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The cycle-complete graph Ramsey number $r(C_6, K_8) \leq 38$

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Abstract. The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer N such that every graph G of order N contains a cycle C_m on m vertices or has independent number $\alpha(G) \geq n$. It has been conjectured by Erdős, Faudree, Rousseau and Schelp that $r(C_m, K_n) = (m-1)(n-1)+1$ for all $m \geq n \geq 3$ (except $r(C_3, K_3) = 6$). In this paper, we show that $r(C_6, K_8) \leq 38$.

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§1. Introduction

Through out this paper we adopt the standard notations, a cycle on m vertices will be denoted by C_m and the complete graph on n vertices by K_n. The minimum degree of a graph G is denoted by $\delta(G)$. An independent set of vertices of a graph G is a subset of $V(G)$ in which no two vertices are adjacent. The independence number of a graph $G, \alpha(G)$, is the size of the largest independent set.

The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer N such that for every graph G of order N, G contains C_m or $\alpha(G) \geq n$. The graph $(n-1)K_{m-1}$ shows that $r(C_m, K_n) \geq (m-1)(n-1)+1$. In one of the earliest contributions to graphical Ramsey theory, Bondy and Erdős [4] proved that for all $m \geq n^2 - 2, r(C_m, K_n) = (m-1)(n-1)+1$. After that, Faudree and Schelp [7] and Rosta [14] proved that for $m \geq 4$, $r(C_m, K_3) = 2(m-1)+1$. Later on, Erdős et al. [6] conjectured that $r(C_m, K_n) = (m-1)(n-1)+1$, for all $m \geq n \geq 3$ except $r(C_3, K_3) = 6$. Nikiforov [12] proved the conjecture for $m \geq 4n+2$.

The conjecture was confirmed by Sheng et al. [17] and Bollobás et al. [3] for $n = 4$ and $n = 5$, respectively. Recently, the conjecture was proved by Schiermeyer [15] for $n = 6$. Most recently, Baniabedalruhman [1], Baniabedalruhman and Jaradat [2] and Cheng et al. [5] independently proved that $r(C_7, K_7) = 37$. Also, In [9] and [10], it was proved that $r(C_8, K_7) = 43$ and $r(C_8, K_8) = 50$. In a related work, Radziszowski and Tse [13] showed that $r(C_4, K_7) = 22$ and $r(C_4, K_8) = 26$ and Cheng et al. [5] proved that $r(C_6, K_7)$ $= 31$. In [11] Jayawardene and Rousseau proved that $r(C_5, K_6) = 21$. Also, Schiermeyer [16] proved that $r(C_5, K_7) = 25$. In this article we prove that $r(C_6, K_8) \leq 38$.

In the rest of this work $N(u)$ stands for the neighbor of the vertex u, which is the set of all vertices of G that are adjacent to u. The symbol $N[u]$ denotes to $N(u) \cup \{u\}$. The symbol $\langle V_1 \rangle_G$ stands for the subgraph of G whose vertex set is $V_1 \subseteq V(G)$ and whose edge set is the set of those edges of G that have both ends in V_1, and is called the subgraph of G induced by V_1.

§2. Main Result

It is known, by taking $G = (n-1)K_{m-1}$, that $r(C_m, K_n) \geq (m-1)(n-1)+1$ and so $r(C_6, K_8) \geq 36$. In this section, we prove that $r(C_6, K_8) \leq 38$. Our proof consists of a series of seven lemmas.

Lemma 2.1. Let G be a graph of order 38 that contains neither C_6 nor an 8-element independent set. Then $\delta(G) \geq 7$.

Proof. Suppose that G contains a vertex of degree less than 7, say u. Then $|V(G) - N[u]| \geq 38 - 7 = 31$. Since $r(C_6, K_7) = 31$, as a result, $G - N[u]$ has an independent set consisting of 7 vertices. This set with the vertex u is an independent set consisting of 8 vertices. This is a contradiction. □

Throughout all Lemmas 2.2 to 2.6, we let G be a graph with minimum degree $\delta(G) \geq 7$ that contains neither C_6 nor an 8-element independent set.

