since the loop goes from q to p in decrements of 1, and $p \le q$, the solution would be $q - p$. Since this loop includes p as part of the end condition ($\ge p$), add one to the solution, making it $q - p + 1$.

(c) Since the loop goes from p to q in increments of k, and $p \le q$, the solution would be $\lfloor (q-p)/k \rfloor + 1$. Since the loop includes q as part of the end condition ($\le q$), add one to the solution, making it $\lfloor (q-p)/k \rfloor + 2$.

(d) Since the loop goes from q to p in decrements of k, and $p \le q$, the solution would be $\lfloor (q-p)/k \rfloor + 1$. Since the loop includes p as part of the end condition ($\ge p$), add one to the solution, making it $\lfloor (q-p)/k \rfloor + 2$.

(e) The loop goes from p to q as follows: $p, kp, k^2p, k^3p, \ldots, q-1, q$. Note that $k^hp = q$ when $h = \log_k(q/p)$ So, the solution would be $\log_k(p/q)+1$. Since the loop includes q as part of the end condition ($\leq q$), add one to the solution, making it $\log_k(p/q)+2$.

(f) The loop goes from q to p as follows: $q, q/k, q/k^2, q/k^3, \ldots, p+1, p$. Note that $q/k^h = p$ when $h = \log_k(q/p)$. So, the solution would be $\log_k(p/q)+1$. Since the loop includes p as part of the end condition ($\geq p$), add one to the solution, making it $\log_k(p/q)+2$.

7.7 Let $T(n)$ be the running time of Algorithm 7.34. Writing the summations that represent the running time of the code in Algorithm 7.34 and solving the summations, we get

$$T(n) = \sum_{i=n^2}^{n^2+5} \sum_{j=4}^{n} c_4 = \sum_{i=n^2}^{n^2+5} c_4(n-3) = c_4(n^2+5-n^2+1)(n-3) = 6c_4(n-3).$$

7.8 Let $T(n)$ be the running time of Algorithm 7.35. Writing the summations that represent the running time of Algorithm 7.35 and solving them by bounding, we get

$$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{3i^3} c = \sum_{i=1}^{n} 3i^3 c = 3c\sum_{i=1}^{n} i^3 \leq 3c\sum_{i=1}^{n} n^3 = 3cn^4.$$

Thus, $T(n) = O(n^4)$.

7.9 For an upper bound, we have

$$f(n) = \sum_{i=n}^{4n^3} \sum_{j=i}^{8n^3} c = \sum_{i=n}^{4n^3} (8n^3 - i + 1)c \leq \sum_{i=n}^{4n^3} (8n^3)c \leq (4n^3 - n + 1)(8n^3)c.$$

It follows that $f(n) \leq (32n^6 - 8n^4 + 8n^3)c$, and therefore $f(n) \in O(n^6)$.

For a lower bound, we have

$$f(n) = \sum_{i=n}^{4n^3} \sum_{j=i}^{8n^3} c \geq \sum_{i=n}^{4n^3} (8n^3 - i)c \geq \sum_{i=n}^{4n^3} (8n^3 - 4n^3)c \geq (4n^3 - n + 1)(4n^3)c.$$

It follows that $f(n) \geq (16n^6 - 4n^4 + 4n^3)c$, and therefore $f(n) \in \Omega(n^6)$.

Thus, from Property 7.1, we have $f(n) \in \Theta(n^6)$.

7.10 From Algorithm 7.30, we have

$$f(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{k=i}^{j} c = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c(j-i+1).$$

For an upper bound, we have

$$f(n) \leq c\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} j \leq c\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} n \leq c\sum_{i=1}^{n-1} (n-i)n \leq c\sum_{i=1}^{n-1} n^2 = cn^2(n-1) \leq cn^3.$$

For a lower bound, we have

$$
\begin{aligned}
f(n) &\geq c\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(j-i)\\
&\geq c\sum_{i=1}^{n-1}\sum_{j=(n-i)/2}^{n}(j-i)\\
&\geq c\sum_{i=1}^{n-1}\sum_{j=(n-i)/2}^{n}\left(\frac{n-i}{2}-i\right)\\
&= c\sum_{i=1}^{n-1}\sum_{j=(n-i)/2}^{n}\left(\frac{n}{2}-\frac{3i}{2}\right)\\
&\geq c\sum_{i=1}^{n-1}\left(\frac{n}{2}-\frac{3i}{2}\right)\left(n-\frac{n-i}{2}\right)\\
&= c\sum_{i=1}^{n-1}\left(\frac{n}{2}-\frac{3i}{2}\right)\left(\frac{n}{2}+\frac{i}{2}\right)\\
&\geq c\sum_{i=1}^{n-1}\left(\frac{n}{2}-\frac{i}{2}\right)\left(\frac{n}{2}+\frac{i}{2}\right)\\
&= c\sum_{i=1}^{n-1}\left(\frac{n^2}{4}-\frac{i^2}{4}\right)\\
&\geq c\sum_{i=1}^{n/2}\left(\frac{n^2}{4}-\frac{i^2}{4}\right)\\
&\geq c\sum_{i=1}^{n/2}\left(\frac{n^2}{4}-\frac{1}{4}\left(\frac{n}{2}\right)^2\right)\\
&= c\left(\frac{3n^2}{16}\right)\left(\frac{n}{2}\right)=\frac{3c}{32}n^3.
\end{aligned}
$$

Thus $\frac{3}{32}cn^3 \leq f(n) \leq cn^3$, and therefore $f(n) = \Theta(n^3)$.

7.11 (a) If $n \geq 1$, then

$$5n^2 - 3n + 20 \leq 5n^2 + 20 \leq 5n^2 + 20n^2 = 25n^2.$$

Therefore, $5n^2 - 3n + 20 \in O(n^2)$ with $c = 25$ and $n_0 = 1$.

(b) If $n \geq 1$, then

$$4n^2 - 12n + 10 \leq 4n^2 + 10 \leq 4n^2 + 10n^2 = 14n^2.$$

Therefore, $4n^2 - 12n + 10 \in O(n^2)$ with $c = 14$ and $n_0 = 1$.

(c) If $n \geq 1$, then

$$5n^5 - 4n^4 - 2n^2 + n \leq 5n^5 + n \leq 5n^5 + n^5 = 6n^5.$$

Therefore, $5n^5 - 4n^4 - 2n^2 + n \in O(n^5)$ with $c = 6$ and $n_0 = 1$.

(d) If $n \geq 2$, then

$$n^{\frac{3}{2}} + \sqrt{n}\sin(n) + n\log(n) \leq n^2 + (n)(n) + (n)(n) \leq 3n^2.$$

Therefore, $n^{\frac{3}{2}} + \sqrt{n}\sin(n) + n\log(n) \in O(n^2)$ with $c = 3$ and $n_0 = 2$.

7.12 (a) If $n \geq 1$, then $4n^2 + n + 1 \geq 4n^2$. Therefore, $4n^2 + n + 1 \in \Omega(n^2)$ with $c = 4$ and $n_0 = 1$.

(b) If $n \geq 1$, then $n\log(n) - 2n + 13 \geq n\log(n) - 2n$. It is enough to find positive constants c and n_0 such that $n\log(n) - 2n \geq cn\log(n)$, or equivalently $1 - 2/\log(n) \geq c$, for all $n \geq n_0$. Note that, for $n \geq 8$, we have $1 - 2/\log(n) \geq 1/3$. Therefore, $n\log(n) - 2n + 13 = \Omega(n\log(n))$ with $c = 1/3$ and $n_0 = 8$.

7.13 (a) If $n \geq 1$, then

$$n^5 \leq n^5 + n^3 + 7n + 1 \leq n^5 + n^5 + 7n^5 + n^5 = 10n^5.$$

Therefore, $n^5 + n^3 + 7n + 1 \in \Theta(n^5)$ with $c_1 = 1, c_2 = 10$ and $n_0 = 1$.

(b) If $n \geq 7$, then

$$\frac{1}{14}n^2 \leq \frac{1}{2}n^2 - 3n \leq \frac{1}{2}n^2,$$

where the first inequality follows by noting that

$$\frac{1}{2}n^2 - 3n \geq c_1 n^2 \iff \frac{1}{2} - \frac{3}{n} \geq c_1. \text{ Hence, for } n \geq 7, \frac{1}{2} - \frac{3}{n} \geq \frac{1}{14} = c_1.$$

Therefore, $\frac{1}{2}n^2 - 3n = \Theta(n^2)$ with $c_1 = 1/14, c_2 = 1/2$ and $n_0 = 7$.

7.14 Given that $f_i(n) = O(g_i(n))$ for each $i = 1, 2, \ldots, k$, then by definition there are positive constants c_i and n_i such that $f_i(n) \leq c_i g_i(n)$ for $n \geq n_i$.

(a) Let $n_0 = \max_{1 \leq i \leq k} n_i$ and $c_0 = c_1 + c_2 + \cdots + c_k$, then for all $n \geq n_0$, we have

$$\sum_{i=1}^{k} f_i(n) \leq \sum_{i=1}^{k}(c_i g_i(n)) \leq \left(\sum_{i=1}^{k} c_i\right) \max_{1 \leq i \leq k} g_i(n) \leq c_0 \max_{1 \leq i \leq k} g_i(n).$$

Thus, $\sum_{i=1}^{k} f_i(n) = O(\max_{1 \leq i \leq k} g_i(n))$.

(b) Let $n_0 = \max_{1 \leq i \leq k} n_i$ and $c_0 = c_1 c_2 \cdots c_k$, then for all $n \geq n_0$, we have

$$\prod_{i=1}^{k} f_i(n) \leq \prod_{i=1}^{k}(c_i g_i(n)) = \left(\prod_{i=1}^{k} c_i\right) \prod_{i=1}^{k} g_i(n) = c_0 \prod_{i=1}^{k} g_i(n).$$

Thus, $\prod_{i=1}^{k} f_i(n) = O(\prod_{i=1}^{k} g_i(n))$.

7.15 We prove Properties 7.1 & 7.6, and prove also Properties 7.2 & 7.7 for Big-O.

Pr 7.1 Pf. By definitions, $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ iff there are positive constants c_1, c_2, n_1 and n_2 such that $f(n) \leq c_1 g(n)$ for $n \geq n_1$, and that $f(n) \geq c_2 g(n)$ for $n \geq n_2$. This is equivalent to $c_2 g(n) \leq f(n) \leq c_1 g(n)$ for all $n \geq \max\{n_1, n_2\}$, which is also equivalent to $f(n) = \theta(g(n))$, with $n_0 = \max\{n_1, n_2\}$, by definition. Thus, we conclude $f(n) = \theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Pr 7.2 Pf. Given that $f(n) = O(g(n))$, then by definition there are positive constants c_1 and n_1 such that $f(n) \le c_1 g(n)$ for $n \ge n_1$. Similarly, given that $g(n) = O(h(n))$, then by definition there are positive constants c_2 and n_2 such that $g(n) \le c_2 h(n)$ for $n \ge n_2$. Let $n_0 = \max\{n_1, n_2\}$, then $f(n) \le c_1 g(n) \le c_1 c_2 h(n)$ for all $n \ge n_0$. Let $c_0 = c_1 c_2$, then $f(n) \le c_0 h(n)$ for all $n \ge n_0$, which means we can conclude that $f(n) = O(h(n))$.

Pr 7.6 Pf. By definition of Big-Oh, $f(n) = O(g(n))$ iff there are positive constants c and n_0 such that $f(n) \le c_1 g(n)$, or equivalently $g(n) \ge \frac{1}{c} f(n)$, for all $n \ge n_0$. By definition of Big-Omega, this equivalent to $g(n) = \Omega(f(n))$ with positive constants $1/c$ and n_0.

Pr 7.7 Pf. Note that $f(n) \le f(n)$ for $n \ge 1$. Thus, $f(n) = O(f(n))$ with $c = n_0 = 1$.

7.16 (a) We have $\sqrt{4n^2+1} = \Theta(n)$ because

$$\lim_{n\to\infty} \frac{\sqrt{4n^2+1}}{n} = \lim_{n\to\infty} \sqrt{\frac{4n^2+1}{n^2}} = \lim_{n\to\infty} \sqrt{4 + \frac{1}{n^2}} = \sqrt{4 + \lim_{n\to\infty} \frac{1}{n^2}} = 2.$$

(b) We have $n^n = \Omega(n!)$ because

$$\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{\overbrace{n \cdot n \cdots n \cdot n}^{n-\text{times}}}{n \cdot (n-1) \cdots 2 \cdot 1} = \lim_{n\to\infty} \left(1 \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdots \frac{n}{2} \cdot n\right) = \infty.$$

(c) We have $\sqrt{n+4} - \sqrt{n} = O(1)$ because

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{n+4} - \sqrt{n}}{1} &= \lim_{n\to\infty} \left(\left(\sqrt{n+4} - \sqrt{n}\right) \cdot \frac{\sqrt{n+4} + \sqrt{n}}{\sqrt{n+4} + \sqrt{n}}\right) \\
&= \lim_{n\to\infty} \frac{(n+4) - n}{\sqrt{n+4} + \sqrt{n}} \\
&= \lim_{n\to\infty} \frac{4}{\sqrt{n+4} + \sqrt{n}} = 0.
\end{aligned}$$

(d) We have $n\log(n^2) + (n-1)^2 \log(\frac{n}{2}) = \Theta(n^2 \log(n))$ because

$$\begin{aligned}
\lim_{n\to\infty} \frac{n\log(n^2) + (n-1)^2 \log(\frac{n}{2})}{n^2 \log n} &= \lim_{n\to\infty} \frac{2n\log n + (n-1)^2(\log n - \log 2)}{n^2 \log n} \\
&= \lim_{n\to\infty} \left(\frac{2}{n} + \frac{(n-1)^2}{n^2} \frac{(\log n - \log 2)}{\log n}\right) \\
&= \lim_{n\to\infty} \left(\frac{2}{n} + \left(1 - \frac{1}{n}\right)^2 \left(1 - \frac{\log 2}{\log n}\right)\right) = 1.
\end{aligned}$$

7.17 Yes, it is true that $2^{(n+10)} = O(2^n)$. To prove this, note that, if $n \geq 1$, then $2^{(n+10)} = 2^{10}2^n \leq (1024)(2^n)$. Thus, $2^{(n+10)} = O(2^n)$ with $c = 1024$ and $n_0 = 1$.

