Conditional Probability.

Many times we know additional information that affects the calculation of a probability:

What is the probability that a person is guilty of murder if you know the crime was committed by a left-handed person?

If the person is right handed, the probability is 0.
If the person is left handed, is the probability 1?

<u>Def</u> The probability that A occurs given that B occurs is called the <u>conditional probability of A given B</u> and is written $P(A|B)$.

<u>Ex(1)</u>. A fair coin is tossed two times. What is the probability that the second coin is a head if you know that at least one head appears.

<u>Soln</u>. The 4 outcomes are

HT
HH
TH
TT

exactly two have heads for the second toss

three satisfy at least one head occurs.

So, $P(\{\text{second toss is a head given at least one head}\}) = \frac{2}{3}$.

Fact: $\boxed{P(A|B) = \dfrac{P(A\cap B)}{P(B)}}$.

In the above example, let:

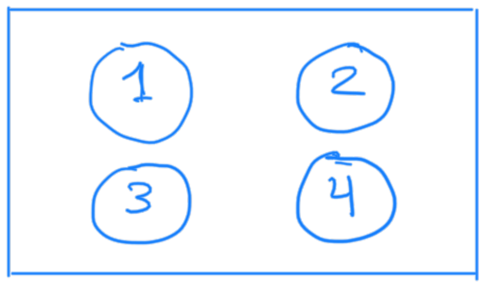
A = the event that the second coin is a head $= \{HH, TH\}$

B = the event that at least one head appears $= \{HH, HT, TH\}$.

Then $A \cap B = \{HH, TH\}$.

$$P(A|B) = \frac{P(\{HH, TH\})}{P(\{HH, HT, TH\})} = \frac{\frac{2}{\cancel{4}}}{\frac{3}{\cancel{4}}} = \frac{2}{3}.$$

Ex(2). Selecting 2 balls randomly with replacement from a box which contains 4 balls numbered 1, 2, 3, 4, what is the probability that an even numbered ball is selected in the second trial given that the first selected ball is also an even numbered?



Soln. $\Omega = \{(x,y) : x, y = 1,2,3,4\}$. $(n(\Omega) = 16)$

Let A be the event that the first ball selected shows an even number, and let B be the event that the second ball selected shows an even number. Then

$A = \{(2,1), (2,2), (2,3), (2,4), (4,1), (4,2), (4,3), (4,4)\}$

$B = \{(1,2), (2,2), (3,2), (4,2), (1,4), (2,4), (3,4), (4,4)\}$.

So $A \cap B = \{(2,2), (2,4), (4,2), (4,4)\}$.

Then $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{4}{\cancel{16}}}{\frac{8}{\cancel{16}}} = \dfrac{4}{8} = \dfrac{1}{2}$.

Remark:

The event B is said to carry:
- negative information about A if $P(A|B) < P(A)$
- positive information about A if $P(A|B) > P(A)$
- no information about A if $P(A|B) = P(A)$

For instance, in Ex (1), B has positive information about A, while in Ex (2), B has no information about A.

From the above fact, we have:

$$P(A \cap B) = P(A)\,P(B|A) = P(B)\,P(A|B)$$

This is called the joint occurrence of two events A & B

Ex. Let A and B be two events in Ω such that $P(A|B) = 0.4$ and $P(A \cap B) = 0.2$, then

$P(B) = 0.2/0.4 = 0.5$.

Independence.

Def. Two events are called independant if the occurrence or nonoccurrence of one event in no way affects the probability of the second event.

- If A and B are not independent, we say that they are dependent or related.

Fact: Two events A and B are independent if

$$P(A|B) = P(A).$$

Thm. Let A and B be two events such that $P(B) > 0$. Then A and B are independent iff

$$P(A \cap B) = P(A)\,P(B).$$

Ex. Rolling a die and letting $A = \{1,2\}$ and $B = \{2,4,5\}$, then A and B ~~are~~

(a) independant (b) dependent (c) mutually disjoint

Soln. $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{6}$.

Then $P(A \cap B) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = P(A)\,P(B)$.

Fact: If A and B are independant, then

1) A and $\bar{B}$ 〃 〃,
2) $\bar{A}$ and B 〃 〃,
3) $\bar{A}$ and $\bar{B}$ 〃 〃.

Ex Assume that $P(A) = 0.4$ and $P(B) = 0.3$. Find

1) $P(A \cup B)$ if A and B are disjoint.
2) $P(\bar{A} \cup B)$ if A and B are independent.
3) $P(\bar{B} \mid \bar{A})$ if A and B are independent.