Lemma 2.2. If G contains $K_5 - S_3$, then $|V(G)| \geq 40$.

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the vertex set of $K_5 - S_3$ where the induced subgraph of $\{u_1, u_2, u_3, u_4\}$ is isomorphic to K_4. With out loss of generality we may assume that $u_1u_5, u_2u_5 \in E(G)$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 5$. Since $\delta(G) \geq 7$, $|U_i| \geq 3$ for all $1 \leq i \leq 5$. Note that between any two vertices of U there is a path of order 5 except possibly between u_1 and u_2. Thus, $U_i \cap U_j = \varnothing$ for all $1 \leq i < j \leq 5$ except possibly for $(i, j) = (1, 2)$. Also note that between any two vertices of U there is a path of order 4. Hence, for all $1 \leq i < j \leq 5$ and for all $x \in U_i$ and $y \in U_j$, $xy \notin E(G)$. Similarly, since between any two vertices of

U there is a path of order 3, $N_R(U_i) \cap N_R(U_j) = \varnothing$ for all $1 \leq i < j \leq 5$. Therefore, $(U_i \cup N_R(U_i)) \cap (U_j \cup N_R(U_j)) = \varnothing$ for all $1 \leq i < j \leq 5$ with $(i, j) \neq (1, 2)$. Moreover, since $u_1u_4u_3u_2$ is a path of length 4, $(U_1 \cap N_R(U_2)) = (U_2 \cup N_R(U_1)) = \varnothing$. Let $A = (\{u_1\} \cup U_1 \cup N_R(U_1)) \cup (\{u_2\} \cup U_2 \cup N_R(U_2))$. Note that $|U_i \cup N_R(U_i) \cup \{u_i\}| \geq \delta(G) + 1 = 8$ for each $3 \leq i \leq 5$. Thus, it is suffices to show that $|A| \geq 16$. If $U_1 - U_2 \neq \varnothing$ and $U_2 - U_1 \neq \varnothing$, then $|A| \geq |\{u_1\} \cup (U_1 - U_2) \cup N_R(U_1 - U_2)| + |\{u_2\} \cup (U_2 - U_1) \cup N_R(U_2 - U_1)| \geq 8 + 8 = 16$. Hence, we may assume that $U_1 - U_2 = \varnothing$ or $U_2 - U_1 = \varnothing$. Then $|U_1 \cap U_2| \geq 3$. Note that for any $x, y \in U_1 \cap U_2$, we have that $N_G(x) \cap N_G(y) = \varnothing$ because otherwise G contains C_6. Hence,

$$\sum_{x \in U_1 \cap U_2} |N_G[x] - \{u_1, u_2\}| \geq 6|U_1 \cap U_2| \geq 18.$$

□

Lemma 2.3. If G contains K_4, then G contains $K_5 - S_3$.
Proof. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of K_4. Let $R = G - U$ and $U_i = N(u_i) \cap V(G)$ for each $1 \leq i \leq 4$. Since $\delta(G) \geq 7$, $|U_i| \geq 4$ for all $1 \leq i \leq 4$. Now we consider the following cases:
Case 1. $U_i \cap U_j \neq \varnothing$ for some $1 \leq i < j \leq 4$. Then it is clear that G contains $K_5 - S_3$. In fact, if we take $w \in U_i \cap U_j$, then the induced subgraph $\langle U \cup \{w\} \rangle_G$ contains $K_5 - S_3$.
Case 2. $U_i \cap U_j = \varnothing$ for each $1 \leq i < j \leq 4$. Note that between any two vertices of U there is a path of order 4. Thus for all $1 \leq i < j \leq 4$ and for all $x \in U_i$ and $y \in U_j$, $xy \notin E(G)$. Therefore, at least one of $\langle U_i \rangle_G$ where $1 \leq i \leq 4$ is a complete graph (otherwise, two independent vertices of $\langle U_i \rangle_G$ for each $1 \leq i \leq 4$ form an 8-element independent set, a contradiction). Now, since $|U_i| \geq 4$ for all $1 \leq i \leq 4$, as a result at least one of the induced subgraph $\langle U_i \cup \{u_i\} \rangle_G$ where $1 \leq i \leq 4$ contains K_5. Hence, G contains $K_5 - S_3$. □