7.18 (a) (iii). (b) (i). (c) (ii). (d) (i).

7.19 The first statement is an assignment statement with constant time. The inside of the for loop is also a simple statement with constant time. So, the part that will affect overall runtime is the runtime and value of $f(n)$. The answers are (*a*) $O(n^2)$ and (*b*) $O(n(n!))$. Below is the reasoning behind these answers, which is similar to that reasoning of Example 7.36 (*b*).

(a) The"test" runtime in the for loop is $O(n)$, since that is how long it takes $f(n)$ to run. The "body" of the for loop takes $O(1)$, as it is just a simple statement. The "reinitialization" runtime still takes $O(1)$. We perform the loop $O(n!)$ times, since the value of $f(n)$ is n. Thus, putting it all together, we have $O(1+(n+1+1)n) = O(n^2)$.

(b) The"test" runtime in the for loop is $O(n)$, since that is how long it takes $f(n)$ to run. The "body" of the for loop takes $O(1)$, as it is just a simple statement. The "reinitialization" runtime still takes $O(1)$. We perform the loop $O(n!)$ times, since the value of $f(n)$ is $n!$. Thus, putting it all together, we have $O(1+(n+1+1)n!) = O(nn!)$.

7.20 The answer is $O(n^3)$. Below is the reasoning behind the answer.

In Figure 7.9, we see how the runtime of for loops is calculated. We add the initialization runtime (usually $O(1)$) plus the cost of going around the loop once multiplied by the number of times we go around the loop, represented $O(1+(f(n)+1)g(n))$. Keep in mind "the cost of going around the loop once" is represented in Figure 7.9 as the runtimes of "test" plus "body" plus "reinitialize", with "test" being the condition of the for loop and "body" obviously being the body. Now, we apply this to our problem.

The "test" runtime in the for loop is $O(n)$, since that is how long it takes the function `cat(.)` to run and, as previously found, `cat(.)` has a runtime of $O(n)$. The "body" of the for loop takes $O(n)$, since, again, that is how long it takes `cat(.)` to run. The "reinitialization" runtime takes $O(1)$.

The "for" loop in line (10) goes from 1 to the value of the function `cat(n,n)`. Thus, we must find what is the value of `cat(n,n)`. The function `cat(.)` takes a number and adds values from 1 to n to the original number, in this case n. So, the value of `cat(n,n)` equals $n + 1 + 2 + 3 + \cdots + n$. In other words,

$$cat(n,n) = n + \sum_{i=1}^{n} i = n + \frac{n(n+1)}{2} = \frac{n^2+3n}{2} \in O(n^2).$$

Thus, the for loop is iterated $O(n^2)$ times (from $i = 1$ to $i = (n^2 + 3n)/2$). Now, putting it all together we have $O(1 + (n + n + 1)n^2) = O(n^3)$.

The first two statements of the function `main(.)` are $O(1)$, as they are simple statements. The statement in line (3) calls the function `cow(.)`, which takes $O(n^3)$ as we found above. The runtime of the statement in line (4), which has a call to `cat(.)`, is $O(n)$ (as previously stated, `cat(.)` has a runtime of $O(n)$). So, the total runtime for `main(.)` is $O(1) + O(1) + O(n^3) + O(n) = O(n^3)$.

7.21 The graph structure of Algorithm 7.38 is shown in Figure A.3.

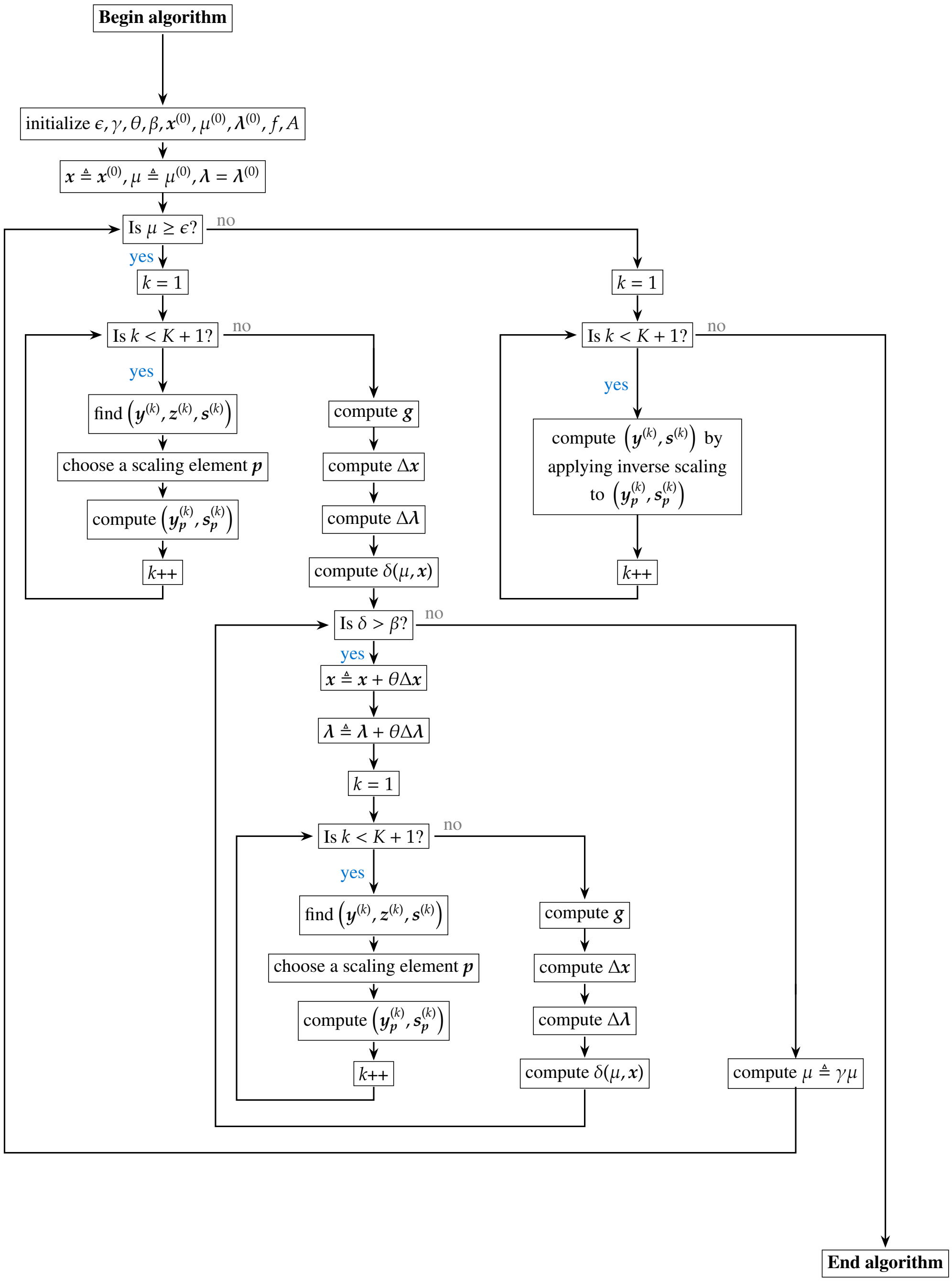


Figure A.3: The graph structure of Algorithm 7.38.

7.22 The running time of the block in lines (17)-(19) is $O(m)$, so the running time of the for-statement in lines (16)-(20) is $O(Km)$. Since the running time of the block in lines (21)-(24) is $O(n)$ and those in the statements in lines (14) and (15) are $O(1)$, the running time of the block in lines (14)-(25) is $O(n + Km)$, and hence the running time of the while-statement in lines (13)-(25) is $O((n + Km)N_{\text{in}})$. Here, N_{in} is the number of times we go around the inner while loop of line (13). The running time of the block in lines (5)-(7) is $O(m)$, so the running time of the for-statement in lines (4)-(8) is $O(Km)$. Since the running time of the block in lines (9)-(12) is $O(n)$, the running time of the block in lines (4)-(12) is $O(n + Km)$, and hence the running time of the block in lines (4)-(26) is $O((n + Km) + (n + Km)N_{\text{in}}) = O((n + Km)N_{\text{in}})$, it follows that the while-statement in lines (3)-(27) is $O((n+Km)N_{\text{in}}N_{\text{out}})$, where N_{out} is the number of times we go around the outer while loop of line (3). As a result, based on the information obtained and because the running time of the for-statement in lines (28)-(30) is $O(Km)$, and all other statements are of constant times $O(1)$, the running time of the algorithm is

$$O(1 + Km + (n + Km)N_{\text{in}}N_{\text{out}}) = O((n + Km)N_{\text{in}}N_{\text{out}}).$$

Bounding N_{out} is exactly similar to bounding N_{out} in Example 7.35. Using a similar argument to that in Example 7.35, we can show that

$$N_{\text{out}} \le \frac{\log(\mu^{(0)}/\epsilon)}{-\log\gamma}.$$

(a) It is given that $N_{\text{in}} = O(n + Km)$. Now, if $\gamma \in (0, 1)$ is an arbitrarily chosen constant, then $\gamma = O(1)$, and hence

$$N_{\text{out}} \le \log\left(\frac{\mu^{(0)}}{\epsilon}\right) O(1).$$

Thus, the running time of the algorithm as a whole is

$$O((n + Km)N_{\text{in}}N_{\text{out}}) = O\left((n + Km)^2 \log\left(\frac{\mu^{(0)}}{\epsilon}\right)\right).$$

(b) It is given that $N_{\text{in}} = O(1)$. Now, if $\gamma = 1 - \sigma/\sqrt{n + Km}$ $(\sigma > 0)$, then[1]

$$\log\gamma = \log\left(1 - \sigma/\sqrt{n + Km}\right) \approx -\sigma/\sqrt{n + Km},$$

and hence

$$N_{\text{out}} \le \frac{\log(\mu^{(0)}/\epsilon)}{-\log\gamma} \approx \frac{\log(\mu^{(0)}/\epsilon)}{\sigma/\sqrt{n + Km}} = \sqrt{n + Km}\log\left(\frac{\mu^{(0)}}{\epsilon}\right) O(1).$$

[1] For small positive values of x, we have $\log(1 + x) \approx (1 + x) - 1 = x$.

Thus, the running time of the algorithm as a whole is

$$O((n + Km)N_{\text{in}}N_{\text{out}}) = O\left((n + Km)^{3/2} \log\left(\frac{\mu^{(0)}}{\epsilon}\right)\right).$$

8.1 (a) (iii). (b) (iv). (c) (iii). (d) (i). (e) (iii). (f) (ii). (g) (iv). (h) (iii). (i) (ii).

8.2 When we write $O(n^{2.376})$, we mean that there is a positive constant c such that the Coppersmith and Winograd's algorithm takes no more than $cn^{2.376}$ flops. For this algorithm, the constant c is so large that it does not beat Strassen's method until n is really enormous.

8.3 The program in Algorithm 8.14 is nonrecursive. Figure A.4 shows a basic scheme of this program. From Algorithm 8.5, the function `linear-search` is $O(n)$. In view of Figure A.4, we analyze the function `karger` before analyzing the function `main` because the later calls the function `karger`. Now, if the function call in the body of a for loop, we add its cost to the bound on the time for each iteration. It follows that the running time of a call to `karger` is $O(mn)$. Next, when the function call is within a simple statement, we add its cost to the cost of that statement. Thus, the function `main` takes $O(mn)$ times. Therefore, the running time of this program is $O(mn)$.

8.4 The program in Algorithm 8.15 is nonrecursive. Figure A.5 shows a basic scheme of this program. From Algorithms 8.6 and 8.9, the functions `linear- search` and `merge-sort` are $O(\log n)$ and $O(n \log n)$ respectively. In view of Figure A.5, we analyze the function `dinic` before analyzing the function `main` because the later calls the first. Now, if the function call in the body of a for loop, we add its cost to the bound on the time for each iteration. It follows that the running time of a call to `dinic` is $O(\log m \log n)$. Next, when the function call is within a simple statement, we add its cost to the cost of that statement. Thus, the function `main` takes $O(\log m \log n + n \log n)$ times. Therefore, the running time of this program is $O((n + \log m) \log n)$.

8.5 If x divides both a and b, then it clearly also divides $a - b\lfloor a/b \rfloor = a \bmod b$. Conversely, if x divides $a \bmod b$ and b, then it also divides $(a \bmod b) + b\lfloor a/b \rfloor = a$.

The function `main` calls the function `karger`, and the function `karger` calls the function `linear-search`. We already analyzed the function `linear-search` in Algorithm 8.5. Thus, we analyze the function `karger`, and then analyze the function `main`.

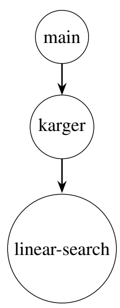


Figure A.4: A basic scheme of the program shown in Algorithm 8.14.

