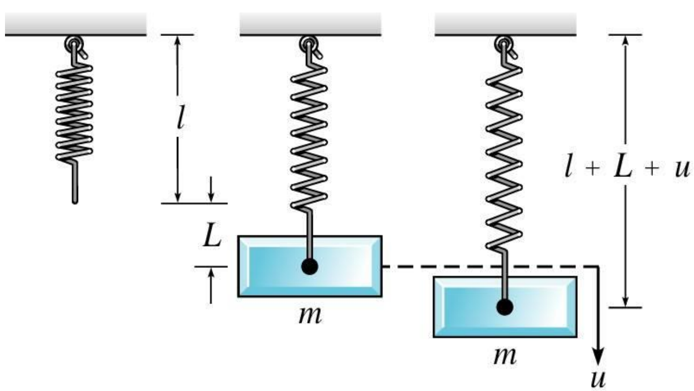
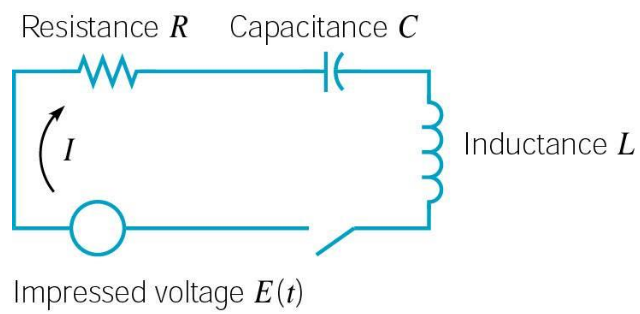
Applications of 2^{nd}-order ODEs: Mechanical & Electrical Vibrations

- There are two important areas of application for second order linear equations with constant coefficients, which are in modeling mechanical and electrical oscillations.
- We will study the motion of a mass on a spring in detail.
- An understanding of the behavior of this simple system is the first step in investigation of more complex vibrating systems.

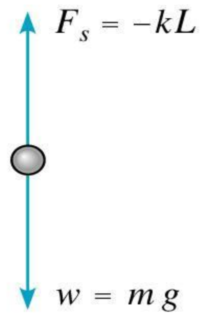


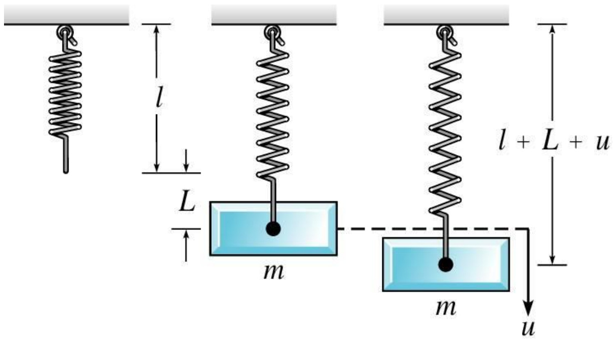


Spring – Mass System

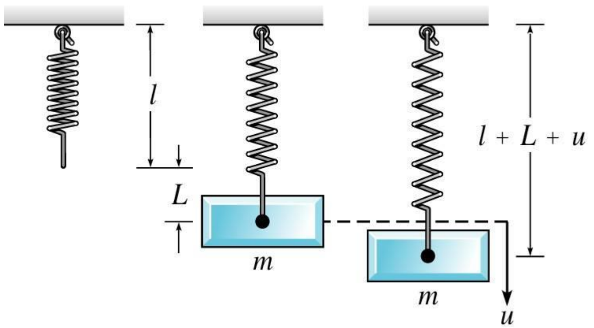
- Suppose a mass *m* hangs from a vertical spring of original length *l*. The mass causes an elongation *L* of the spring.
- The force F_G of gravity pulls the mass down. This force has magnitude *mg*, where *g* is acceleration due to gravity.
- The force F_S of the spring stiffness pulls the mass up. For small elongations *L*, this force is proportional to *L*.

  That is, $F_s = kL$ (Hooke's Law).
- When the mass is in equilibrium, the forces balance each other: $mg = kL$