Lemma 2.4. If G contains $K_1 + P_4$, then G contains K_4.
Proof: Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the vertex set of $K_1 + P_4$, where u_1 is a K_1 and $P_4 = u_2u_3u_4u_5$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 5$. Then as in Lemma 2.2, $|U_i| \geq 3$ for all $1 \leq i \leq 5$. Note that between any two vertices of $U - \{u_1\}$ there are paths of order 5 and 4. Thus, $U_i \cap U_j = \varnothing$ and $xy \notin E(G)$ for all $x \in U_i$ and $y \in U_j$ for any $2 \leq i < j \leq 5$. Therefore, $\langle U_i \rangle_G$ is complete graph for some $2 \leq i \leq 5$ (otherwise, two independent vertices of $\langle U_i \rangle_G$ for each $2 \leq i \leq 5$ form an 8-element independent set, a contradiction). Now, since $|U_i| \geq 3$ for all $2 \leq i \leq 5$, at least one induced subgraph $\langle U_i \cup \{u_i\} \rangle_G$ where $2 \leq i \leq 5$ contains K_4. Hence, G contains K_4.
□

Lemma 2.5. If G contains K_1+P_3, then G contains K_1+P_4 or K_4

Proof. Let $U=\{u_1,u_2,u_3,u_4\}$ be the vertex set of K_1+P_3 where $K_1=u_1$ and $P_3=u_2u_3u_4$. Now, if $u_2u_4\in E(G)$, then $\langle U\rangle_G$ is K_4. Thus, in the rest of this lemma we may assume that $u_2u_4\notin E(G)$. Let $R=G-U$ and $U_i=N(u_i)\cap V(R)$ for each $1\le i\le 4$. Since $\delta(G)\ge 7$, $|U_i|\ge 4$ for $i=1,3$ and $|U_i|\ge 5$ for $i=2,4$. We now consider the following cases:

Case 1. $U_i\cap U_j=\varnothing$ for all $2\le i<j\le 4$. Note that between any two vertices of U there is a path of order 4. Thus, for all $2\le i<j\le 5$, if $x\in U_i$ and $y\in U_j$, then $xy\notin E(G)$. Hence, either $\alpha(\langle U_2\rangle_G)\le 2$ or $\alpha(\langle U_4\rangle_G)\le 2$ or $\langle U_3\rangle_G$ is a complete graph (otherwise, three independent vertices of $\langle U_i\rangle_G$ for each $i=2,4$ and two independent vertices of $\langle U_3\rangle_G$ form an 8-element independent set, a contradiction). Now, if $\langle U_3\rangle_G$ is a complete graph, then $\langle U_3\rangle_G$ contains K_4 because $|U_3|\ge 4$. Also, if $\alpha(\langle U_2\rangle_G)\le 2$ or $\alpha(\langle U_4\rangle_G)\le 2$, say $\alpha(\langle U_2\rangle_G)\le 2$, then $\langle U_2\rangle_G$ contains either K_3 or P_4. And so $\langle U_2\cup\{u_2\}\rangle_G$ contains either K_4 or K_1+P_4.

Case 2. $U_2\cap U_3\ne\varnothing$, say $u_5\in U_2\cap U_3$. Then G contains K_1+P_4, where u_3 is a K_1 and $P_4=u_5u_2u_1u_4$.

Case 3. $U_3\cap U_4\ne\varnothing$, say $u_5\in U_3\cap U_4$. Then G contains K_1+P_4, where u_3 is a K_1 and $P_4=u_5u_4u_1u_2$.