The function `main` calls the functions `dinic` and `merge-sort`, and the function `dinic` calls the function `binary-search`. We already analyzed the functions `binary- search` and `merge-sort` in Algorithms 8.6 and 8.9, respectively. So, we analyze the function `dinic`, and then analyze the function `main`.

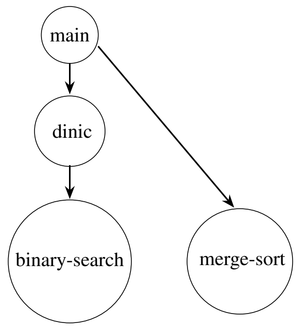


Figure A.5: A basic scheme of the program shown in Algorithm 8.15.

8.6 We need to show that if $a > b > 0$ and Algorithm 8.11 performs $n \geq 1$ iterations of the while loop, then $a \geq f_{n+2}$ and $b \geq f_{n+1}$.

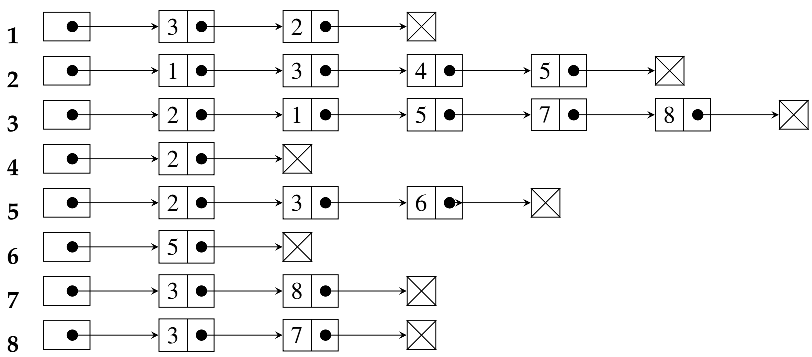
The proof is by induction on n. The statement holds for $n = 1$ because $b \geq 1 = f_2$ and $a > b$ imply that $a \geq 2 = f_3$. Now assume that $n \geq 2$. Because $a > b \geq 1$ and $n \geq 2$, the next iteration will use the number b and $a \bmod b$. To these, we can apply the induction hypothesis, because $n - 1$ iterations remain and $b > a \bmod b > 0$ (as $n - 1 > 0$). Thus, $b \geq f_{n+1}$ and $a \bmod b \geq f_n$. Thus, $a = \lfloor a/b \rfloor \cdot b + (a \bmod b) \geq b + (a \bmod b) \geq f_{n+1} + f_n = f_{n+2}$.

8.7 This is an implementation exercise. Applying Newton's method, we obtain the root 3.15145 if we start with $x_0 = 1$.

9.1 (a) (iv). (b) (ii). (c) (iii). (d) (ii). (e) (iii). (f) (i). (g) (iv). (h) (iv). (i) (iii). (j) (iii).

9.2 (a) True. (b) True. (c) True. (d) False. (e) True. (f) False. (g) True. (h) True. (i) False. (j) False.

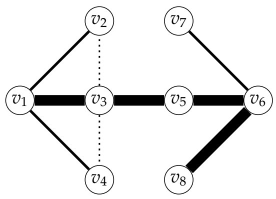
9.3 (a) The adjacency list representation for the graph is:



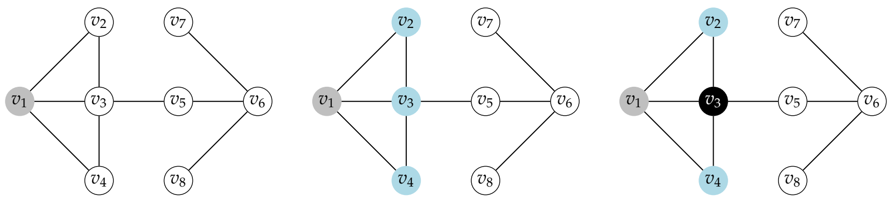
(b) The adjacency matrix representation for the graph is:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	0	0	0	0
5	0	1	1	0	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

9.4 (*a*) Running a breadth-first search, we obtain the breadth-first tree shown to the right. It follows that $(v_1, v_2, v_5, v_6, v_8)$ is the shortest path from vertex v_1 to vertex v_8. This shortest path is clearly unique.

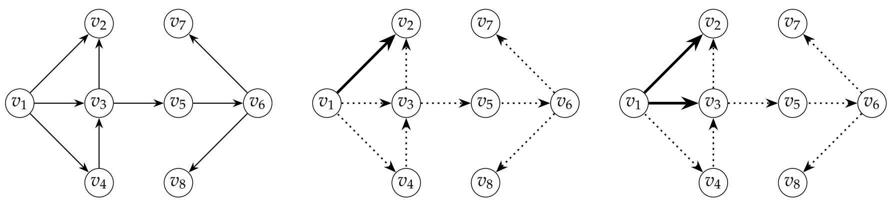


(*b*) As shown below, running a breadth-first search, we find that there is a conflicting assignment which occurs when we color the vertex v_3. Hence this is not a bipartite graph.

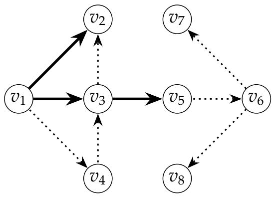


(*i*) (*ii*) (*iii*)

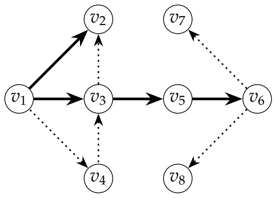
(*c*) As shown below, running a depth-first search, we get the following depth-first tree.



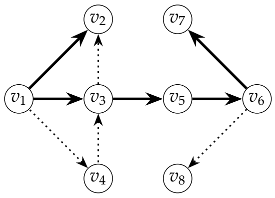
(*i*) (*ii*) (*iii*)



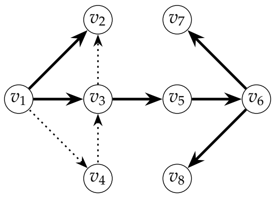
(*iv*)



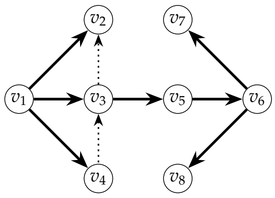
(*v*)



(*vi*)



(*vii*)

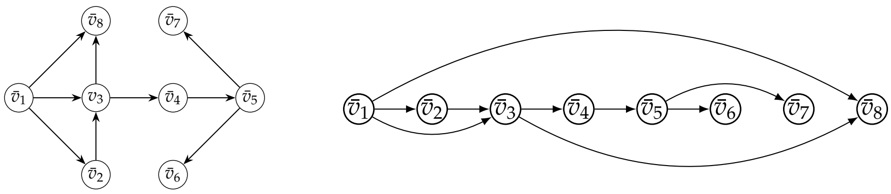


(*viii*)

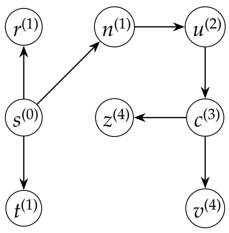
In (*viii*), the solid edges are the tree edges, and dotted edges are the cross edges.

(*d*) Because we did not find a back edge we performed a depth-first search in item (*c*), the directed graph has no a directed cycle.

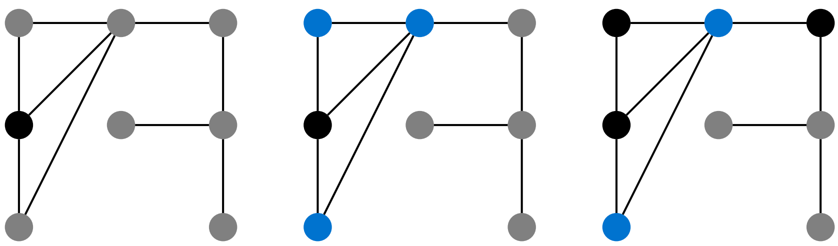
(*e*) A topological ordering is computed and shown below.



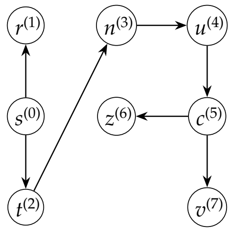
9.5 (*a*) Running a breadth-first search on the graph, we obtain the following breadth-first tree.



(*b*) Running a breadth-first search on the given graph, we find that the vertices *s* and *r* have the same color as shown below. Thus, this is not a bipartite graph.



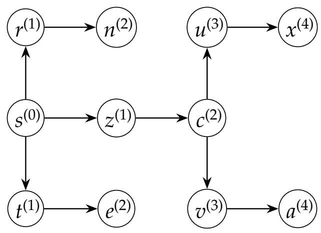
(*c*) Running a depth-first search on the graph, we obtain the following depth-first tree.



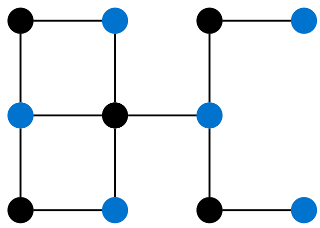
(*d*) Because we did not find a back edge we performed a depth-first search in item (*c*), the directed graph has no a directed cycle.

(*e*) See the solution of item (*e*) in Exercise 9.4. This graph and that in Exercise 9.4 are isomorphic, and they have the same topological ordering.

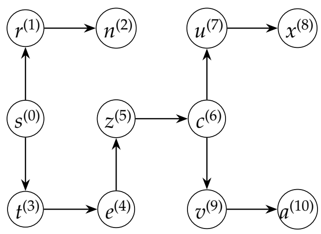
9.6 (*a*) Running a breadth-first search on the graph, we obtain the following breadth-first tree.



(*b*) The undirected version of the graph is bipartite as shown below.

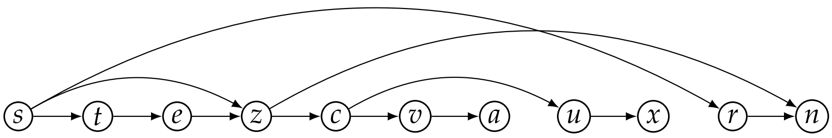


(*c*) Running a depth-first search on the graph, we obtain the following depth-first tree.



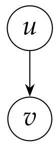
(*d*) Because we did not find a back edge we performed a depth-first search in item (*c*), the directed graph has no a directed cycle.

(*e*) A topological ordering is computed and shown below.



9.7 This is a false statement. The depth-first search may produce different depth-first trees with different numbers of tree edges depending on the starting vertex and upon the order in which vertices are searched.

As a counterexample, consider the graph shown to the right. If the depth-first search starts at u, then it will visit v next, and (u, v) will become a tree edge. But if the depth-first search starts at v, then u and v become separate trees in the depth-first forest, and (u, v) becomes a cross edge.



10.1 (*a*) (ii).
(*b*) (ii).
(*c*) (ii).
(*d*) (ii).
(*e*) (v).
(*f*) (iv).
(*g*) (v).
(*h*) (i).
(*i*) (i).
(*j*) (iii).
(*k*) (ii).
(*l*) (v).
(*m*) (i).
(*n*) (v).
(*o*) (ii).
(*p*) (i).
(*q*) (i).
(*r*) (iii).
(*s*) (v).
(*t*) (ii).
(*u*) (ii).

10.2 The decision variables are:

$$x_1 = \text{ kg of food } F_1;$$
$$x_2 = \text{ kg of food } F_2.$$

Minimizing the cost of the mixtures, we have the following LP problem.

$$\begin{array}{ll} \min & 60x_1 + 80x_2 \\ \text{s.t.} & 5x_1 + 2x_2 \geq 11, \\ & 3x_1 + 4x_2 \geq 8. \end{array}$$

10.3 The decision variables are:

x_1: The number of loaves of bread baked;

x_2: The change in the supply of flour through financial transactions (in ounces).

Measuring profits in cents, we have the following LP problem.

$$\begin{array}{ll} \max & 30x_1 - 4x_2 \\ \text{s.t.} & 5x_1 - x_2 \leq 30, \\ & x_1 \leq 5, \\ & x_1 \geq 0. \end{array}$$

10.4 The decision variables are:

x_1: Acres of radishes produced;
x_2: Acres of onions produced;
x_3: Acres of potatoes produced.

We wish to maximize the agriculturist's profit (in \$). Subtracting labor and water costs, we obtain the following LP problem.

$$\begin{array}{lrrrl} \max & 7072.5x_1 + 6775x_2 + 4812.5x_3 & & & \\ \text{s.t.} & x_1 + & x_2 + & x_3 \leq & 126, \\ & 6x_1 + & 5x_2 + & 6x_3 \leq & 500, \\ & & & x_1, x_2, x_3 \geq & 0. \end{array}$$

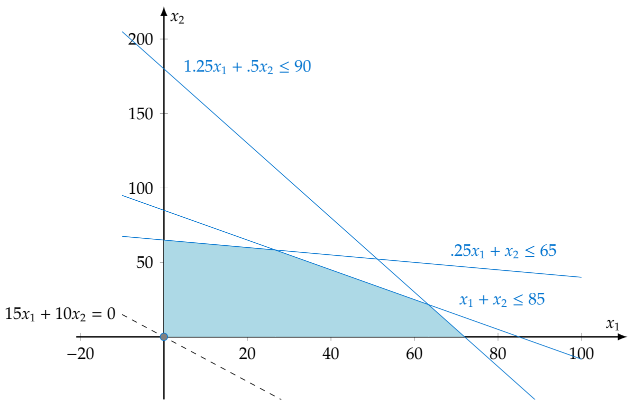


Figure A.6: Graphical solution of the optimization problem in Exercise 10.5.