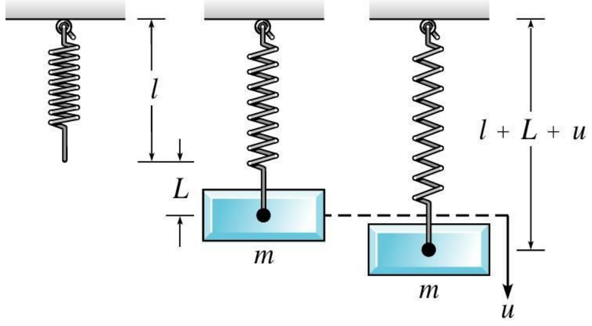


Spring Model



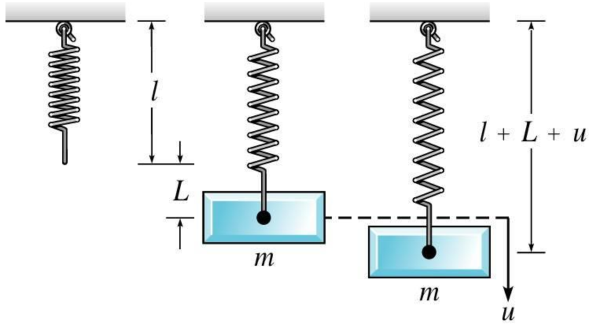
- We will study the motion of a mass when it is acted on by an external force (forcing function) and/or is initially displaced.
- Let $u(t)$ denote the displacement of the mass from its equilibrium position at time t, measured downward.
- Let f be the net force acting on the mass. We will use Newton's 2nd Law: $$mu''(t) = f(t)$$
- In determining f, there are four separate forces to consider:
  - Weight: $w = mg$ (downward force)
  - Spring force: $F_s = -k(L + u)$ (up or down force, see next slide)
  - Damping force: $F_d(t) = -\gamma u'(t)$ (up or down, see following slide)
  - External force: $F(t)$ (up or down force, see text)

Spring Model:
Spring Force Details



- The spring force F_s acts to restore a spring to the natural position, and is proportional to $L + u$. If $L + u > 0$, then the spring is extended and the spring force acts upward. In this case
$$F_s = -k(L+u)$$
- If $L + u < 0$, then spring is compressed a distance of $|L + u|$, and the spring force acts downward. In this case
$$F_s = k|L+u| = k\left[-\left(L+u\right)\right] = -k\left(L+u\right)$$
- In either case,
$$F_s = -k(L+u)$$

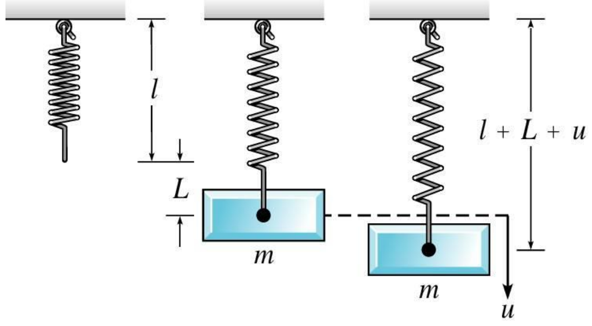
Spring Model: Damping Force Details



- The damping or resistive force F_d acts in the opposite direction as the motion of the mass. This can be complicated to model. F_d may be due to air resistance, internal energy dissipation due to action of spring, friction between the mass and guides, or a mechanical device (dashpot) imparting a resistive force to the mass.
- We simplify this and assume F_d is proportional to the velocity.
- In particular, we find that
  - If $u' > 0$, then u is increasing, so the mass is moving downward. Thus F_d acts upward and hence $F_d = -\gamma u'$, where $\gamma > 0$.
  - If $u' < 0$, then u is decreasing, so the mass is moving upward. Thus F_d acts downward and hence $F_d = -\gamma u'$, $\gamma > 0$.
- In either case,

$$F_d(t) = -\gamma u'(t), \quad \gamma > 0$$

Spring Model: Differential Equation



- Taking into account these forces, Newton's Law becomes:

$$mu''(t) = mg + F_s(t) + F_d(t) + F(t)$$
$$= mg - k[L + u(t)] - \gamma u'(t) + F(t)$$

- Recalling that *mg = kL*, this equation reduces to

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

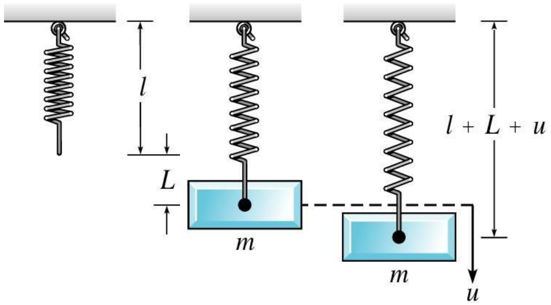
where the constants *m*, γ, and *k* are positive.

- We can prescribe initial conditions also:

$$u(0) = u_0, \; u'(0) = v_0$$

- It follows from Theorem 3.2.1 that there is a unique solution to this initial value problem. Physically, if the mass is set in motion with a given initial displacement and velocity, then its position is uniquely determined at all future times.