Case 4. $U_2\cap U_4\ne\varnothing$, say $u_5\in U_2\cap U_4$. If $u_5u_3\in E(G)$, then G contains K_1+P_4 where $K_1=u_3$ and $P_4=u_5u_4u_1u_2$. Thus, in the rest of this lemma we assume that $u_5u_3\notin E(G)$. Now let $U^{'}=\{u_1,u_2,u_3,u_4,u_5\}$, $R^{'}=G-U^{'}$ and $U_i^{'}=N(u_i)\cap V(R^{'})$ for each $1\le i\le 5$. We now have the following: (1) If $U_2^{'}\cap U_3^{'}\ne\varnothing$, then we get a similar case to Case 2 and so G contains K_1+P_4. Therefore, in the rest of this case we can assume that $U_2^{'}\cap U_3^{'}=\varnothing$. (2) $U_2^{'}\cap U_5^{'}=\varnothing$ (otherwise, if $x\in U_2^{'}\cap U_5^{'}$, then $xu_2u_1u_3u_4u_5x$ is a C_6, a contradiction). (3) $U_3^{'}\cap U_5^{'}=\varnothing$ (otherwise, if $x\in U_3^{'}\cap U_5^{'}$, then $xu_3u_2u_1u_4u_5x$ is a C_6, a contradiction). Note that u_1 and u_3 are symmetric in the role. Thus, as in the above, we may assume that $u_5u_1\notin E(G)$. This means $|U_5^{'}|\ge 5$. Also, as in the above, we may assume that $U_1^{'}\cap U_2^{'}=\varnothing$ and we have that $U_1^{'}\cap U_5^{'}=\varnothing$. Similarly, $U_1^{'}\cap U_3^{'}=\varnothing$ (because, if $x\in U_1^{'}\cap U_3^{'}$, then $xu_1u_2u_5u_4u_3x$ is a C_6, a contradiction). Consequently there is no edge joining u_1 to $U_2^{'}\cup U_3^{'}\cup U_5^{'}$. Also, note the following: (I) for each $x\in U_2^{'}$ and $y\in U_3^{'}$, $xy\notin E(G)$ (Otherwise, $xyu_2u_1u_4u_3x$ is a C_6, a contradiction). (II) for each $x\in U_2^{'}$ and $y\in U_5^{'}$, $xy\notin E(G)$ (Otherwise, $xyu_2u_3u_4u_5x$ is a C_6, a contradiction). (III) for each $x\in U_3^{'}$ and $y\in U_5^{'}$, $xy\notin E(G)$ (Otherwise, $xyu_3u_1u_4u_5x$ is a C_6, a contradiction). Therefore, either $\left\langle U_2^{'}\right\rangle_G$ is complete or $\left\langle U_3^{'}\right\rangle_G$ is complete or $\alpha(\left\langle U_5^{'}\right\rangle_G)\le 2$ (Otherwise, $\alpha(\left\langle\{u_1\}\cup U_2^{'}\cup U_3^{'}\cup U_5^{'}\right\rangle_G)\ge 1+2+2+3=8$, a contradiction). Now, if $\left\langle U_2^{'}\right\rangle_G$ is complete or $\left\langle U_3^{'}\right\rangle_G$ is complete, then G contains K_4. Also, if $\alpha(\left\langle U_5' \right\rangle_G) \leq 2$, then as above, $\left\langle U_5' \right\rangle_G$ contains either K_3 or P_4. And so $\left\langle U_5' \cup \{u_5\} \right\rangle_G$ contains either K_4 or $K_1 + P_4$. □

Lemma 2.6. If G contains K_3, then G contains $K_1 + P_3$ or K_4

Proof. Let $U = \{u_1, u_2, u_3\}$ be the vertex set of K_3. Let $R = G - U$ and $U_i = N(u_i) \cap V(G)$ for each $1 \leq i \leq 3$. Since $\delta(G) \geq 7$, $|U_i| \geq 5$ for all $1 \leq i \leq 3$. Now we split our work into the following two cases:

Case 1: $U_i \cap U_j \neq \varnothing$ for some $1 \leq i < j \leq 3$. Then G contains $K_1 + P_3$. The result is obtained.