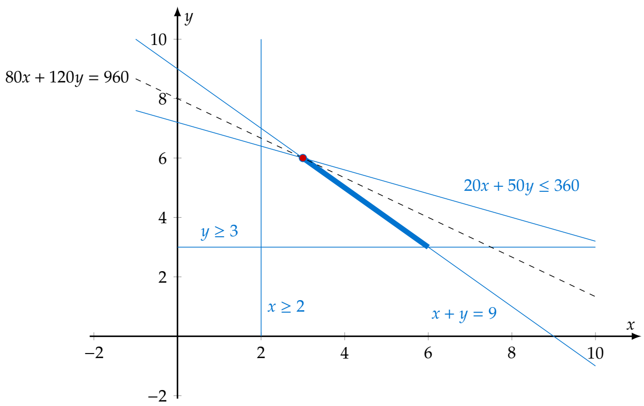


Figure A.7: Graphical solution of the optimization problem in Exercise 10.6.

10.5 The graphical representation of the given LP problem is shown in Figure A.6, with the feasible region shaded in cyan. From the graph, we find that the minimum value for z is 0 at $\boldsymbol{x} = (0, 0)$.

10.6 (*a*) The LP problem that can be used to maximize the profit is seen below:

$$\begin{array}{lrrrr} \max & 80x+120y & & & \\ \text{s.t.} & x & & \geq & 2, \\ & & y & \geq & 3, \\ & x+ & y & = & 9, \\ & 20x+ & 50y & \leq & 360. \end{array}$$

(*b*) The feasible region is the thick black line segment shown in the graph in Figure A.7 (b). We find that the maximum value for the objective function is 960 at $\boldsymbol{x} = (3, 6)$.

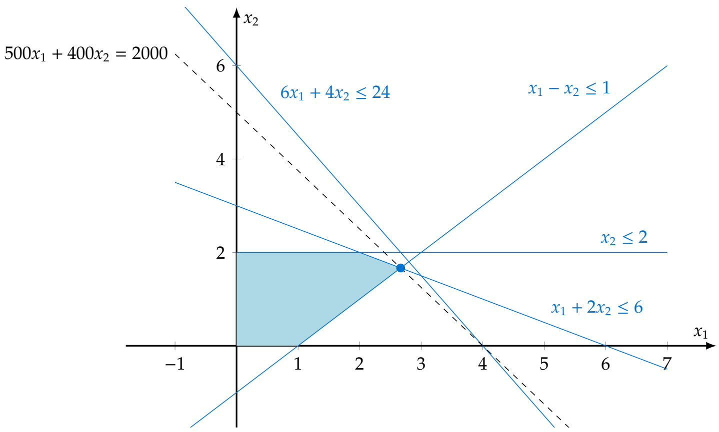


Figure A.8: Graphical solution of the optimization problem in Exercise 10.7 (b).

10.7 (*a*) Define decision variables: Letting the objective function be the maximum total daily profit (in \$), we obtain the following LP:

$$\begin{array}{lrrl} \max & 500x_1 + 400x_2 & & \\ \text{s.t.} & 6x_1 + & 4x_2 & \leq 24, \\ & x_1 + & 2x_2 & \leq 6, \\ & x_1 - & x_2 & \leq 1, \\ & & x_2 & \leq 2, \\ & x_1, & x_2 & \geq 0. \end{array}$$

(*b*) The graphical representation of the given LP problem is shown in Figure A.8, with the feasible region shaded in cyan. From the graph, we find that the maximum value for the objective function is 2000 at $\boldsymbol{x} = (8/3, 5/3)$.

(*c*) Restricting x_2 to be integer-valued changes the feasible region, we find that the feasible region is the thick black line segments shown in the graph in Figure A.9. We also find that the optimal solution is now 1800 at $\boldsymbol{x} = (2, 2)$.

(*d*) Restricting both x_1 and x_2 to be integer-valued again changes the feasible region to become the black bullets shown in the graph in Figure A.10. The optimal solution remains the same.

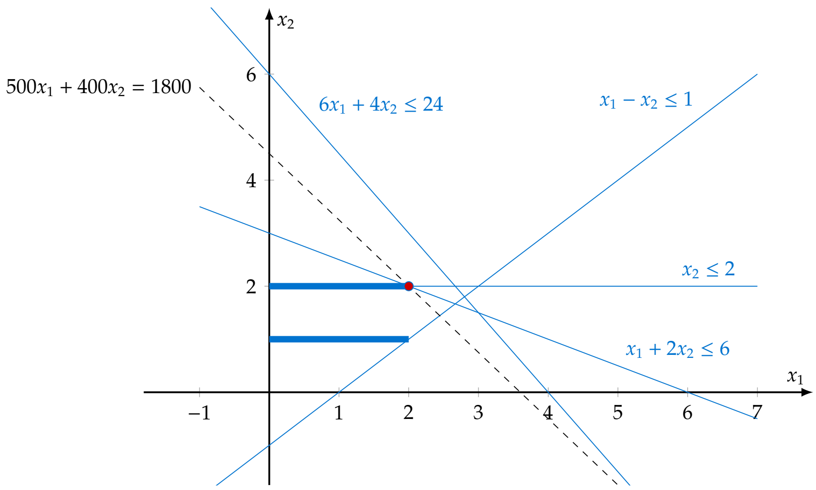


Figure A.9: Graphical solution of the optimization problem in Exercise 10.7 (c).

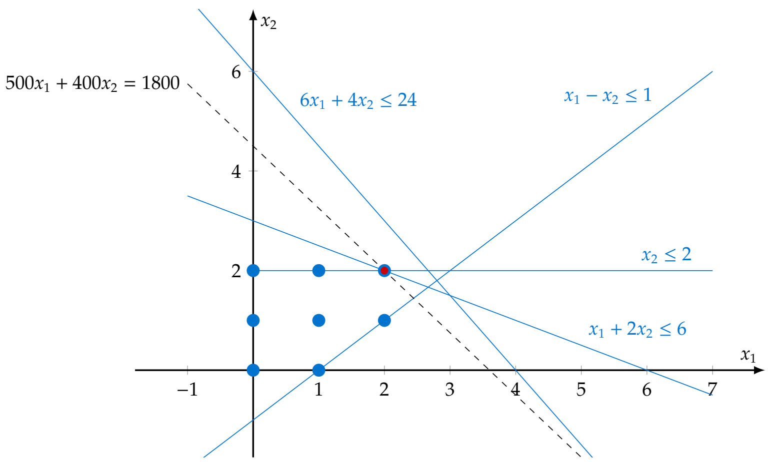


Figure A.10: Graphical solution of the optimization problem in Exercise 10.7 (d).

10.8 (*a*) The graphical representation of the given LP problem is shown in Figure A.11, with the feasible region shaded in cyan. From the graph, we find that the minimum value for the objective function is $230/13$ at $\boldsymbol{x} = (18/13, 20/13)$.

(*b*) The feasible region is the thick black line segment shown in the graph in Figure A.12. From the graph, we find that the maximum value for the objective function is 18 at $\boldsymbol{x} = (2, 2)$.

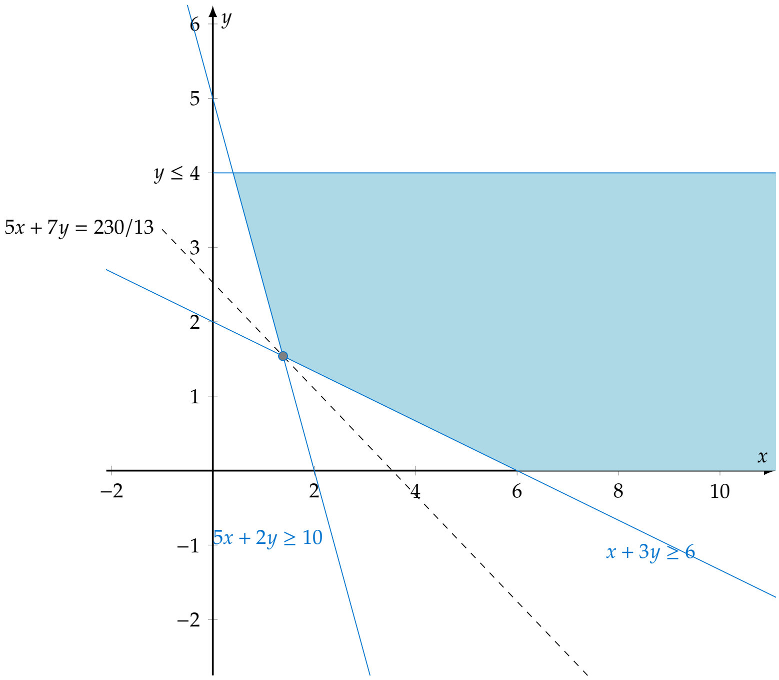


Figure A.11: Graphical solution of the optimization problem in Exercise 10.8 (a).

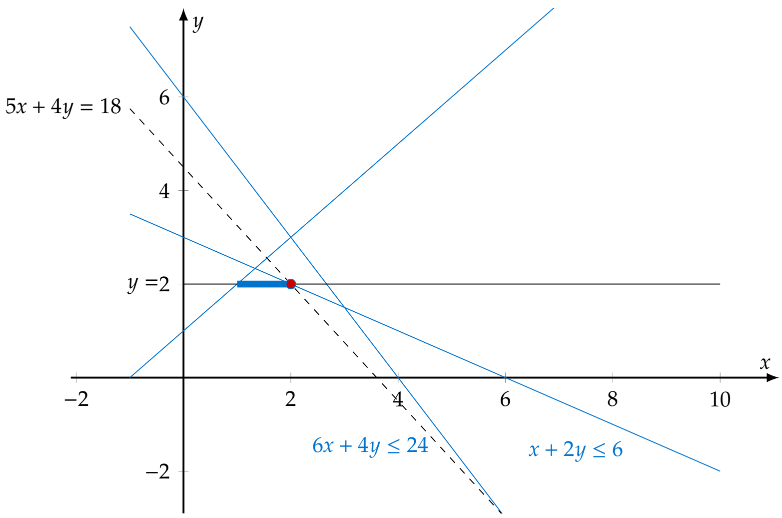


Figure A.12: Graphical solution of the optimization problem in Exercise 10.8 (b).

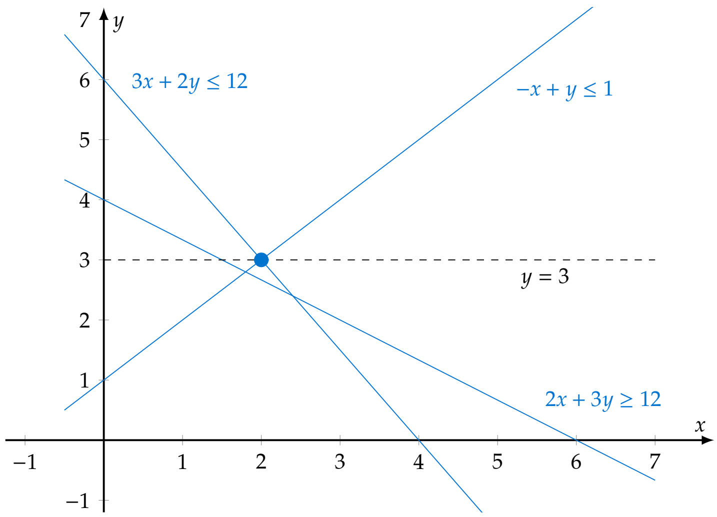


Figure A.13: Graphical solution of the optimization problem in Exercise 10.8 (c).

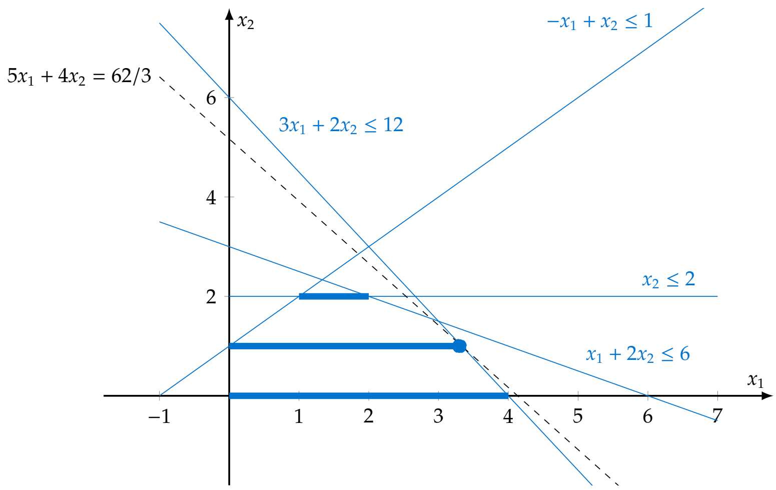


Figure A.14: Graphical solution of the optimization problem in Exercise 10.9 (a).

(*c*) The feasible region is the black bullet shown in the graph in Figure A.13. From the graph, we find that the maximum value for the objective function is 3 at $\boldsymbol{x} = (2, 3)$.

10.9 (*a*) Restricting x_2 to be integer-valued changes the feasible region to become the thick black line segments shown in the graph in Figure A.14. We find that the optimal solution is $62/3$ at $\boldsymbol{x} = (10/3, 1)$.

