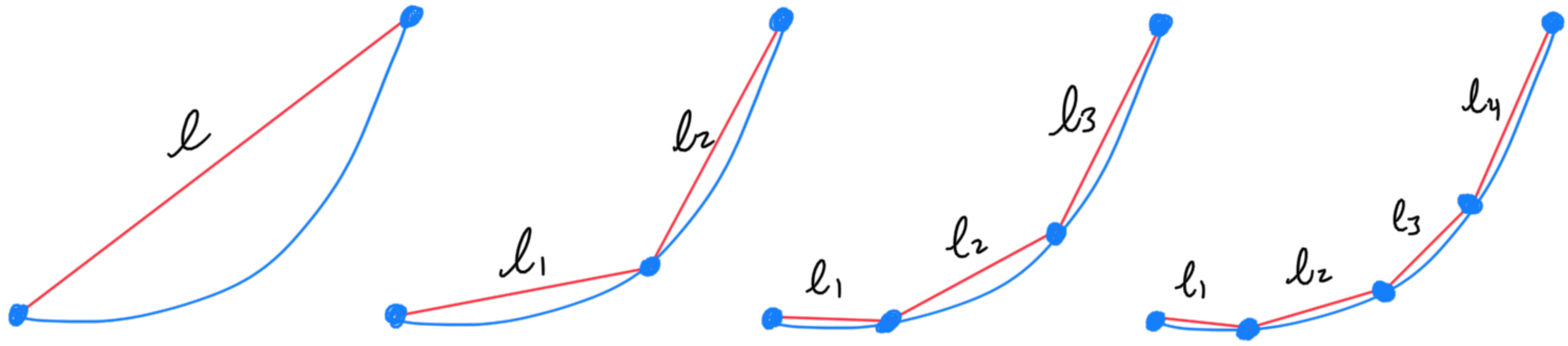
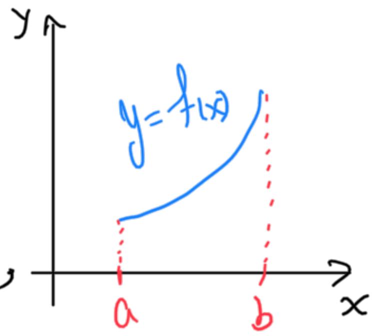
Arc Length.



$$\mathcal{L} = \lim_{n \to \infty} \sum_{i=1}^{n} l_i .$$



Fact: If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$\mathcal{L} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx .$$

Ex. Consider the func. $f(x) = \frac{1}{6} x^3 + \frac{1}{2} x^{-1}$, $1 \le x \le 3$. Find the length of the graph of f between the points $(1, \frac{2}{3})$ and $(3, \frac{14}{3})$.

Soln. $f'(x) = \frac{1}{2} x^2 - \frac{1}{2} x^{-2}$.

$$1 + (f'(x))^2 = \left(\frac{1}{2} x^2 - \frac{1}{2} x^{-2}\right)^2 + 1$$

$$= \frac{1}{4} x^4 + \frac{1}{2} + \frac{1}{4} x^{-4}$$

$$= \left(\frac{1}{2} x^2 + \frac{1}{2} x^{-2}\right)^2 .$$

$$\mathcal{L} = \int_1^3 \sqrt{1+\left(f'(x)\right)^2}\, dx = \int_1^3 \sqrt{\left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^{\cancel{2}}}$$

$$= \int_1^3 \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right) dx = \frac{1}{6}x^3 - \frac{1}{2}x^{-1}\Big|_1^3 = \frac{14}{3}.$$

Fact: If a curve has the equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then

$$\mathcal{L} = \int_c^d \sqrt{1+\left(g'(y)\right)^2}\, dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy.$$

Ex. Find the length of the arc of the parabola $y^2 = x$ from $(0,0)$ to $(1,1)$.

Soln. $\mathcal{L} = \int_0^1 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1+(2y)^2}\, dy$

$$= \int_0^1 \sqrt{1+4y^2}\, dy$$

let $y = \frac{1}{2}\tan\theta$, then $dy = \frac{1}{2}\sec^2\theta\, d\theta$

and $\sqrt{1+4y^2} = \sqrt{1+\tan^2\theta} = \sec\theta$.

$y = 0 \Longrightarrow \tan\theta = 0 \Longrightarrow \theta = 0$

$y = 1 \Longrightarrow \tan\theta = 2 \Longrightarrow \theta = \tan^{-1}2$. Then

$$I = \int_0^{\tan^{-1}2} \sec\theta \cdot \left(\tfrac{1}{2}\sec^2\theta\right) d\theta = \frac{1}{2}\int_0^{\tan^{-1}2} \sec^3\theta \, d\theta$$

$$\overset{\text{Exc}}{=} \frac{1}{2}\cdot\frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\tan^{-1}2}$$

$$= \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5}+2)}{4}.$$

The Arc Length Function

The arc length func. for $y = f(x)$ taking $(a, f(a))$ as the starting point is

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt.$$

Ex. Find the arc length func. for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1,1)$ as the starting point.

Soln. $f'(x) = 2x - \frac{1}{8x}$

$$1 + (f'(x))^2 = 1 + \left(2x - \frac{1}{8x}\right)^2$$

$$= 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2}$$

$$= \left(2x + \frac{1}{8x}\right)^2.$$

So $\sqrt{1+(f'(t))^2} = 2t + \frac{1}{8t}$. Then

$$s(x) = \int_1^x \left(2t + \frac{1}{8t}\right) dt = \left[t^2 + \frac{1}{8}\ln t\right]_1^x$$

$$= x^2 + \frac{1}{8}\ln x - 1.$$

For instance, the arc length between $1 \le x \le 3$ is

$s(x) = 3^2 + \frac{1}{8}\ln 3 - 1 \approx 8.1373$.

This lecture: Arc length.

Next lecture: Area of a surface of revolution.

Searching keywords:

- Find the arc length احسب طول القوس
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

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