Soln.

1) $P(A \cup B) = P(A) + P(B) - \cancel{P(A \cap B)}$ 0 as disjoint

$= 0.4 + 0.3$

$= 0.7$

2) $P(\bar{A} \cup B) = P(\bar{A}) + P(B) - P(\bar{A} \cap B)$

$= P(\bar{A}) + P(B) - P(\bar{A})P(B)$

$= 0.6 + 0.3 - 0.6 * 0.3$

$= 0.72.$

3) $P(\bar{B} \mid \bar{A}) = P(\bar{B})$

$= 0.7.$

Ex- Assume that $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cap B) = 0.3$. Find (1) $P(\bar{A} \mid \bar{B})$ (2) $P(\overline{A \cap B})$.

Soln.

1) $P(\bar{A} \mid \bar{B}) = \dfrac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$

$= \dfrac{0.4}{0.6}$

$= \dfrac{2}{3}.$

	A	$\bar{A}$	Total
B	$A \cap B$ 0.3	$\bar{A} \cap B$ 0.1	0.4
$\bar{B}$	$A \cap \bar{B}$ 0.2	$\bar{A} \cap \bar{B}$ 0.4	0.6
Total	0.5	0.5	always 1

2) $P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - 0.3 = 0.7.$

Def. The events $A_1, A_2, \ldots, A_n$ are mutually (or totally or complete) independent if $P\left(\bigcap_{j=1}^{k} A_j\right) = \prod_{j=1}^{n} P(A_j)$ $\forall k = 1, 2, \ldots, n$

In particular, A_1, A_2 and A_3 are mutually independent iff $P(A_i \cap A_j) = P(A_i)P(A_j)$ for each $i, j = 1, 2, 3$ $(i \neq j)$ and $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

Fact: If n events are mutually independent, then their complements are also mutually independent.

Thm If $A_1, A_2, \ldots, A_n$ are mutually independent events, then $P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \prod_{i=1}^{n} P(\overline{A_i})$.

Pf.
$$P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\overline{\bigcap_{i=1}^{n} \overline{A_i}}\right) \quad \text{DeMorgan law}$$
$$= 1 - P\left(\bigcap_{i=1}^{n} \overline{A_i}\right)$$
$$\overset{\text{Fact}}{=} 1 - \prod_{i=1}^{n} P(\overline{A_i}).$$

Ex. Find the probability that in 6 independent rolls of a fair die, the outcome 4 will appear at least once.

Soln Let A_i be the event that the outcomes is 4, for $i = 1, 2, \ldots, 6$. Then

$$P\left(\left\{\begin{matrix}\text{the outcome 4 will}\\ \text{appear at least once}\end{matrix}\right\}\right) = P\left(\bigcup_{i=1}^{6} A_i\right)$$

$$= 1 - \prod_{i=1}^{6} P(\overline{A}_i)$$

$$= 1 - \prod_{i=1}^{6} \left(\frac{5}{6}\right)$$

$$= 1 - \left(\frac{5}{6}\right)^6.$$

Remark: If a collection of events are pairwise independent then it is not necessarily mutually independent.

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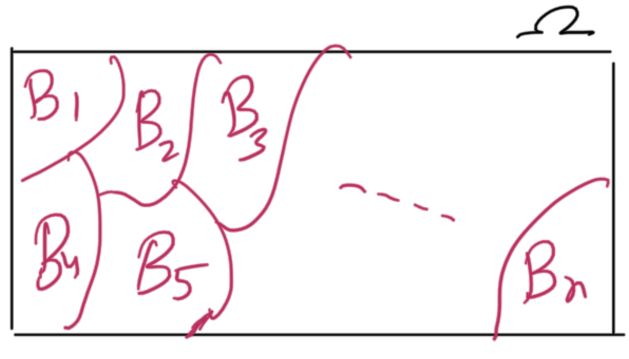
See, for instance, Example 2.3.7 in the text.

Events $B_1, B_2, \ldots, B_n$ are called a partition of the sample space Ω.