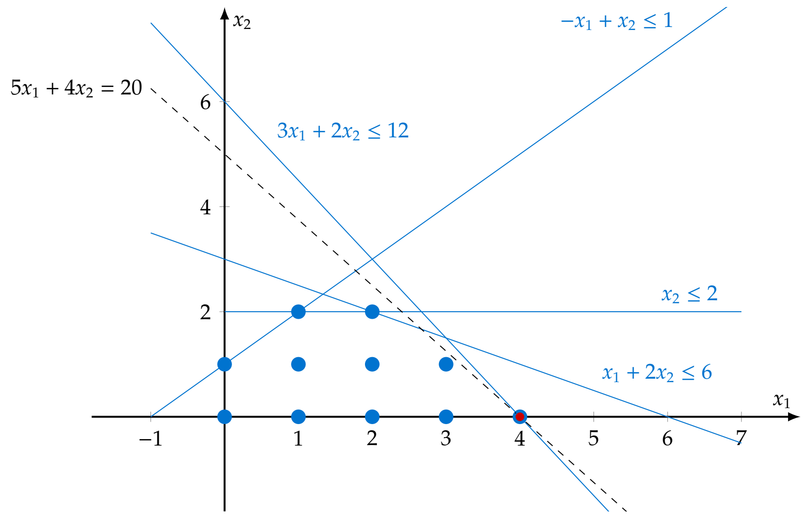


Figure A.15: Graphical solution of the optimization problem in Exercise 10.9 (b).

(b) Restricting both x_1 and x_2 to be integer-valued again changes to become the black bullets shown in the graph in Figure A.15. We find that the optimal solution is 20 at $\boldsymbol{x} = (4, 0)$.

10.10 Letting $x_i = x_i^+ - x_i^-$ for $i = 1, 2, 3$, and introducing the excess variable x_4 and the slack variables x_5 and x_6, the given LP problem is equivalent to the standard form problem:

$$
\begin{array}{lrl}
\min & z = & 2x_1^+ - 2x_1^- - 4x_2^+ + 4x_2^- + 5x_3^+ - 5x_3^- - 30 \\
\text{s.t.} & & 3x_1^+ - 3x_1^- + 2x_2^+ - 2x_2^- - x_3^+ + x_3^- - x_4 = 10, \\
& & -2x_1^+ + 2x_1^- + 4x_3^+ - 4x_3^- + x_5 = 35, \\
& & 4x_1^+ - 4x_1^- - x_2^+ + x_2^- + x_6 = 20, \\
& & x_1^+ - x_1^- + x_7 = 6, \\
& & x_2^+ - x_2^- + x_8 = 8, \\
& & x_3^+ - x_3^- + x_9 = 10, \\
& & x_1^+, \ x_1^-, \ x_2^+, \ x_2^-, \ x_3^+, \ x_3^-, \ x_4, \ x_5, \ x_6, x_7, x_8, x_9 \geq 0.
\end{array}
$$

10.11 We need to show that if (D|LP) is infeasible, then (P|LP) is either infeasible or unbounded. The possibility that Problems (P|LP) and (D|LP) could be both infeasible has been grounded in Exercise 10.19. So, to prove the desired result, it remains to show that if (D|LP) is infeasible and (P|LP) is feasible, then (P|LP) must be unbounded. Assume that (D|LP) is infeasible and let $\bar{\boldsymbol{x}}$ be a feasible solution for (P|LP). Due to the infeasibility of (D|LP), there does not exist $\boldsymbol{y}$ satisfying $A^{\mathsf{T}}\boldsymbol{y} \leq \boldsymbol{c}$. Using Farkas' lemma (Version II); see Theorem 3.16, there is a vector $\hat{\boldsymbol{x}}$ satisfying $A\hat{\boldsymbol{x}} = \boldsymbol{0}$, $\boldsymbol{c}^{\mathsf{T}}\hat{\boldsymbol{x}} < 0$, and $\hat{\boldsymbol{x}} \geq \boldsymbol{0}$. Due to the feasibility of $\bar{\boldsymbol{x}}$ in (P|LP), we have $A\bar{\boldsymbol{x}} = \boldsymbol{b}$ and $\bar{\boldsymbol{x}} \geq \boldsymbol{0}$. Define $\boldsymbol{x}_\alpha \triangleq \bar{\boldsymbol{x}} + \alpha\hat{\boldsymbol{x}}$ for $\alpha \geq 0$. Then

$$A\boldsymbol{x}_\alpha = A(\bar{\boldsymbol{x}} + \alpha\hat{\boldsymbol{x}}) = A\bar{\boldsymbol{x}} + \alpha A\hat{\boldsymbol{x}} = \boldsymbol{b} + \alpha A\hat{\boldsymbol{x}} = \boldsymbol{b}, \text{ and } \boldsymbol{x}_\alpha = \bar{\boldsymbol{x}} + \alpha\hat{\boldsymbol{x}} \geq \boldsymbol{0}.$$

This means that $\boldsymbol{x}_\alpha$ is feasible in (P|LP). Note that, because $\boldsymbol{c}^\mathsf{T}\hat{\boldsymbol{x}} < 0$, we have

$$\boldsymbol{c}^\mathsf{T}\boldsymbol{x}_\alpha = \boldsymbol{c}^\mathsf{T}(\bar{\boldsymbol{x}} + \alpha\hat{\boldsymbol{x}}) = \boldsymbol{c}^\mathsf{T}\bar{\boldsymbol{x}} + \alpha\boldsymbol{c}^\mathsf{T}\hat{\boldsymbol{x}} \longrightarrow \boldsymbol{c}^\mathsf{T}\bar{\boldsymbol{x}} - \infty = -\infty,$$

as $\alpha \longrightarrow \infty$, which implies that Problem (P|LP) is unbounded.

10.12 $(x_1, x_2) = (4, -1/3)$.

10.13 (*a*) (ii). (*b*) (i). (*c*) (iv). (*d*) (ii). (*e*) (iii). (*f*) (ii). (*g*) (iii).

10.14 (*a*) After introducing slack variables, say x_3 and x_4, we obtain the following standard form problem:

$$\begin{array}{ll} \max & z = x_1 + 1.5x_2 \\ \text{s.t.} & 2x_1 + 4x_2 + x_3 = 12, \\ & 3x_1 + 2x_2 + x_4 = 10, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

Note that $\boldsymbol{x} = (0, 0, 12, 10)$ is a basic feasible solution. Hence, we have the following initial tableau:

	rhs	x_1	x_2	x_3	x_4
	0	1	1.5	0	0
$x_3 =$	12	2	4	1	0
$x_4 =$	10	3	2	0	1

Since we are maximizing the objective function, we select a nonbasic variable with the greatest positive reduced cost to be the one that enters the basis. Indicating the pivot element with a circled number, we obtain the following two tableaux:

	rhs	x_1	x_2	x_3	x_4
	$-9/2$	$1/4$	0	$-1/4$	0
$x_2 =$	3	$1/2$	1	$1/4$	0
$x_4 =$	4	2	0	$-1/2$	1

	rhs	x_1	x_2	x_3	x_4
	-20	0	0	$-3/4$	0
$x_2 =$	2	0	1	$3/8$	$-1/2$
$x_1 =$	2	1	0	$-1/4$	1

The reduced costs in the zeroth row of the tableau are all nonpositive, so the current basic feasible solution is optimal. In terms of the original variables x_1 and x_2, this solution is $\boldsymbol{x} = (2, 2)$.

(*b*) After introducing the slack variables, say s_1 and s_2, we obtain the following standard form problem:

$$\begin{array}{ll} \max & z = 3x_1 + 5x_2 + 4x_3 \\ \text{s.t.} & 2x_1 + 3x_2 + s_1 = 8, \\ & 2x_2 + 5x_3 + s_2 = 10, \\ & 3x_1 + 2x_2 + 4x_3 + s_3 = 15, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

We then obtain the following sequence of tableaux when using the circled pivot point.

x_1	x_2	x_3	s_1	s_2	s_3	rhs
2	3	0	1	0	0	8
0	2	5	0	1	0	10
3	2	4	0	0	1	15
−3	−5	−4	0	0	0	0

x_1	x_2	x_3	s_1	s_2	s_3	rhs
2/3	1	0	1/3	0	0	8/3
−4/3	0	5	−2/3	1	0	14/3
5/3	0	4	−2/3	0	1	29/3
1/3	0	−4	5/3	0	0	40/3

x_1	x_2	x_3	s_1	s_2	s_3	rhs
2/3	1	0	1/3	0	0	8/3
−4/15	0	1	−2/15	1/5	0	14/15
41/15	0	0	−2/15	−4/5	1	98/15
−11/15	0	0	17/15	4/5	0	256/15

x_1	x_2	x_3	s_1	s_2	s_3	rhs
0	1	0	15/41	8/41	−10/41	50/41
0	0	1	−6/41	5/41	4/41	62/41
1	0	0	−2/41	−12/41	−15/41	89/41
0	0	0	45/41	24/41	11/41	765/41

The reduced costs in the zeroth row of the tableau are all positive, so the basic feasible solution is optimal. In terms of the original variables, x_1, x_2, and x_3 this solutions is $x = (89/41, 50/41, 62/41)$.

(c) After introducing the slack variables, say x_4, x_5 and x_6, we obtain the following standard form problem:

$$\begin{array}{lrcrcrcrcrcrcl}
\max & 2x_1 & - & x_2 & + & x_3 & & & & & & & & \\
\text{s.t.} & 3x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & = & 6, \\
 & x_1 & - & x_2 & + & 2x_3 & & & + & x_5 & & & = & 1, \\
 & x_1 & + & x_2 & - & x_3 & & & & & + & x_6 & = & 2, \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0.
\end{array}$$

Note that $\boldsymbol{x} = (0, 0, 0, 6, 1, 2)$ is a basic feasible solution. Hence, we have the following initial tableau:

	rhs	x_1	x_2	x_3	x_4	x_5	x_6
	0	2	−1	1	0	0	0
$x_4 =$	6	3	1	1	1	0	0
$x_5 =$	1	1	−1	2	0	1	0
$x_6 =$	2	1	1	−1	0	0	1

Since we are maximizing the objective function, we choose a nonbasic variable with the greatest positive reduced cost to be the one that enters the basis. Indicating the pivot variable with a circled number, we obtain the following tableaux:

	rhs	x_1	x_2	x_3	x_4	x_5	x_6
	−2	0	1	−3	0	−2	0
$x_4 =$	3	0	4	−5	1	−3	0
$x_1 =$	1	1	−1	2	0	1	0
$x_6 =$	1	0	2	−3	0	−1	1

	rhs	x_1	x_2	x_3	x_4	x_5	x_6
	−2.5	0	0	−1.5	0	−1.5	−0.5
$x_4 =$	1	0	0	1	1	−1	−2
$x_1 =$	1.5	1	0	0.5	0	0.5	0.5
$x_2 =$	0.5	0	1	−1.5	0	−0.5	0.5

The reduced costs in the zeroth row of the tableau are all nonpositive, so the current basic feasible solution is optimal. In terms of the original variables x_1, x_2, and x_3, this solution is $\boldsymbol{x} = (1.5, 0.5, 0)$.

(d) After introducing slack variables, say x_4, x_5, x_6 and x_7, we obtain the following standard form problem:

$$
\begin{array}{llllllllll}
\max z = & 60x_1 & + 30x_2 & + 20x_3 & & & & & & \\
\text{s.t.} & 8x_1 & + 6x_2 & + x_3 & + x_4 & & & & = 48, \\
& 4x_1 & + 2x_2 & + 1.5x_3 & & + x_5 & & & = 20, \\
& 2x_1 & + 1.5x_2 & + 0.5x_3 & & & + x_6 & & = 8, \\
& & x_2 & & & & & + x_7 & = 5, \\
& x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq 0.
\end{array}
$$

Note that $\boldsymbol{x} = (0, 0, 0, 48, 20, 8, 5)$ is a basic feasible solution. Hence, we have the following initial tableau:

	rhs	x_1	x_2	x_3	x_4	x_5	x_6	x_7
	0	60	30	20	0	0	0	0
$x_4 =$	48	8	6	1	1	0	0	0
$x_5 =$	20	4	2	1.5	0	1	0	0
$x_6 =$	8	2	1.5	0.5	0	0	1	0
$x_7 =$	5	0	1	0	0	0	0	1

Again indicating the pivot element with a circled number, we obtain the following tableaux:

	rhs	x_1	x_2	x_3	x_4	x_5	x_6	x_7
	−240	0	−15	5	0	0	−30	0
$x_4 =$	16	0	0	−1	1	0	−4	0
$x_5 =$	4	0	−1	0.5	0	1	−2	0
$x_1 =$	4	1	0.75	0.25	0	0	0.5	0
$x_7 =$	5	0	1	0	0	0	0	1

	rhs	x_1	x_2	x_3	x_4	x_5	x_6	x_7
	−280	0	−5	0	0	−10	−10	0
$x_4 =$	24	0	−2	0	1	2	−8	0
$x_3 =$	8	0	−2	1	0	2	−4	0
$x_1 =$	2	1	1.25	0	0	−0.5	1.5	0
$x_7 =$	5	0	1	0	0	0	0	1

The reduced costs in the zeroth row of the tableau are all nonpositive, so the current basic feasible solution is optimal. In terms of the original variables x_1, x_2, and x_3, this solution is $\boldsymbol{x} = (2, 0, 8)$.

10.15 (*a*) We have the following tableau:

x_1	x_2	s_1	s_2	s_3	rhs
0	0	17	−7	0	10
1	0	3	−1	0	2
0	1	4	−2	0	2
0	0	1	0	1	6

Since every element of the s_2 column is less than or equal to zero, this LP is unbounded. The LP does not have any optimal solution.

(*b*) We have the following tableau:

x_1	x_2	s_1	s_2	s_3	rhs
0	0	17	1	0	10
1	0	3	−1	0	2
0	1	4	2	0	2
0	0	1	−1	1	6

The LP has only one optimal solution.