Example 1:
Find Coefficients (1 of 2)



- A 4 lb mass stretches a spring 2". The mass is displaced an additional 6" and then released; and is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/sec. Formulate the IVP that governs the motion of this mass:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t),\ \ u(0) = u_0,\ \ u'(0) = v_0$$

- Find *m*:

$$w = mg \Rightarrow m = \frac{w}{g} \Rightarrow m = \frac{4\,\text{lb}}{32\text{ft}/\sec^2} \Rightarrow m = \frac{1}{8}\frac{\text{lb}\sec^2}{\text{ft}}$$

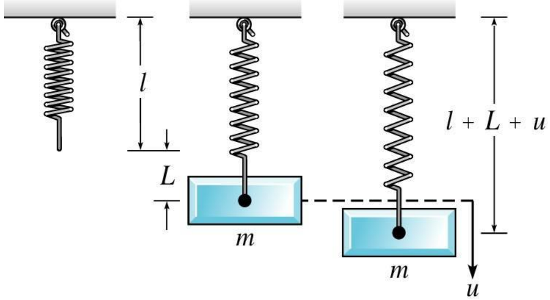
- Find γ:

$$\gamma u' = 6\,\text{lb} \Rightarrow \gamma = \frac{6\,\text{lb}}{3\text{ft}/\sec} \Rightarrow \gamma = 2\frac{\text{lb}\sec}{\text{ft}}$$

- Find *k*:

$$F_s = -k\,L \Rightarrow k = \frac{4\,\text{lb}}{2\,\text{in}} \Rightarrow \ k = \frac{4\,\text{lb}}{1/6\,\text{ft}} \Rightarrow k = 24\frac{\text{lb}}{\text{ft}}$$

Example 1: Find IVP (2 of 2)



- Thus our differential equation becomes

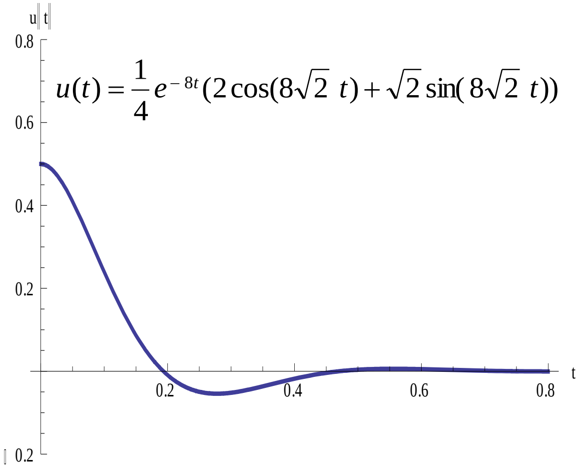
$$\frac{1}{8}u''(t)+2u'(t)+24u(t)=0$$

and hence the initial value problem can be written as

$$u''(t)+16u'(t)+192u(t)=0$$
$$u(0)=\frac{1}{2},\quad u'(0)=0$$

- This problem can be solved using the methods of Chapter 3.3 and yields the solution

$$u(t)=\frac{1}{4}e^{-8t}(2\cos(8\sqrt{2}\ t)+\sqrt{2}\sin(8\sqrt{2}\ t))$$



Spring Model:
Undamped Free Vibrations (1 of 4)

- Recall our differential equation for spring motion:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

- Suppose there is no external driving force and no damping. Then $F(t) = 0$ and $\gamma = 0$, and our equation becomes

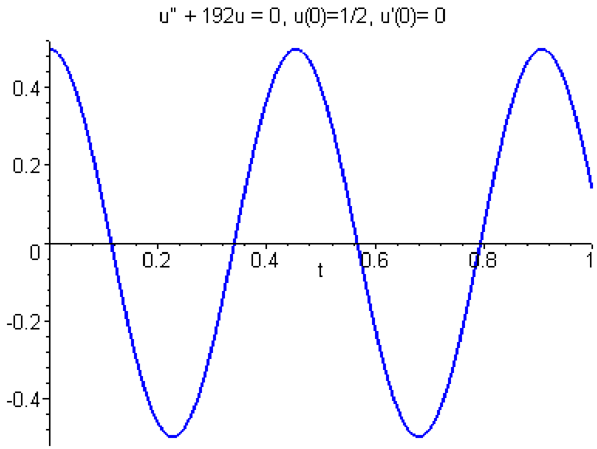
$$mu''(t) + ku(t) = 0$$

- The general solution to this equation is

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t,$$

where

$$\omega_0^2 = k/m$$



Spring Model: Undamped Free Vibrations (2 of 4)

- Using trigonometric identities, the solution

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t, \quad \omega_0^2 = k/m$$

can be rewritten as follows:

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t \quad \Leftrightarrow \quad u(t) = R\cos(\omega_0 t - \delta)$$

$$\Leftrightarrow u(t) = R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t,$$

where

$$A = R\cos\delta, \;\; B = R\sin\delta \;\; \Rightarrow \;\; R = \sqrt{A^2 + B^2}, \;\; \tan\delta = \frac{B}{A}$$

- Note that in finding δ, we must be careful to choose the correct quadrant. This is done using the signs of cos δ and sin δ.

