Integration by parts

The rule $\boxed{\int u\,dv = uv - \int v\,du}$

dv		u	
	$\to$	u	$\oplus$
v	$\to$	du	$\ominus$

Explanation $(uv)' = u\,dv + v\,du$, so $uv = \int u\,dv + \int v\,du$.

Integration by parts is useful with integrals of the form

$$\int (\text{Polynomial}) * \left(\text{Trig}(ax+b) \text{ or } \text{Exp}(ax+b)\right) dx.$$

Ex. Evaluate

(1) $I = \int x \sin 2x \, dx$.

Soln. $I = -\frac{x}{2}\cos 2x + \frac{1}{2}\int \cos 2x \, dx$

$= -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C.$

dv		u	
$\sin 2x$	$\to$	x	$\oplus$
$-\frac{1}{2}\cos 2x$	$\to$	dx	$\ominus$

---

(2) $I = \int x^2 \sin 2x \, dx$.

اجزاء
مرتين

Soln

$I = \frac{-x^2}{2}\cos 2x + \frac{2x}{4}\sin 2x - \frac{1}{2}\int \sin 2x \, dx$

$= \frac{-x^2}{2}\cos 2x + \frac{2x}{4}\sin 2x - \frac{1}{4}\cos 2x + C.$

dv		u	
$\sin 2x$	$\to$	x^2	$\oplus$
$-\frac{1}{2}\cos 2x$		$2x$	$\ominus$
$-\frac{1}{4}\sin 2x$	$\to$	2	$\oplus$

---

(3) $I = \int e^{\sqrt{x}} dx$.

Soln. let $y = \sqrt{x}$, then $y^2 = x$ and $2y\,dy = dx$.

$$I = \int 2y\, e^{y}\, dy$$
$$= 2y\, e^{y} - 2e^{y} + C.$$
$$= 2\sqrt{x}\, e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

dv		u	
e^{y}	$\rightarrow$	$2y$	$\oplus$
e^{y}	$\rightarrow$	$2dy$	$\ominus$

---

(4) $I = \int x \sec^2 x \tan x\, dx$.

Soln.

dv		u	
$\sec^2 x \tan x$	$\rightarrow$	x	$\oplus$
$\frac{1}{2}\sec^2 x$	$\rightarrow$	dx	$\ominus$

To integrate $\int \sec^2 x \tan x\, dx$, let $w = \sec x$ then $dw = \sec x \tan x\, dx$, and hence

$$\int \sec^2 x \tan x\, dx = \int w^2 \cancel{\tan x}\, \frac{dw}{\underbrace{\sec x}_{w} \cancel{\tan x}}$$
$$= \int w\, dw = \frac{1}{2} w^2 + C = \frac{1}{2}\sec^2 x.$$

Then $I = \frac{x}{2}\sec^2 x - \frac{1}{2}\int \sec^2 x\, dx = \frac{x}{2}\sec^2 x - \frac{1}{2}\tan x + C.$

---

(5) $\int x \sin^2 x \cos x\, dx$ Exc.

---

(6) $I = \int (3x+1)(\cos x + \sin x)^2\, dx$

Soln. Note that

$$(\cos x + \sin x)^2 = \cos^2 x + \sin^2 x + 2\sin x \cos x$$
$$= 1 + \sin 2x.$$

dv		u	
$(\cos x + \sin x)^2$	$\rightarrow$	$3x+1$	$\oplus$
$x - \frac{1}{2}\cos 2x$	$\rightarrow$	3	$\ominus$

It follows that

$$I = (3x+1)\left(x - \frac{\cos 2x}{2}\right) - \frac{3}{2}x^2 + \frac{3}{4}\sin 2x + C.$$

Remark: For $\int (\text{trig.})^2$ use
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\tan^2 x = \sec^2 x - 1$$
$$\cot^2 x = \csc^2 x - 1$$

(7) $\int (x+3)\cot^2(2x)\,dx$.

Soln. Note that

$$\int \cot^2(2x)\,dx = \int \left(\csc^2(2x) - 1\right)dx$$

$$= -\frac{1}{2}\cot(2x) - x.$$

dv	u
$\cot^2(2x)$	$x+3$ $\oplus$
$-\frac{1}{2}\cot(2x) - x$	dx $\ominus$

So, $I = (x+3)\left(-\frac{1}{2}\cot(2x) - x\right) + \frac{1}{4}\ln|\sin(2x)| + \frac{x^2}{2} + C.$

Remark: Integrals of the form

$$\int e^{ax+b} * \left(\sin(ax+b) \text{ or } \cos(ax+b)\right)dx$$

repeat when integrating by parts twice.