(*c*) We have the following tableau:

x_1	x_2	s_1	s_2	s_3	rhs
0	0	17	0	0	10
1	0	3	−1	0	2
0	1	4	3/2	0	2
0	0	1	1	1	6

There is a nonbasic column we can pivot to, so this LP has many optimal solutions.

10.16 (*a*) The basic variables are x_1, x_3, and x_6. For x_3 to be basic, $a_3 = 1$ and $a_4 = 0$. For x_6 to be basic, $c_3 = 0$, $a_7 = 0$, and $a_8 = 1$.

(*b*) Setting $b \geq 0$ makes the LP feasible. Setting $c_1 \geq 0$, $c_2 \geq 0$, and $c_3 \geq 0$ makes the LP optimal.

(*c*) Setting $b \geq 0$, there are three variables we can introduce into the basis to obtain alternative optimal solutions:

(*i*) Variable x_2: To do this, at least one of a_1, a_2 must be greater than zero to pivot into Column 2.

(*ii*) Variable x_4: To do this, $c_1 = 0$ and $a_5 \geq 0$ to pivot into Column 4.

(*iii*) Variable x_5: To do this, $c_2 = 0$ and $a_6 \geq 0$ to pivot into Column 5.

10.17 After introducing slack variables, we obtain the following standard form problem:

$$\begin{array}{llrcl}
\max & z = 5x_1 - x_2 & & & \\
\text{s.t.} & x_1 - 3x_2 + x_3 & & = & 1, \\
 & x_1 - 4x_2 & + x_4 & = & 3, \\
 & x_1, \quad x_2, \quad x_3, & x_4 & \geq & 0.
\end{array}$$

We form the initial tableau:

	rhs	x_1	x_2	x_3	x_4
	0	5	−1	0	0
$x_3 =$	1	−3	1	1	0
$x_4 =$	1	-4	−1	0	1

The tableau has a nonbasic variable x_1 that could enter and improve the value of z, but there are no candidates for the minimum ratio test. Suppose x_1 entered the basis. The equations representing the constraints are

$$\begin{aligned} x_3 &= 1 + 3x_1 \\ x_4 &= 1 + 4x_1. \end{aligned}$$

As x_1 increases, both x_3 and x_4 stay positive. Thus we could keep increasing x_1 (improving z) without ever encountering infeasibility. Hence the LP is unbounded.

10.18 We can easily see that $\boldsymbol{x}^\star$ and $\boldsymbol{y}^\star$ are feasible in the primal and dual problems, respectively. One can also easily see that $\boldsymbol{b}^\mathsf{T}\boldsymbol{y}^\star = 19$ and $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}^\star = 19$. Based on the strong duality property (Theorem 10.9), since $\boldsymbol{b}^\mathsf{T}\boldsymbol{y}^\star = \boldsymbol{c}^\mathsf{T}\boldsymbol{x}^\star$, we conclude that $\boldsymbol{x}^\star$ and $\boldsymbol{y}^\star$ are are optimal in the primal and dual problems, respectively, and their optimal value is 19.

10.19 The problem max $y_1 + y_2$, subject to $y_1 - y_2 \le -1$, $-y_1 + y_2 \le -1$, and $y_1, y_2 \ge 0$, and its dual problem are a pair with such a property.

10.20 (*a*) The dual problem is:

$$\begin{array}{lrcrcrcr} \max & w = 4y_1 & + & 9y_2 & + & 5y_3 & & \\ \text{s.t.} & y_1 & + & 2y_2 & & & \ge & 5, \\ & y_1 & + & 3y_2 & - & y_3 & \ge & 3, \\ & y_1 & - & y_2 & + & 3y_3 & \le & -2, \\ & y_1, & & y_2, & & y_3 & \ge & 0. \end{array}$$

(*b*) (*i*) The optimal solution to the primal problem is $\boldsymbol{x} = (0; 3.5; 1.5)$. The optimal value to the primal problem is 7.5.

(*ii*) The optimal solution to the dual problem is $\boldsymbol{y} = (0; 1.25; -7.5)$. The optimal value to the dual problem is 7.5.

10.21 (*a*)

```
function [x, optimal_cost, iters, Running_time] = rsm (A, b, c, N,
    M)

t1=cputime;

c=c';
[m n] = size(A);
bfs=[zeros(1,n-m)';b];
B_indices = find(bfs);
N_indices = find(ones(1,n) - abs(sign(bfs))');

rsm_nnz = zeros(5000,2);

% disp('Please determine the way of choosing the entering
    variable as follows: ');
% fprintf(1,'Input 1 if you want to choose the variable with the
    smallest reduced cost to enter the basis.\n');
```

```
% fprintf(1,'Or input 0 if you want to choose the variable that first gives a negative reduced cost to enter the basis.\n');
% N = input(' ');
%
% disp('Please determine the way of choosing the leaving varialble as follows: ');
% fprintf(1,'Input "1" if you want to choose the smallest index rule.\n');
% fprintf(1,'Or input "0" if you want to choose the lexicographic .\n');
% M = input(' ');

iters=0;
while 1==1
iters=iters+1;

Binv = inv(A(:,B_indices));

rsm_nnz(iters,1) = nnz(A(:,B_indices));
rsm_nnz(iters,2) = nnz(Binv);

d = Binv * b;

if N == 1
    c_tilde = zeros(1,n);
    c_tilde(:,N_indices) = c(:,N_indices) - c(:,B_indices) * Binv * A(:,N_indices);
    [cj j]=min(c_tilde);

    if cj >= 0
        x = zeros(n,1);
        x(B_indices,:) = d;
        optimal_cost = c*x;
        x=x';
        break;
    end;
end

if N == 0
    c_tilde = zeros(1,n);
    for k=1:length(N_indices)
        c_tilde(:,N_indices(k)) = c(:,N_indices(k)) - c(:,B_indices) * Binv * A(:,N_indices(k));
        cj = c_tilde(:,N_indices(k));
        if cj < 0
            j = N_indices(k);
            break
        end;
```

```
    end;
    if cj >= 0
        x = zeros(n,1);
        x(B_indices,:) = d;
        optimal_cost = c*x;
        x=x';
        break;
    end;
end

u = Binv * A(:,j);
mn = inf;
i=0;
zz = find (u > 0)';

if (length(zz) == 0)
    x='The LP is unbounded';
    optimal_cost='The LP is unbounded';
    break
else
    [yy, ii] = min (d(zz) ./ u (zz)) ;
    i = zz(ii(1));
    k = B_indices(i);
    B_indices(i) = j;
    N_indices(j == N_indices) = k;
end;

end;

t2=cputime;
Running_time=t2-t1;
```

10.21 (*b*)

```
function [x, optimal_cost, iters, Running_time] = tsm (A, b, c,N,M)

t1=cputime;

c=c';
[m n] = size(A);
bfs=[zeros(1,n-m)';b];
B_indices = find(bfs);

% disp('Please determine the way of choosing the entering varialble as follows: ');
% fprintf(1,'Input "1" if you want to choose the variable with the smallest reduced cost to enter the basis.\n');
```

```
% fprintf(1,'Or input "0" if you want to choose the variable with
    the smallest index with a negative reduced cost to enter the
    basis.\n');
% N = input(' ');
%
% disp('Please determine the way of choosing the leaving varialble
     as follows: ');
% fprintf(1,'Input "1" if you want to choose the smallest index
    rule.\n');
% fprintf(1,'Or input "0" if you want to choose the lexicographic
    .\n');
% M = input(' ');

Binv = inv(A(:,B_indices));
x_B=Binv * b;
c_tilde = c - c(:,B_indices) * Binv * A;
T_1=[- c(:,B_indices)*x_B, c_tilde; x_B, Binv * A];

T=T_1;

[m,n] = size(T);

m=m-1;
n=n-1;

iters=0;
while 1==1
iters=iters+1;


if N==0
    y=find(T(1,:)<0);
    if length(y)>0
        j=y(1)-1;
    else
        x=zeros(1,n);
        for k=2:n+1
            z=find(T(:,k));
            if length(z)==1
                x(k-1)=T(z(1),1);
            end
        end
        optimal_cost = -T(1,1);
        break
    end
end


```

```
if N==1
    [x_j j]=min(T(1,:));
    if x_j<0
        j=j-1;
    else
        x=zeros(1,n);
        for k=2:n+1
            z=find(T(:,k));
            if length(z)==1
                x(k-1)=T(z(1),1);
            end
        end
        optimal_cost = -T(1,1);
        break
    end
end

u = Binv * A(:,j);
zz = find (u > 0)' ;
if (length(zz) == 0)
    x='The LP is unbounded';
    optimal_cost='The LP is unbounded';
    break
end

dind = find(T(2:end,j+1)>0);

if M== 1
[thetast,l] = min( T(1+dind,1)./ T(1+dind,j+1) );
l=dind(l);
end

if M==0
T(dind+1,:)=T(dind+1,:)./repmat(T(dind+1,j+1),1,n+1);
[Ts,sind]=sortrows(T(dind+1,:));
l=dind(sind(1));
thetast = T(l+1,1)/T(l+1,j+1);
end


T(l+1,:)=T(l+1,:)/T(l+1,j+1);
for i=setdiff( (1:m+1), l+1 )
    T(i,:) = T(i,:) - T(i,j+1)*T(l+1,:);
end

for k=2:n+1
    z=find(T(:,k));
    if length(z)==1 && T(z(1),k) ~= 1
```

```
            T(z(1),:)=T(z(1),:)/T(z(1),k);
        end
end

if norm(T - T_1)<1.0e-2
    disp('The simplex method cycles'),break
end

end

t2=cputime;
Running_time=t2-t1;
```

10.22 Note that from the first three equations in (10.21) we have that

$$\begin{aligned} (A\boldsymbol{x})^{\mathsf{T}}\boldsymbol{y} - \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}\tau &= \mathbf{0}, \\ -\boldsymbol{x}^{\mathsf{T}}A^{\mathsf{T}}\boldsymbol{y} - \boldsymbol{x}^{\mathsf{T}}\boldsymbol{s} + \boldsymbol{x}^{\mathsf{T}}\boldsymbol{c}\tau &= \mathbf{0}, \\ -\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}\tau + \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}\tau - \kappa\tau &= 0. \end{aligned}$$

Now we get the desired equation by adding the above three equations.

11.1 For each $i = 1, 2, \ldots, k$, it is known that the matrix $\bar{M}_i \succeq 0$ if and only if every principle minor of $\bar{M}_i$ is nonnegative. Since

$$\det(\tau_i\Lambda_i - I) = \begin{vmatrix} \tau_1\lambda_{i1} - 1 & 0 & \ldots & 0 \\ 0 & \tau_2\lambda_{i2} - 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tau_k\lambda_{ik} - 1 \end{vmatrix} = \prod_{j=1}^{k}(\tau_j\lambda_{ij} - 1).$$

It follows that $\bar{M}_i \succeq 0$ if and only if $\Pi_{j=1}^{s}(\tau_j\lambda_{ij} - 1) \geq 0$, for all $s = 1, 2, \ldots, k$ and $\det(\bar{M}_i) \geq 0$. Thus, $\bar{M}_i \succeq 0$ if and only if $\tau_j\lambda_{ij} \geq 1$, for all $j = 1, 2, \ldots, k$ and $\det(\bar{M}_i) \geq 0$.

Notice that

$$\det(\bar{M}_i) = \left(\prod_{j=1}^{k}(\tau_i\lambda_{ij} - 1)\right)(\tau_i v_i + d_2 - \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}) - \sum_{j=1}^{k} u_{ij}^2.$$

This means the inequality $\det(\bar{M}_i) \geq 0$ strictly holds for each $i \leq j \leq k$ such that $\tau_j\lambda_{ij} = 1$. Hence, $\det(\bar{M}_i) \geq 0$ if and only if

$$(\tau_i v_i + d_2 - \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}) - \mathbf{1}^{\mathsf{T}}\boldsymbol{s}_i = (\tau_i v_i + d_2 - \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}}) - \sum_{\tau_i\lambda_{ij}>1}(u_{ij}^2/(\tau_j\lambda_{ij} - 1)) \geq 0.$$

Therefore, $\bar{M}_i \succeq 0$ if and only if $\tau_i\lambda_{\min}(H_i) \geq 1$ and $d_2 \geq \bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{x}} - \tau_i v_i + \mathbf{1}^{\mathsf{T}}\boldsymbol{s}_i$.

11.2 Let $p \in [1, \infty]$. We first prove that $\mathcal{P}_q^n \subseteq \mathcal{P}_p^{n\star}$. Let $\boldsymbol{x} = (x_0; \widetilde{\boldsymbol{x}}) \in \mathcal{P}_q^n$, we show that $\boldsymbol{x} \in \mathcal{P}_p^{n\star}$ by verifying that $\boldsymbol{x}^\mathsf{T}\boldsymbol{y} \geq 0$ for any $\boldsymbol{y} \in \mathcal{P}_p^n$. So let $\boldsymbol{y} = (y_0; \widetilde{\boldsymbol{y}}) \in \mathcal{P}_p^n$, then

$$\boldsymbol{x}^\mathsf{T}\boldsymbol{y} = x_0\, y_0 + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} \geq \|\widetilde{x}\|_q \|\widetilde{\boldsymbol{y}}\|_p + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} \geq |\widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}}| + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} \geq 0,$$

where the first inequality follows from the fact that $\boldsymbol{x} \in \mathcal{P}_q^n$ and $\boldsymbol{y} \in \mathcal{P}_p^n$, and the second one from Hölder's inequality. Thus, $\mathcal{P}_q^n \subseteq \mathcal{P}_p^{n\star}$.