Spring Model:
Undamped Free Vibrations (3 of 4)

- Thus our solution is

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t \quad = R\cos(\omega_0 t - \delta)$$

where

$$\omega_0 = \sqrt{k/m}$$

- The solution is a shifted cosine (or sine) curve, that describes simple harmonic motion, with period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

- The circular frequency ω_0 (radians/time) is the **natural frequency** of the vibration, *R* is the **amplitude** of the maximum displacement of mass from equilibrium, and δ is the **phase** or phase angle (dimensionless).

Spring Model: Undamped Free Vibrations (4 of 4)

- Note that our solution

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta), \quad \omega_0 = \sqrt{k/m}$$

  is a shifted cosine (or sine) curve with period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- Initial conditions determine *A* & *B*, hence also the amplitude *R*.
- The system always vibrates with the same frequency ω_0, regardless of the initial conditions.
- The period *T* increases as *m* increases, so larger masses vibrate more slowly. However, *T* decreases as *k* increases, so stiffer springs cause a system to vibrate more rapidly.

Example 2: Find IVP (1 of 3)

- A 10 lb mass stretches a spring 2". The mass is displaced an additional 2" and then set in motion with an initial upward velocity of 1 ft/sec. Determine the position of the mass at any later time, and find the period, amplitude, and phase of the motion.

$$mu''(t) + ku(t) = 0,\ \ u(0) = u_0,\ \ u'(0) = v_0$$

- Find m:

$$w = mg \Rightarrow m = \frac{w}{g} \Rightarrow m = \frac{10\text{ lb}}{32\text{ft}/\sec^2} \Rightarrow m = \frac{5}{16}\,\frac{\text{lb}\sec^2}{\text{ft}}$$

- Find k:

$$F_s = -k\,L \Rightarrow k = \frac{10\text{ lb}}{2\text{ in}} \Rightarrow\ k = \frac{10\text{ lb}}{1/6\text{ ft}} \Rightarrow k = 60\,\frac{\text{lb}}{\text{ft}}$$

- Thus our IVP is

$$5/16\,u''(t) + 60u(t) = 0,\ \ u(0) = 1/6,\ \ u'(t) = -1$$

Example 2: Find Solution (2 of 3)

- Simplifying, we obtain

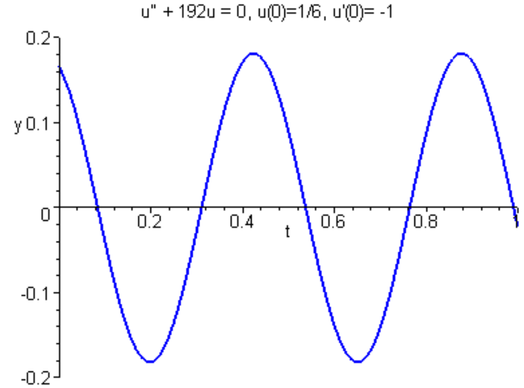
$$u''(t) + 192u(t) = 0, \quad u(0) = 1/6, \quad u'(0) = -1$$

- To solve, use methods of Ch 3.3 to obtain

$$u(t) = \frac{1}{6}\cos\sqrt{192}\,t - \frac{1}{\sqrt{192}}\sin\sqrt{192}\,t$$

or

$$u(t) = \frac{1}{6}\cos 8\sqrt{3}\,t - \frac{1}{8\sqrt{3}}\sin 8\sqrt{3}\,t$$



Example 2: Find Period, Amplitude, Phase (3 of 3)

$$u(t) = \frac{1}{6}\cos 8\sqrt{3}t - \frac{1}{8\sqrt{3}}\sin 8\sqrt{3}t$$

- The natural frequency is

$$\omega_0 = \sqrt{k/m} = \sqrt{192} = 8\sqrt{3} \cong 13.856 \text{ rad/sec}$$

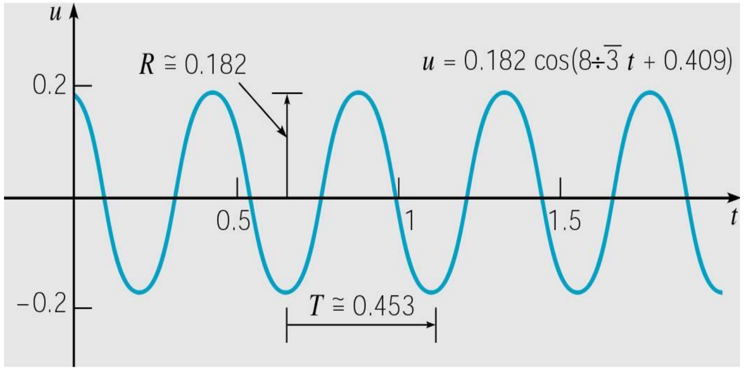
- The period is

$$T = 2\pi/\omega_0 \cong 0.45345 \text{ sec}$$

- The amplitude is

$$R = \sqrt{A^2 + B^2} \cong 0.18162 \text{ ft}$$

- Next, determine the phase δ:



$$A = R\cos\delta, \;\; B = R\sin\delta, \;\; \tan\delta = B/A$$

$$\tan\delta = \frac{B}{A} \Rightarrow \tan\delta = \frac{-\sqrt{3}}{4} \Rightarrow \delta = \tan^{-1}\left(\frac{-\sqrt{3}}{4}\right) \cong -0.40864 \text{ rad}$$

Thus $u(t) = 0.182\cos\left(8\sqrt{3}t + 0.409\right)$

Spring Model: Damped Free Vibrations (1 of 8)

- Suppose there is damping but no external driving force *F*(*t*):

$$mu''(t) + \gamma u'(t) + ku(t) = 0$$

- What is effect of the damping coefficient γ on system?
- The characteristic equation is

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{\gamma}{2m}\left[-1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}}\right]$$

- Three cases for the solution:

$$\gamma^2 - 4mk > 0: \quad u(t) = Ae^{r_1 t} + Be^{r_2 t}, \text{ where } r_1 < 0,\ r_2 < 0;$$

$$\gamma^2 - 4mk = 0: \quad u(t) = (A + Bt)e^{-\gamma t/2m}, \text{ where } \gamma/2m > 0;$$

$$\gamma^2 - 4mk < 0: \quad u(t) = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t),\ \mu = \frac{\sqrt{4mk - \gamma^2}}{2m} > 0.$$

Note: In all three cases, $\lim_{t\to\infty} u(t) = 0$, as expected from the damping term.

Damped Free Vibrations: Small Damping (2 of 8)

- Of the cases for solution form, the last is most important, which occurs when the damping is small:

$$\gamma^2 - 4mk > 0: \ u(t) = Ae^{r_1 t} + Be^{r_2 t}, \quad r_1 < 0, \ r_2 < 0$$

$$\gamma^2 - 4mk = 0: \ u(t) = (A + Bt)e^{-\gamma t/2m}, \quad \gamma / 2m > 0$$

$$\gamma^2 - 4mk < 0: \ u(t) = e^{-\gamma t/2m}(A\cos \mu t + B\sin \mu t), \ \mu > 0$$

- We examine this last case. Recall

$$A = R\cos\delta, \ B = R\sin\delta$$

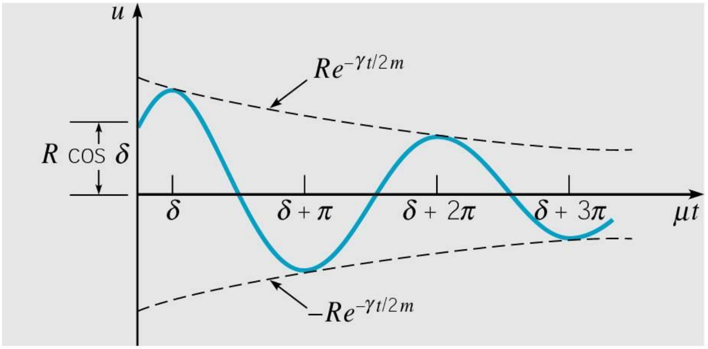
- Then

$$u(t) = Re^{-\gamma t/2m}\cos(\mu t - \delta)$$

and hence

$$|u(t)| \le Re^{-\gamma t/2m}$$

(damped oscillation)



Damped Free Vibrations: Quasi Frequency

- Thus we have damped oscillations:

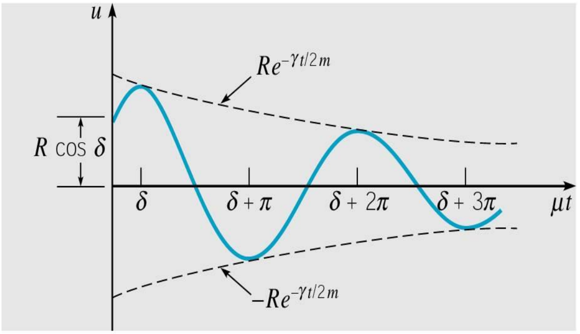
$$u(t) = Re^{-\gamma t/2m}\cos(\mu t - \delta) \quad \Rightarrow \quad |u(t)| \le Re^{-\gamma t/2m}$$