---

(8) $I = \int e^x \sin x\,dx$.

Soln. $I = -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx$

dv	u
$\sin x$	e^x $\oplus$
$-\cos x$	e^x $\ominus$
$-\sin x$	e^x $\oplus$

This is I !!

It follows that

$$I = -e^x\cos x + e^x\sin x - I. \text{ Thus}$$

$$I = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

Question: How to choose u and dv?

Ans: $\ln x \rightsquigarrow x^k \rightsquigarrow e^x \rightsquigarrow \sin x,\ \cos x$

u $\quad$ dv

---

(9) $I = \int x^3 \ln(2x)\, dx$.

Soln. $I = \frac{1}{4} x^4 \ln(2x) - \int \frac{x^3}{4}\, dx$

$= \frac{1}{4} x^4 \ln(2x) - \frac{1}{16} x^4 + C$.

dv	u	
x^3	$\ln(2x)$	$\oplus$
$\frac{1}{4}x^4$	$\frac{dx}{x}$	$\ominus$

---

(10) $I = \int \cos x \ln(\sin x)\, dx$.

Soln. $I = \sin x \ln(\sin x) - \int \cos x\, dx$

$= \sin x \ln(\sin x) - \sin x + C$.

dv	u	
$\cos x$	$\ln(\sin x)$	$\oplus$
$\sin x$	$\frac{\cos x}{\sin x}$	$\ominus$

---

(11) $I = \int \frac{\ln x}{x}\, dx$.

Soln. $I = (\ln x)^2 - \int \frac{\ln x}{x}\, dx \leftarrow I$

Thus, $I = \frac{1}{2}(\ln x)^2 + C$.

dv	u	
$\frac{1}{x}$	$\ln x$	$\oplus$
$\ln x$	$\frac{dx}{x}$	$\ominus$

---

(12) $I = \int \ln x\, dx$.

Soln. $I = x \ln x - \int \left(\frac{1}{x}\right)(x)\, dx$

$= x \ln x - \int dx$

$= x \ln x - x + C$.

dv	u	
dx	$\ln x$	$\oplus$
x	$\frac{dx}{x}$	$\ominus$

---

(13) $\int \ln \sqrt{x}\, dx$. Exc.

---

(14) $I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$.

Soln. $I = \underbrace{\int \frac{1}{x} e^x \, dx}_{\text{let } II} - \underbrace{\int \frac{1}{x^2} e^x \, dx}_{III}$.

Now, $II = \frac{1}{x} e^x + \int \frac{1}{x^2} e^x \, dx$. This is III again!

du	u	
e^x	$\frac{1}{x}$	$+$
e^x	$-\frac{1}{x^2}$	$-$

Then

$$I = \frac{1}{x} e^x + \cancel{III} - \cancel{III} + C = \frac{1}{x} e^x + C.$$

---

(15) $I = \int \sin^{-1} x \, dx$.

Soln. $I = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, dx$

let $w = 1 - x^2$, then $dw = -2x\,dx$

Then $I = x \sin^{-1} x + \frac{1}{2} \int \frac{dw}{\sqrt{w}}$

$$= x \sin^{-1} x + \sqrt{w} + C$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + C.$$

du	u	
dx	$\sin^{-1} x$	$+$
x	$\frac{dx}{\sqrt{1-x^2}}$	$-$

Facts:

(1) $\int \ln x \, dx = x \ln x - x + C$.

(2) $\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} + C$.

(3) $\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$.

(4) $\int \sec^{-1} x \, dx = x \sec^{-1} x - \ln\left|x + \sqrt{x^2-1}\right| + C$.

This lecture: Integration by parts.

Next lecture: Integration of trigonometric functions.

Searching keywords:

- Evaluate the integral احسب التكامل
- Integration by parts التكامل بالأجزاء
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

د. بهاء محمود الزالق
The University of Jordan
Dr. Baha Alzalg
baha2math@gmail.com

This material might not be used for commercial purposes.

A Copyright: All Rights Reserved.
B. Alzalg, 2020, Amman, Jordan.