Now we show $\mathcal{P}_p^{n\star} \subseteq \mathcal{P}_q^n$. Let $\boldsymbol{y} = (y_0; \widetilde{\boldsymbol{y}}) \in \mathcal{P}_p^{n\star}$, we need to show that $\boldsymbol{y} \in \mathcal{P}_q^n$. This is trivial if $\widetilde{\boldsymbol{y}} = \boldsymbol{0}$ or $p = \infty$. If $\widetilde{\boldsymbol{y}} \neq \boldsymbol{0}$ and $1 \leq p < \infty$, let $\boldsymbol{u} \triangleq ({y_1}^{p/q}; {y_2}^{p/q}; \ldots; {y_{n-1}}^{p/q})$ and consider $\boldsymbol{x} \triangleq (\|\boldsymbol{u}\|_p; -\boldsymbol{u}) \in \mathcal{P}_p^n$. Then by using Hölder's inequality, where the equality is attained, we obtain

$$0 \leq \boldsymbol{x}^\mathsf{T}\boldsymbol{y} = \|\boldsymbol{u}\|_p\, y_0 - \boldsymbol{u}^\mathsf{T}\widetilde{\boldsymbol{y}} = \|\boldsymbol{u}\|_p\, y_0 - \|\boldsymbol{u}\|_p \|\widetilde{\boldsymbol{y}}\|_q = \|\boldsymbol{u}\|_p\, (y_0 - \|\widetilde{\boldsymbol{y}}\|_q).$$

This implies that $y_0 \geq \|\widetilde{\boldsymbol{y}}\|_q$, and therefore means that $\boldsymbol{y} \in \mathcal{P}_q^n$. Thus, $\mathcal{P}_p^{n\star} \subseteq \mathcal{P}_q^n$. The proof is complete.

11.3 We first show that $\mathcal{K}^n_{(M^{-1})^\mathsf{T}} \subseteq (\mathcal{K}_M^n)^\star$. Let $\boldsymbol{x} = (x_0; \widetilde{\boldsymbol{x}}) \in \mathcal{K}^n_{(M^{-1})^\mathsf{T}}$, we need to show that $\boldsymbol{x} \in (\mathcal{K}_M^n)^\star$. For any $\boldsymbol{y} = (y_0; \widetilde{\boldsymbol{y}}) \in \mathcal{K}_M^n$, we have

$$\begin{array}{rcl}
\boldsymbol{x}^\mathsf{T}\boldsymbol{y} & = & x_0 y_0 + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} \\
& \geq & \left\|\left(M^{-1}\right)^\mathsf{T}\widetilde{x}\right\| \left\|M\widetilde{\boldsymbol{y}}\right\| + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} \\
& \geq & \left|\widetilde{\boldsymbol{x}}^\mathsf{T} M^{-1} M \widetilde{\boldsymbol{y}}\right| + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} \\
& = & |\widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}}| + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} \;\geq\; 0,
\end{array}$$

where we used the assumptions that $\boldsymbol{x} \in \mathcal{K}^n_{(M^{-1})^\mathsf{T}}$ and $\boldsymbol{y} \in \mathcal{K}_M^n$ to obtain the first inequality, and we used Cauchy-Schwartz inequality to obtain the second inequality. This means that $x \in (\mathcal{K}_M^n)^\star$ and hence $\mathcal{K}^n_{(M^{-1})^\mathsf{T}} \subseteq (\mathcal{K}_M^n)^\star$.

To prove the reverse inclusion, let $\boldsymbol{y} \in (\mathcal{K}_M^n)^\star$, we need to show that $\boldsymbol{y} \in \mathcal{K}^n_{(M^{-1})^\mathsf{T}}$, which is trivial if $\widetilde{\boldsymbol{y}} = 0$. If $\widetilde{\boldsymbol{y}} \neq \boldsymbol{0}$, let $x \triangleq \left(\left\|M\widetilde{\boldsymbol{y}}\right\|; -\widetilde{\boldsymbol{y}}\right) \in \mathcal{K}_M^n$. Then, we have

$$\begin{array}{rcl}
\boldsymbol{x}^\mathsf{T}\boldsymbol{y} & = & x_0 y_0 + \widetilde{\boldsymbol{x}}^\mathsf{T}\widetilde{\boldsymbol{y}} \\
& = & y_0 \left\|M\widetilde{\boldsymbol{y}}\right\| - \widetilde{\boldsymbol{y}}^\mathsf{T}\widetilde{\boldsymbol{y}} \\
& = & y_0 \left\|M\widetilde{\boldsymbol{y}}\right\| - \widetilde{\boldsymbol{y}}^\mathsf{T} M^\mathsf{T} \left(M^{-1}\right)^\mathsf{T} \widetilde{\boldsymbol{y}} \;=\; y_0 \left\|M\widetilde{\boldsymbol{y}}\right\| - \left\|M\widetilde{\boldsymbol{y}}\right\| \left\|\left(M^{-1}\right)^\mathsf{T} \widetilde{\boldsymbol{y}}\right\|,
\end{array}$$

where we used Cauchy-Schwartz inequality, where the equality is attained, to obtain the last equality. Since x belongs to the elliptic cone $\mathcal{K}_M^n$ and $\boldsymbol{y}$ belongs to its dual, it follows that

$$0 \leq \boldsymbol{x}^\mathsf{T}\boldsymbol{y} = \left\|M\widetilde{\boldsymbol{y}}\right\| \left(y_0 - \left\|\left(M^{-1}\right)^\mathsf{T} \widetilde{\boldsymbol{y}}\right\|\right).$$

As $\widetilde{\boldsymbol{y}} \neq \mathbf{0}$, this implies that $y_0 \geq \left\| \left(M^{-1}\right)^{\mathsf{T}} \widetilde{\boldsymbol{y}} \right\|$. That is, $\boldsymbol{y} \in \mathcal{K}^n_{\left(M^{-1}\right)^{\mathsf{T}}}$ and hence $(\mathcal{K}^n_M)^{\star} \subseteq \mathcal{K}^n_{\left(M^{-1}\right)^{\mathsf{T}}}$. The proof is complete.

11.4 To prove item (c) of Lemma 11.8, note that

$$\begin{array}{rcl}
\overline{\boldsymbol{x}}^{+\mathsf{T}} \underline{\boldsymbol{s}}^{+} & = & \left(\overline{\boldsymbol{x}} + \alpha \overline{\Delta \boldsymbol{x}}\right)^{\mathsf{T}} \left(\underline{\boldsymbol{s}} + \alpha \underline{\Delta \boldsymbol{s}}\right) \\
& = & \overline{\boldsymbol{x}}^{\mathsf{T}} \underline{\boldsymbol{s}} + \alpha \left(\overline{\Delta \boldsymbol{x}}^{\mathsf{T}} \underline{\boldsymbol{s}} + \overline{\boldsymbol{x}}^{\mathsf{T}} \underline{\Delta \boldsymbol{s}}\right) + \alpha^2 \overline{\Delta \boldsymbol{x}}^{\mathsf{T}} \underline{\Delta \boldsymbol{s}} \\
& = & \overline{\boldsymbol{x}}^{\mathsf{T}} \underline{\boldsymbol{s}} + \frac{1}{2} \alpha \operatorname{trace}\left(\sigma \mu \boldsymbol{e} - \overline{\boldsymbol{x}} \circ \underline{\boldsymbol{s}}\right) \\
& = & \overline{\boldsymbol{x}}^{\mathsf{T}} \underline{\boldsymbol{s}} + \frac{1}{2} \alpha \sigma \mu \operatorname{trace}(\boldsymbol{e}) - \frac{1}{2} \alpha \operatorname{trace}\left(\overline{\boldsymbol{x}} \circ \underline{\boldsymbol{s}}\right) \\
& = & \overline{\boldsymbol{x}}^{\mathsf{T}} \underline{\boldsymbol{s}} + \alpha \sigma \mu r - \alpha \overline{\boldsymbol{x}}^{\mathsf{T}} \underline{\boldsymbol{s}} \\
& = & \overline{\boldsymbol{x}}^{\mathsf{T}} \underline{\boldsymbol{s}} + \frac{1}{2} \alpha \sigma \overline{\boldsymbol{x}}^{\mathsf{T}} \underline{\boldsymbol{s}} - \alpha \overline{\boldsymbol{x}}^{\mathsf{T}} \underline{\boldsymbol{s}} \; = \; \left(1 - \alpha\left(1 - \frac{\sigma}{2}\right)\right) \overline{\boldsymbol{x}}^{\mathsf{T}} \underline{\boldsymbol{s}},
\end{array}$$

where the third equality follows from items (a) and (b) of Lemma 11.8.

11.5 Given $\alpha \in \mathbb{R}$, using item (c) of Lemma 11.8, we have

$$\boldsymbol{x}(\alpha)^{\mathsf{T}} \boldsymbol{s}(\alpha) = (1 - \alpha + \alpha\sigma) \overline{\boldsymbol{x}}^{\mathsf{T}} \underline{\boldsymbol{s}}, \quad \text{and hence} \quad \mu(\alpha) = (1 - \alpha + \alpha\sigma)\mu.$$

Thus, we get

$$\begin{array}{rcl}
V(\alpha) & = & \boldsymbol{x}(\alpha) \circ \boldsymbol{s}(\alpha) - \mu(\alpha) \boldsymbol{e} \\
& = & \left(\overline{\boldsymbol{x}} + \alpha \overline{\Delta \boldsymbol{x}}\right) \circ \left(\underline{\boldsymbol{s}} + \alpha \underline{\Delta \boldsymbol{s}}\right) - (1 - \alpha + \alpha\sigma) \mu \boldsymbol{e} \\
& = & (1 - \alpha)(\overline{\boldsymbol{x}} \circ \underline{\boldsymbol{s}} - \mu \boldsymbol{e}) + \alpha(\overbrace{\overline{\boldsymbol{x}} \circ \underline{\boldsymbol{s}} - \sigma \mu \boldsymbol{e}}^{-\boldsymbol{h}}) + \alpha \overbrace{\left(\overline{\boldsymbol{x}} \circ \underline{\Delta \boldsymbol{s}} + \overline{\Delta \boldsymbol{x}} \circ \underline{\boldsymbol{s}}\right)}^{\boldsymbol{h}} \\
& & + \alpha^2 \, \overline{\Delta \boldsymbol{x}} \circ \underline{\Delta \boldsymbol{s}} \\
& = & (1 - \alpha)(\overline{\boldsymbol{x}} \circ \underline{\boldsymbol{s}} - \mu \boldsymbol{e}) + \alpha^2 \, \overline{\Delta \boldsymbol{x}} \circ \underline{\Delta \boldsymbol{s}}.
\end{array}$$

This completes the proof.

11.6 By the last equation of system (11.15) and from the operator commutativity, we have

$$\boldsymbol{h} = \overline{\boldsymbol{x}} \circ \underline{\Delta \boldsymbol{s}} + \overline{\Delta \boldsymbol{x}} \circ \underline{\boldsymbol{s}} = \overline{\boldsymbol{x}} \circ \underline{\Delta \boldsymbol{s}} + \mu \overline{\Delta \boldsymbol{x}} \circ \overline{\boldsymbol{x}}^{-1} + \left(\overline{\Delta \boldsymbol{x}} \circ \overline{\boldsymbol{x}}^{-1}\right) \circ \left(\overline{\boldsymbol{x}} \circ \underline{\boldsymbol{s}} - \mu \boldsymbol{e}\right).$$

It immediately follows that

$$\begin{array}{rcl}
\|\boldsymbol{h}\|_F & \geq & \left\| \overline{\boldsymbol{x}} \circ \underline{\Delta \boldsymbol{s}} + \mu \, \overline{\Delta \boldsymbol{x}} \circ \overline{\boldsymbol{x}}^{-1} \right\|_F - \left\| \overline{\Delta \boldsymbol{x}} \circ \overline{\boldsymbol{x}}^{-1} \right\|_F \left\| \overline{\boldsymbol{x}} \circ \underline{\boldsymbol{s}} - \mu \boldsymbol{e} \right\| \\
& \geq & \left\| \overline{\boldsymbol{x}} \circ \underline{\Delta \boldsymbol{s}} + \mu \, \overline{\Delta \boldsymbol{x}} \circ \overline{\boldsymbol{x}}^{-1} \right\|_F - \theta \delta_x \\
& = & \sqrt{\left\| \overline{\boldsymbol{x}} \circ \underline{\Delta \boldsymbol{s}} \right\|_F^2 + \left\| \mu \, \overline{\Delta \boldsymbol{x}} \circ \overline{\boldsymbol{x}}^{-1} \right\|_F^2} - \theta \delta_x \\
& = & \sqrt{\delta_x^2 + \delta_s^2} - \theta \delta_x \; \geq \; (1 - \theta) \sqrt{\delta_x^2 + \delta_s^2},
\end{array} \qquad \text{(A.6)}$$

where the second inequality follows from the assumption that $\left\|\bar{x} \circ \underline{s} - \mu e\right\| \leq \theta\mu$, and the first equality follows from (11.2) and the fact that

$$\begin{aligned}\left(\bar{x} \circ \underline{\Delta s}\right)^{\mathsf{T}} \left(\overline{\Delta x} \circ \bar{x}^{-1}\right) &= \operatorname{trace}\left(\left(\bar{x} \circ \underline{\Delta s}\right) \circ \left(\overline{\Delta x} \circ \bar{x}^{-1}\right)\right) \\ &= \operatorname{trace}\left(\underline{\Delta s} \circ \overline{\Delta x}\right) = \overline{\Delta x}^{\mathsf{T}} \underline{\Delta s},\end{aligned}$$

which is essentially zero due to item (a) of Lemma 11.8.