- The amplitude *R* depends on the initial conditions, since

$$u(t) = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t), \;\; A = R\cos\delta, \;\; B = R\sin\delta$$

- Although the motion is not periodic, the parameter μ determines the mass oscillation frequency.
- Thus μ is called the **quasi frequency**.
- Recall

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m}$$



Damped Free Vibrations: Quasi Period (4 of 8)

- Compare μ with ω_0, the frequency of undamped motion:

$$\frac{\mu}{\omega_0} = \frac{\sqrt{4km-\gamma^2}}{2m\sqrt{k/m}} = \frac{\sqrt{4km-\gamma^2}}{\sqrt{4m^2}\sqrt{k/m}} = \frac{\sqrt{4km-\gamma^2}}{\sqrt{4km}} = \sqrt{1-\frac{\gamma^2}{4km}}$$

For small γ

$$= \sqrt{1-\frac{\gamma^2}{4km}+\frac{\gamma^4}{64k^2m^2}} = \sqrt{\left(1-\frac{\gamma^2}{8km}\right)^2} = 1-\frac{\gamma^2}{8km}$$

- Thus, small damping reduces oscillation frequency slightly.
- Similarly, the **quasi period** is defined as $T_d = 2\pi/\mu$. Then

$$\frac{T_d}{T} = \frac{2\pi/\mu}{2\pi/\omega_0} = \frac{\omega_0}{\mu} = \left(1-\frac{\gamma^2}{4km}\right)^{-1/2} \cong \left(1-\frac{\gamma^2}{8km}\right)^{-1} \cong 1+\frac{\gamma^2}{8km}$$

- Thus, small damping increases quasi period.

Damped Free Vibrations: Neglecting Damping for Small $\gamma^2/4km$

- Consider again the comparisons between damped and undamped frequency and period:

$$\frac{\mu}{\omega_0} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2}, \quad \frac{T_d}{T} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2}$$

- Thus it turns out that a small γ is not as telling as a small ratio $\gamma^2/4km$.
- For small $\gamma^2/4km$, we can neglect the effect of damping when calculating the quasi frequency and quasi period of motion. But if we want a detailed description of the motion of the mass, then we cannot neglect the damping force, no matter how small it is.

Damped Free Vibrations: Frequency, Period (6 of 8)

- Ratios of damped and undamped frequency, period:

$$\frac{\mu}{\omega_0} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2}, \quad \frac{T_d}{T} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2}$$

- Thus

$$\lim_{\gamma \to 2\sqrt{km}} \mu = 0 \text{ and } \lim_{\gamma \to 2\sqrt{km}} T_d = \infty$$

- The importance of the relationship between γ^2 and $4km$ is supported by our previous equations:

$$\gamma^2 - 4mk > 0: \quad u(t) = Ae^{r_1 t} + Be^{r_2 t}, \quad r_1 < 0, \; r_2 < 0$$

$$\gamma^2 - 4mk = 0: \quad u(t) = (A + Bt)e^{-\gamma t/2m}, \quad \gamma/2m > 0$$

$$\gamma^2 - 4mk < 0: \quad u(t) = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t), \quad \mu > 0$$

Damped Free Vibrations: Critical Damping Value (7 of 8)

- Thus the nature of the solution changes as γ passes through the value $2\sqrt{km}$.
- This value of γ is known as the **critical damping** value, and for larger values of γ the motion is said to be **overdamped**.
- Thus for the solutions given by these cases,

$$\gamma^2 - 4mk > 0: \quad u(t) = Ae^{r_1 t} + Be^{r_2 t}, \quad r_1 < 0,\ r_2 < 0 \qquad (1)$$

$$\gamma^2 - 4mk = 0: \quad u(t) = (A + Bt)e^{-\gamma t/2m}, \quad \gamma/2m > 0 \qquad (2)$$

$$\gamma^2 - 4mk < 0: \quad u(t) = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t), \quad \mu > 0 \qquad (3)$$

  we see that the mass creeps back to its equilibrium position for solutions (1) and (2), but does not oscillate about it, as it does for small γ in solution (3).
- Soln (1) is overdamped and soln (2) is critically damped.