Case 2: $U_i \cap U_j = \varnothing$ for all $1 \leq i < j \leq 3$. Let $z_i \in U_i$ for each $1 \leq i \leq 3$. Let $Z = \{z_1, z_2, z_3\}$ and $R' = G - (U \cup Z)$. Let $Z_i = N(z_i) \cap V(R')$. If $|E \langle Z \rangle_G| \geq 2$, then $\langle Z \cup U \rangle_G$ contains a cycle of order 6. Thus we may assume that $|E \langle Z \rangle_G| \leq 1$. Then $|Z_i| \geq 5$ for each $1 \leq i \leq 3$. Note that between any two vertices of U there are paths of order 2 and 3. Hence $Z_i \cap Z_j = \varnothing$ and $xy \notin E(G)$ for each $x \in Z_i$, $y \in Z_j$ and $1 \leq i < j \leq 3$. Now, $\alpha(\langle Z_i \rangle_G) \leq 2$ for some $1 \leq i \leq 3$ (Otherwise, if $\alpha(\langle Z_i \rangle_G) \geq 3$ for each $1 \leq i \leq 3$, $\alpha(\langle Z_1 \cup Z_2 \cup Z_3 \rangle_G) \geq 3+3+3 = 9$. Thus $\alpha(G) \geq 9$, a contradiction). Without loss of generality we may assume that $\alpha(\langle Z_1 \rangle_G) \leq 2$. By an argument similar to the above and since $|Z_1| \geq 5$, $\langle Z_1 \cup \{z_1\} \rangle_G$ contains either $K_1 + P_4$ or K_4. Hence, G contains $K_1 + P_3$ or K_4. □

Lemma 2.7. Let G be a graph of order 38 that contains neither C_6 nor an 8-element independent set. Then G contains K_3.

Proof. Suppose that G does not contain K_3. Then $|N(u)| \leq 7$ for any $u \in V(G)$ (because otherwise, if u is a vertex with $|N(u)| \geq 8$, then the induced subgraph $< N(u) >_G$ does not contain P_2. Hence, the induced subgraph $< N(u) >_G$ is a null graph, and so , $\alpha(G) \geq 8$. This is a contradiction).

Now, for any two independent vertices u_1 and u_2, $|N[u_1] \cup N[u_2]| \geq 13$ (To see that, suppose that $|N[u_1] \cup N[u_2]| \leq 12$. Then $|V(G)| - |N[u_1] \cup N[u_2]| \geq 38 - 12 = 26$. But, $r(C_6, K_6) = 26$, hence $G - \{N[u_1] \cup N[u_2]\}$ contains an independent set of 6 vertices. Thus, this independent set with u_1 and u_2 is an independent set of 8 vertices, a contradiction).

Now, by Lemma 2.1, $\delta(G) \geq 7$. Thus $|N(u_1)| = 7$ and $N(u_1)$ is independent. Similarly, $|N(u_2)| = 7$ and $N(u_2)$ is independent. Hence $|N(u_1) \cap N(u_2)| = |N[u_1] \cap N[u_2]| = |N[u_1]| + |N[u_2]| - |N[u_1] \cup N[u_2]| \leq 3$. Let $N'(u_1) = N(u_1) - (N(u_2) \cap N(u_1))$ and $N'(u_2) = N(u_2) - (N(u_1) \cap N(u_2))$. Then $|N'(u_1)| = |N'(u_2)| \geq 4$. Since $\alpha(G) \leq 7$, we have $|N(X) \cap N'(u_2)| \geq |X|$ for each $X \subseteq N'(u_1)$. Therefore by the Matching Theorem of Hall, there is a perfect matching between $N'(u_1)$ and $N'(u_2)$, which implies that $\left\langle N'[u_1] \cup N'[u_2] \right\rangle_G$ contains C_6 where $N'[u_1] = N'(u_1) \cup \{u_1\}$ and $N'[u_2] = N'(u_2) \cup \{u_2\}$. This is a contradiction. □

Theorem 2.8. The cycle-complete Ramsey number $r(C_6, K_8) \leq 38$.
Proof : We prove it by contradiction. Suppose that G is a graph of order 38 which contains neither C_6 nor an 8-element independent set. Then by Lemma 2.7, G contains K_3. Also, by Lemma 2.1, $\delta(G) \geq 7$. Thus, by Lemmas 2.6, 2.5, 2.4, 2.3, and 2.2, $|V(G)| \geq 40$. This is a contradiction. Thus, $r(C_6, K_8) \leq 38$.
$\square$

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