The right-hand side inequality in (11.18) follows by noting that $(\delta_x - \delta_s)^2 \geq 0$, and the left-hand side inequality in (11.18) follows from the last inequality in (A.6). The proof is complete.

11.7 This is an implementation exercise. As a sample answer, see [Alzalg, 2018, Example 7.1].

11.8 The homogeneous model for the pair (11.29) and (11.30) is as follows:

$$\begin{aligned}
&W_0 \boldsymbol{x}_0 - \boldsymbol{h}_0 \tau = \boldsymbol{0},\\
&B_k \boldsymbol{x}_0 + W_k \boldsymbol{x}_k - \boldsymbol{h}_k \tau = \boldsymbol{0}, k = 1, 2, \ldots, K,\\
&-W_0{}^{\mathsf{T}} \boldsymbol{y}_0 - \sum_{k=1}^{K} B_k{}^{\mathsf{T}} \boldsymbol{y}_k + \tau \boldsymbol{c}_0 - \boldsymbol{s}_0 = \boldsymbol{0},\\
&-W_k{}^{\mathsf{T}} \boldsymbol{y}_k + \tau \boldsymbol{c}_k - \boldsymbol{s}_k = \boldsymbol{0}, k = 1, 2, \ldots, K,\\
&\sum_{k=0}^{K} \boldsymbol{h}_k^{\mathsf{T}} \boldsymbol{y}_k - \sum_{k=0}^{K} \boldsymbol{c}_k{}^{\mathsf{T}} \boldsymbol{x}_k - \kappa = 0,\\
&\boldsymbol{x}_k \succeq \boldsymbol{0}, \boldsymbol{s}_k \succeq \boldsymbol{0}, k = 0, 1, \ldots, K,\\
&\tau \geq 0, \kappa \geq 0.
\end{aligned} \tag{A.7}$$

11.9 The search direction system corresponding to (A.7) is defined by the following system:

$$\begin{aligned}
&W_0 \Delta \boldsymbol{x}_0 - \boldsymbol{h}_0 \Delta\tau = \eta \boldsymbol{r}_{p0},\\
&B_k \Delta \boldsymbol{x}_0 + W_k \Delta \boldsymbol{x}_k - \boldsymbol{h}_k \Delta\tau = \eta \boldsymbol{r}_{pk}, k = 1, 2, \ldots, K,\\
&-W_0{}^{\mathsf{T}} \Delta \boldsymbol{y}_0 - \sum_{k=1}^{K} B_k{}^{\mathsf{T}} \Delta \boldsymbol{y}_k + \Delta\tau \boldsymbol{c}_0 - \Delta \boldsymbol{s}_0 = \eta \boldsymbol{r}_{d0},\\
&-W_k{}^{\mathsf{T}} \Delta \boldsymbol{y}_k + \Delta\tau\, \boldsymbol{c}_k - \Delta \boldsymbol{s}_k = \eta \boldsymbol{r}_{dk}, k = 1, 2, \ldots, K,\\
&\sum_{k=0}^{K} \boldsymbol{h}_k^{\mathsf{T}} \Delta \boldsymbol{y}_k - \sum_{k=0}^{K} \boldsymbol{c}_k{}^{\mathsf{T}} \Delta \boldsymbol{x}_k - \Delta\kappa = \eta r_g,\\
&\kappa \Delta\tau + \tau \Delta\kappa = \gamma\mu - \tau\kappa,\\
&\Delta \boldsymbol{x}_0 \circ \boldsymbol{s}_0 + \boldsymbol{x}_0 \circ \Delta \boldsymbol{s}_0 = \gamma\mu \boldsymbol{e}_0 - \boldsymbol{x}_0 \circ \boldsymbol{s}_0,\\
&\Delta \boldsymbol{x}_k \circ \boldsymbol{s}_k + \boldsymbol{x}_k \circ \Delta \boldsymbol{s}_k = \gamma\mu \boldsymbol{e}_k - \boldsymbol{x}_k \circ \boldsymbol{s}_k, k = 1, 2, \ldots, K,
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{r}_{p0} &\triangleq \boldsymbol{h}_0\tau - W_0\boldsymbol{x}_0;\\
\boldsymbol{r}_{pk} &\triangleq \boldsymbol{h}_k\tau - B_k\boldsymbol{x}_0 - W_k\boldsymbol{x}_k;\\
\boldsymbol{r}_{d0} &\triangleq {W_0}^{\mathsf{T}}\boldsymbol{y}_0 + \sum_{k=1}^{K} {B_k}^{\mathsf{T}}\boldsymbol{y}_k + \boldsymbol{s}_0 - \tau\boldsymbol{c}_0;\\
\boldsymbol{r}_{dk} &\triangleq {W_k}^{\mathsf{T}}\boldsymbol{y}_k + \boldsymbol{s}_k - \tau\boldsymbol{c}_k;\\
r_g &\triangleq \kappa - \sum_{k=0}^{K} \boldsymbol{h}_k^{\mathsf{T}}\boldsymbol{y}_k + \sum_{k=0}^{K} {\boldsymbol{c}_k}^{\mathsf{T}}\boldsymbol{x}_k;\\
\mu &\triangleq \frac{1}{2r(K+1)+1}\left(\sum_{k=0}^{K} {\boldsymbol{x}_k}^{\mathsf{T}}\boldsymbol{s}_k + \tau\kappa\right),
\end{aligned}$$

and η and γ are two parameters.

12.1 To prove item (c) of Lemma 12.6, note that

$$\begin{aligned}
X^+ \bullet S^+ &= (X + \alpha\Delta X)\bullet(S + \alpha\Delta S)\\
&= X\bullet S + \alpha\,(\Delta X\bullet S + X\bullet\Delta S) + \alpha^2\Delta X\bullet\Delta S\\
&= X\bullet S + \tfrac{1}{2}\alpha\ \mathrm{trace}\,(\sigma\mu I - X\circ S)\\
&= X\bullet S + \tfrac{1}{2}\alpha\sigma\mu\ \mathrm{trace}\,(I) - \tfrac{1}{2}\alpha\ \mathrm{trace}\,(X\circ S)\\
&= X\bullet S + \alpha\sigma\mu r - \alpha X\bullet S\\
&= X\bullet S + \tfrac{1}{2}\alpha\sigma X\bullet S - \alpha X\bullet S \;=\; \left(1-\alpha\left(1-\tfrac{\sigma}{2}\right)\right)\,X\bullet S,
\end{aligned}$$

where the third equality follows from items (a) and (b) of Lemma 12.6.

12.2 Given $\alpha\in\mathbb{R}$, using item (c) of Lemma 12.6, we have

$$X(\alpha)\bullet S(\alpha) = (1-\alpha+\alpha\sigma)\,X\bullet S, \quad \text{and hence} \quad \mu(\alpha) = (1-\alpha+\alpha\sigma)\mu.$$

Thus, we get

$$\begin{aligned}
V(\alpha) &= X(\alpha)\circ \boldsymbol{s}(\alpha) - \mu(\alpha)I\\
&= (X+\alpha\Delta X)\circ(S+\alpha\Delta S) - (1-\alpha+\alpha\sigma)\mu I\\
&= (1-\alpha)(X\circ S-\mu I) + \alpha\overbrace{(X\circ S - \sigma\mu I)}^{-H} + \alpha\,\overbrace{(X\circ\Delta S + \Delta X\circ S)}^{H}\\
&\quad +\alpha^2\,\Delta X\circ\Delta S\\
&= (1-\alpha)(X\circ S-\mu I) + \alpha^2\,\Delta X\circ\Delta S.
\end{aligned}$$

This completes the proof.

12.3 Using the last equation of system (12.18) and the commutativity property, we have

$$H = X\circ\Delta S + \Delta X\circ S = X\circ\Delta S + \mu\Delta X\circ X^{-1} + \left(\Delta X\circ X^{-1}\right)\circ(XS-\mu I)\,.$$

It immediately follows that

$$\begin{array}{rcl}\|H\|_F & \geq & \left\|X \circ \Delta S + \mu\, \Delta X \circ X^{-1}\right\|_F - \left\|\Delta X X^{-1}\right\|_F \left\|XS - \mu I\right\|_2 \\ & \geq & \left\|X \circ \Delta S + \mu\, \Delta X \circ X^{-1}\right\|_F - \theta\delta_X \\ & = & \sqrt{\|X\Delta S\|_F^2 + \left\|\mu\, \Delta X X^{-1}\right\|_F^2} - \theta\delta_X \\ & = & \sqrt{\delta_X^2 + \delta_S^2} - \theta\delta_X \;\geq\; (1-\theta)\sqrt{\delta_X^2 + \delta_S^2},\end{array} \tag{A.8}$$

where the second inequality follows from the assumption that $\left\|XS - \mu I\right\|_2 \leq \theta\mu$, and the first equality follows from (12.2) and the fact that

$$\begin{array}{rcl}(X \circ \Delta S) \bullet \left(\Delta X \circ X^{-1}\right) & = & \operatorname{trace}\left((X \circ \Delta S) \circ \left(\Delta X \circ X^{-1}\right)\right) \\ & = & \operatorname{trace}\left(\Delta S \circ \Delta X\right) \;=\; \Delta X \bullet \Delta S,\end{array}$$

which is essentially zero due to item (a) of Lemma 12.6.

The right-hand side inequality in (12.20) follows by noting that $(\delta_X - \delta_S)^2 \geq 0$, and the left-hand side inequality in (12.20) follows from the last inequality in (A.8). The proof is complete.

12.4 This is an implementation exercise. As a sample answer, see [Touil et al., 2017, Example 2].

12.5 The homogeneous model for the pair (P|SDP) and (D|SDP) is as follows (see also Potra and Sheng [1998], Jin et al. [2012]):

$$\begin{array}{ccccccc}\mathcal{A}X & & & -\boldsymbol{b}\tau & & = & \boldsymbol{0}, \\ & -\mathcal{A}^{\star}\boldsymbol{y} & -S & +C\tau & & = & 0, \\ -C \bullet X & +\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & & -\kappa & = & 0, \\ X & & & & & \succeq & 0, \\ & & S & & & \succeq & 0, \\ & & & \tau & & \geq & 0, \\ & & & & \kappa & \geq & 0.\end{array} \tag{A.9}$$

12.6 The search direction system corresponding to (A.9) is defined by the following system (see also Potra and Sheng [1998], Jin et al. [2012]):

$$\begin{array}{cccccccccc}\mathcal{A}\,\Delta X & & & & & -\boldsymbol{b}\,\Delta\tau & & & = & \eta r_p, \\ & - & \mathcal{A}^{\star}\,\Delta\boldsymbol{y} & - & \Delta S & +C\,\Delta\tau & & & = & \eta R_d, \\ -C \bullet\, \Delta X & + & \boldsymbol{b}^{\mathsf{T}}\,\Delta\boldsymbol{y} & & & & - & \Delta\kappa & = & \eta r_g, \\ & & & & & \kappa\,\Delta\tau & + & \tau\,\Delta\kappa & = & \gamma\mu - \tau\kappa, \\ \mathcal{H}_P(\Delta X S) & & & + & \mathcal{H}_P(X\Delta S) & & & & = & \gamma\mu I - \mathcal{H}_P(XS),\end{array} \tag{A.10}$$

where $\mathcal{H}_P(\cdot)$ is the symmetrization operator $\mathcal{H}_P : \mathbb{R}^{n\times n} \longrightarrow \mathbb{S}^n$ defined in (12.5), η and γ are two parameters, and

$$\begin{array}{lcl} \boldsymbol{r}_p & \triangleq & \boldsymbol{b}\tau - \mathcal{A}X, \\ R_d & \triangleq & \mathcal{A}^\star \boldsymbol{y} + S - \tau C, \\ r_g & \triangleq & C \bullet X - \boldsymbol{b}^\mathsf{T}\boldsymbol{y} + \kappa, \\ \mu & \triangleq & \frac{1}{n+1}(X \bullet S + \tau\kappa). \end{array}$$

Algorithm A.3: Generic homogeneous self-dual algorithm for SDP

Input: Data in Problems (P|SDP) and (D|SDP) $(X, \boldsymbol{y}, S, \tau, \kappa) \triangleq (I, \boldsymbol{0}, I, 1, 1)$
Output: An approximate optimal solution to Problem (P|SDP)

1: **while** a stopping criterion is not satisfied **do**
2: choose η, γ
3: compute the solution $(\Delta X, \Delta\boldsymbol{y}, \Delta S, \Delta\tau, \Delta\kappa)$ of the linear system (A.10)
4: compute a step length θ so that
$X + \theta\Delta X \succ 0$
$S + \theta\Delta S \succ 0$
$\tau + \theta\Delta\tau > 0$
$\kappa + \theta\Delta\kappa > 0$
5: set the new iterate according to
$(X, \boldsymbol{y}, S, \tau, \kappa) \triangleq (X, \boldsymbol{y}, S, \tau, \kappa) + \theta(\Delta X, \Delta\boldsymbol{y}, \Delta S, \Delta\tau, \Delta\kappa)$
6: **end**

12.7 We state the generic homogeneous algorithm for solving the pair (P|SDP) and (D|SDP) in Algorithm A.3 (see also [Jin et al., 2012, Algorithm 1]).

12.8 Under Assumptions 12.2 and 12.1, if the pair (P|SDP) and (D|SDP) has a solution $(X^\star, \boldsymbol{y}^\star, S^\star)$, then Algorithm A.3 finds an ϵ-approximate solution in at most

$$O\left(\sqrt{n}\ln\left(\text{trace}\,(X^\star + S^\star)\left(\frac{\epsilon_0}{\epsilon}\right)\right)\right)$$

iteration (see also [Potra and Sheng, 1998, Theorems 5.2 and 6.2]).

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