Damped Free Vibrations: Characterization of Vibration (8 of 8)

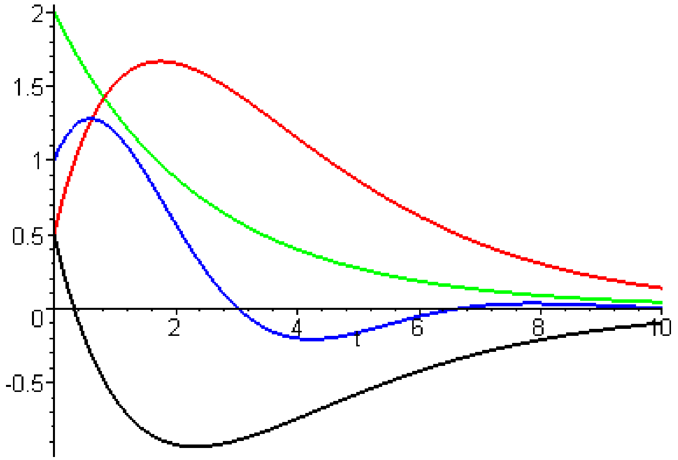
- The mass creeps back to the equilibrium position for solutions (1) & (2), but does not oscillate about it, as it does for small γ in solution (3).

$$\gamma^2 - 4mk > 0: \quad u(t) = Ae^{r_1 t} + Be^{r_2 t}, \quad r_1 < 0,\ r_2 < 0 \qquad \text{(Green)} \qquad (1)$$

$$\gamma^2 - 4mk = 0: \quad u(t) = \left(A + Bt\right)e^{-\gamma t/2m}, \quad \gamma/2m > 0 \qquad \text{(Red, Black)} \qquad (2)$$

$$\gamma^2 - 4mk < 0: \quad u(t) = e^{-\gamma t/2m}\left(A\cos\mu t + B\sin\mu t\right) \qquad \text{(Blue)} \qquad (3)$$

- Solution (1) is overdamped and
- Solution (2) is critically damped.
- Solution (3) is underdamped



Example 3: Initial Value Problem (1 of 4)

- Suppose that the motion of a spring-mass system is governed by the initial value problem
$$u'' + 0.125u' + u = 0, \quad u(0) = 2, \;\; u'(0) = 0$$
- Find the following:
  (a) quasi frequency and quasi period;
  (b) time at which mass passes through equilibrium position;
  (c) time τ such that $|u(t)| < 0.1$ for all $t > \tau$.
- For Part (a), using methods of this chapter we obtain:
$$u(t) = e^{-t/16}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right) = \frac{32}{\sqrt{255}}e^{-t/16}\cos\left(\frac{\sqrt{255}}{16}t - \delta\right)$$
where
$$\tan\delta = \frac{1}{\sqrt{255}} \Rightarrow \delta \cong 0.06254 \quad (\text{recall } A = R\cos\delta,\; B = R\sin\delta)$$

Example 3: Quasi Frequency & Period (2 of 4)

- The solution to the initial value problem is:

$$u(t) = e^{-t/16}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right) = \frac{32}{\sqrt{255}}e^{-t/16}\cos\left(\frac{\sqrt{255}}{16}t - \delta\right)$$

- The graph of this solution, along with solution to the corresponding undamped problem, is given below.
- The quasi frequency is