Theorem (Law of total probability)

Let $B_1, B_2, \ldots, B_n$ be events of Ω such that $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$ and $B_i \cap B_j = \phi$ for every $i \neq j$ $(i, j = 1, 2, \ldots, n)$. Then for any event A, we have

$$\boxed{P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_n)P(B_n)} \quad (*)$$

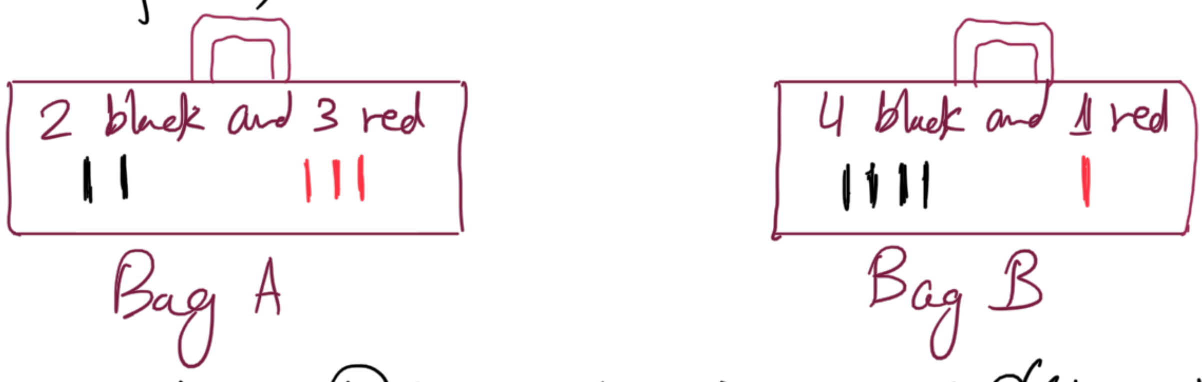
Bayes Theorem

Let $B_1, B_2, \dots, B_n$ be as in the above theorem, then, for any event A, we have

$$\boxed{P(B_k \mid A) = \frac{P(A \mid B_k)\, P(B_k)}{P(A)}, \quad k = 1, 2, \dots, n}$$

where $P(A)$ can be computed using $(*)$.

Ex. We have 2 bags, each contains black and red pens, as follows:



Assume that $P(\text{Selecting bag } A) = 0.4$ & $P(\text{Selecting bag } B) = 0.6$

1) Find the probability that the selected pen is red.
2) If the selected pen is red, find the probability that it was selected from bag B.

Soln. 1) $P(\text{Pen is red}) = P(\text{Pen is red} \mid \text{Bag is A})\, P(\text{Bag is A})$
$+ P(\text{Pen is red} \mid \text{Bag is B})\, P(\text{Bag is B})$

$$= \frac{3}{5}(0.4) + \frac{1}{5}(0.6)$$

$$= 0.24 + 0.02$$

$$= 0.26.$$

2) $$P(\text{Bag is B} \mid \text{Pen is red}) = \frac{P(\text{Pen is Red} \mid \text{Bag is B})\, P(\text{Bag is B})}{P(\text{Pen is red})}$$

$$= \frac{\left(\frac{1}{5}\right)\left(\frac{6}{10}\right)}{0.26}$$

$$\simeq 0.46.$$

Ex. In a city, 50% of the residents are Orange, 20% Umniah and 30% Zain. Assume that 20% of Orange, 10% of Umniah and 40% of Zain use iPhones.

1) If a person is randomly selected from this city, find the probability that this person uses iPhone.

2) If the selected person uses iPhone, find the probability that the person is Orange.

iPhone

Soln. 1) $P(\text{Person uses iPhone}) = P(\text{iPhone}|\text{Orange})\,P(\text{Orange})$

$$+\,P(\text{iPhone}|\text{Umniah})\,P(\text{Umniah})$$

$$+\,P(\text{iPhone}|\text{Zain})\,P(\text{Zain})$$

$$= (0.20)(0.50)$$

$$+\,(0.10)(0.20)$$

$$+\,(0.40)(0.30)$$

$$= 0.1 + 0.02 + 0.12$$

$$= 0.24.$$

2) $$P(\text{Orange}|\text{iPhone}) = \frac{P(\text{iPhone}|\text{Orange})\,P(\text{Orange})}{P(\text{iPhone})}$$

$$= \frac{(0.20)(0.50)}{0.24}$$

$$= 0.4167.$$

Searching keywords:

- Conditional probability, independence.
- Low of total probability, Bayes theorem.
- The University of Jordan الجامعة الأردنية
- Principles of Statistics مبادئ الإحصاء
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/131.aspx

For any comments or concerns, please use my email to contact me.

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