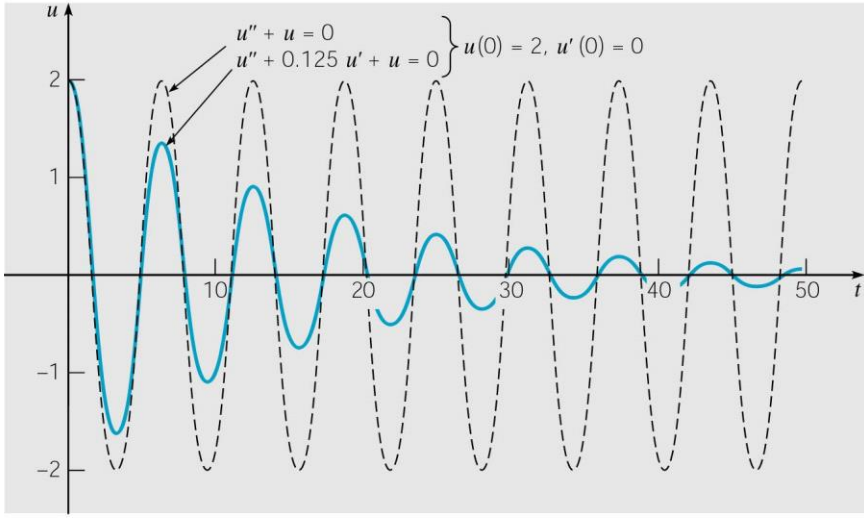
$$\mu = \sqrt{255}/16 \cong 0.998$$

and quasi period is

$$T_d = 2\pi/\mu \cong 6.295$$

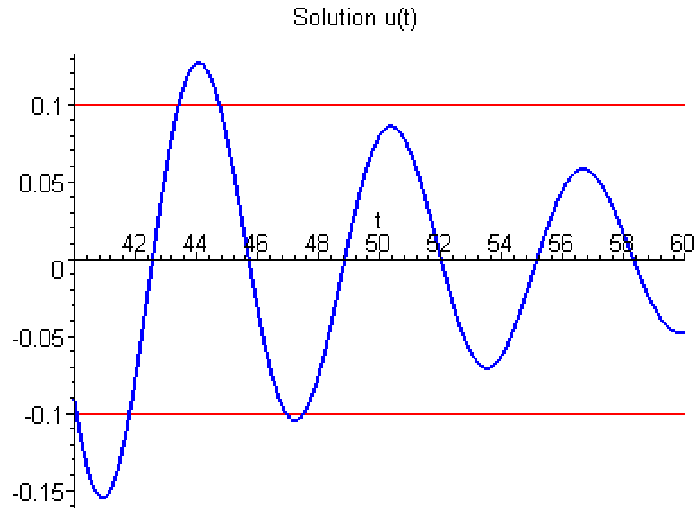
- For the undamped case:

$$\omega_0 = 1,\ T = 2\pi \cong 6.283$$



Example 3: Quasi Frequency & Period (3 of 4)

- The damping coefficient is γ = 0.125 = 1/8, and this is 1/16 of the critical value $2\sqrt{km} = 2$
- Thus damping is small relative to mass and spring stiffness. Nevertheless the oscillation amplitude diminishes quickly.
- Using a solver, we find that $|u(t)| < 0.1$ for $t > \tau \approx 47.515$ sec



Example 3: Quasi Frequency & Period (4 of 4)

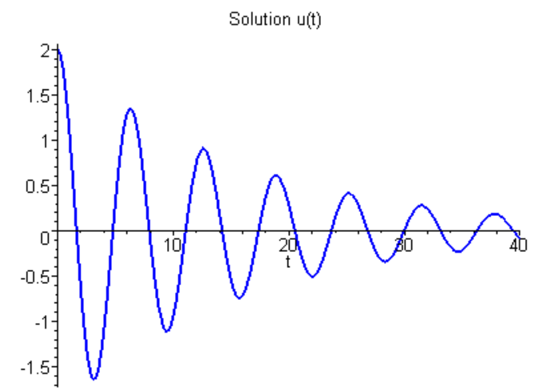
- To find the time at which the mass first passes through the equilibrium position, we must solve

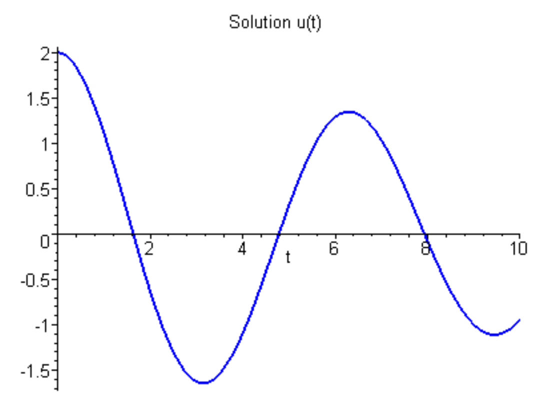
$$u(t) = \frac{32}{\sqrt{255}} e^{-t/16} \cos\left(\frac{\sqrt{255}}{16} t - \delta \right) = 0$$

- Or more simply, solve

$$\frac{\sqrt{255}}{16} t - \delta = \frac{\pi}{2}$$

$$\Rightarrow t = \frac{16}{\sqrt{255}} \left(\frac{\pi}{2} + \delta \right) \cong 1.637 \text{ sec}$$



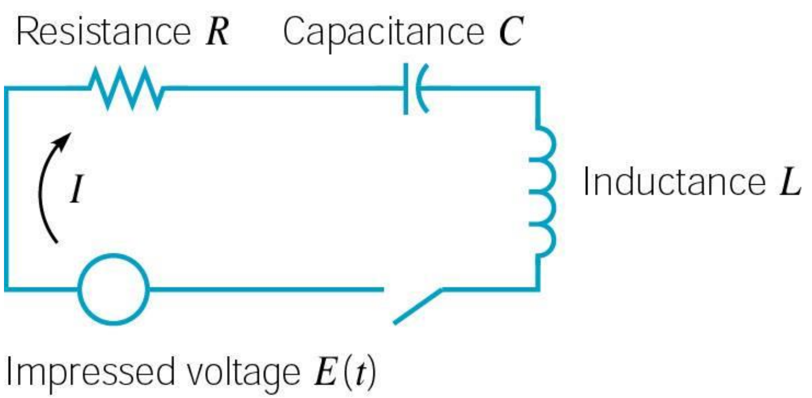


Electric Circuits

- The flow of current in certain basic electrical circuits is modeled by second order linear ODEs with constant coefficients:

$$L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = E'(t)$$

$$I(0) = I_0,\;\; I'(0) = I_0'$$



- It is interesting that the flow of current in this circuit is mathematically equivalent to motion of spring-mass system.
- For more details, see text.