

# Integration by parts

The rule  $\int u dv = uv - \int v du$



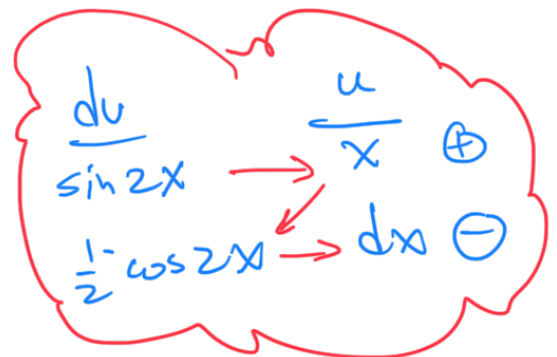
Explanation  $(uv)' = u dv + v du$ , so  $uv = \int u dv + \int v du$ .

Integration by parts is useful with integrals of the form  $\int (\text{Polynomial}) * (\text{Trig.}(ax+tb) \text{ or } \text{Exp}(ax+tb)) dx$ .

Ex. Evaluate

(1)  $I = \int x \sin 2x dx$

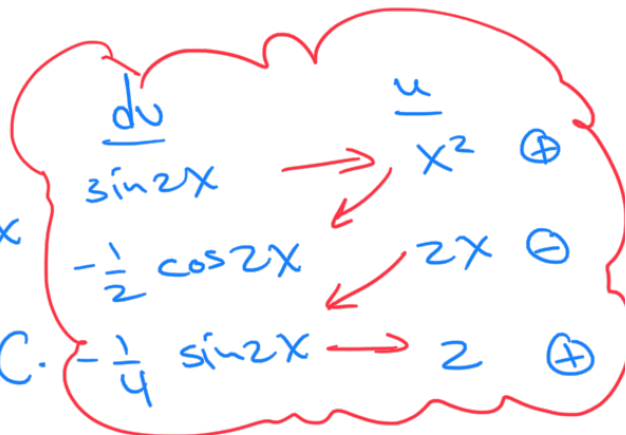
Soln.  $I = -\frac{x^2}{2} \cos 2x + \frac{1}{2} \int \cos 2x dx$   
 $= -\frac{x^2}{2} \cos 2x + \frac{1}{4} \sin 2x + C$



(2)  $I = \int x^2 \sin 2x dx$

Soln

$I = -\frac{x^2}{2} \cos 2x + \frac{2x}{4} \sin 2x - \frac{1}{2} \int \sin 2x dx$   
 $= -\frac{x^2}{2} \cos 2x + \frac{2x}{4} \sin 2x - \frac{1}{4} \cos 2x + C$



(3)  $I = \int e^{\sqrt{x}} dx$

Soln. let  $y = \sqrt{x}$ , then  $y^2 = x$  and  $2y dy = dx$ .

$$\begin{aligned}
 I &= \int zy e^y dy \\
 &= zy e^y - 2e^y + C. \\
 &= 2\sqrt{x} e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.
 \end{aligned}$$

$$\begin{array}{l}
 \frac{du}{e^y} \rightarrow \frac{u}{zy} \oplus \\
 e^y \leftarrow \rightarrow z dy \ominus
 \end{array}$$

(4)  $I = \int x \sec^2 x \tan x dx$ .

Soln.  $\frac{du}{\sec^2 x \tan x} \rightarrow \frac{u}{x} \oplus$   
 $\frac{1}{2} \sec^2 x \leftarrow \rightarrow dx \ominus$

To integrate  $\int \sec^2 x \tan x dx$ , let  $w = \sec x$   
 then  $dw = \sec x \tan x dx$ , and hence  
 $\int \sec^2 x \tan x dx = \int w^2 \tan x \frac{dw}{\sec x \tan x}$   
 $= \int w dw = \frac{1}{2} w^2 + C = \frac{1}{2} \sec^2 x$ .

Then  $I = \frac{x}{2} \sec^2 x - \frac{1}{2} \int \sec^2 x dx = \frac{x}{2} \sec^2 x - \frac{1}{2} \tan x + C$ .

(5)  $\int x \sin^2 x \cos x dx$  Exc.

(6)  $I = \int (3x+1) (\cos x + \sin x)^2 dx$

Soln. Note that

$$\begin{aligned}
 (\cos x + \sin x)^2 &= \cos^2 x + \sin^2 x + 2 \sin x \cos x \\
 &= 1 + \sin 2x.
 \end{aligned}$$

It follows that

$$I = (3x+1) \left( x - \frac{\cos 2x}{2} \right) - \frac{3}{2} x^2 + \frac{3}{4} \sin 2x + C.$$

$$\begin{array}{l}
 \frac{du}{(\cos x + \sin x)^2} \rightarrow \frac{u}{3x+1} \oplus \\
 x - \frac{1}{2} \cos 2x \leftarrow \rightarrow 3 \ominus
 \end{array}$$

Remark: For  $\int (\text{trig})^2$  use  $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\tan^2 x = \sec^2 x - 1$$

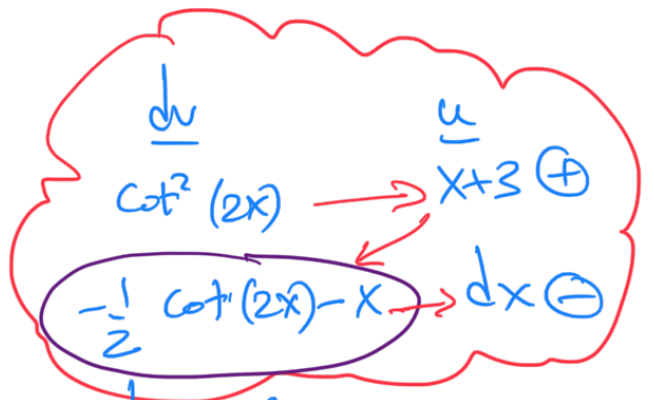
$$\cot^2 x = \csc^2 x - 1$$

(7)  $\int (x+3) \cot^2(2x) dx.$

Soln Note that

$$\int \cot^2(2x) dx = \int (\csc^2(2x) - 1) dx$$

$$= -\frac{1}{2} \cot(2x) - x.$$



So,  $I = (x+3) \left(-\frac{1}{2} \cot(2x) - x\right) + \frac{1}{4} \ln|\sin(2x)| + \frac{x^2}{2} + C.$

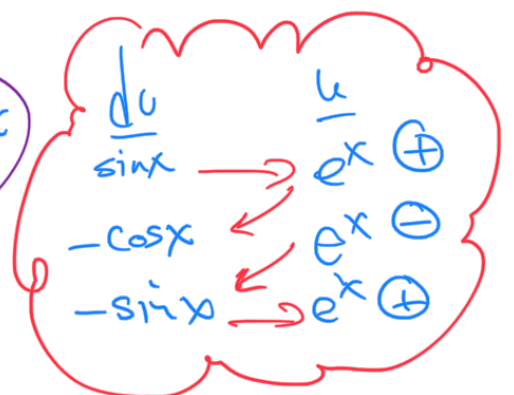
Remark: Integrals of the form

$$\int e^{ax+b} * (\sin(ax+b) \text{ or } \cos(ax+b)) dx$$

repeat when integrating by parts twice.

(8)  $I = \int e^x \sin x dx.$

Soln  $I = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$



It follows that

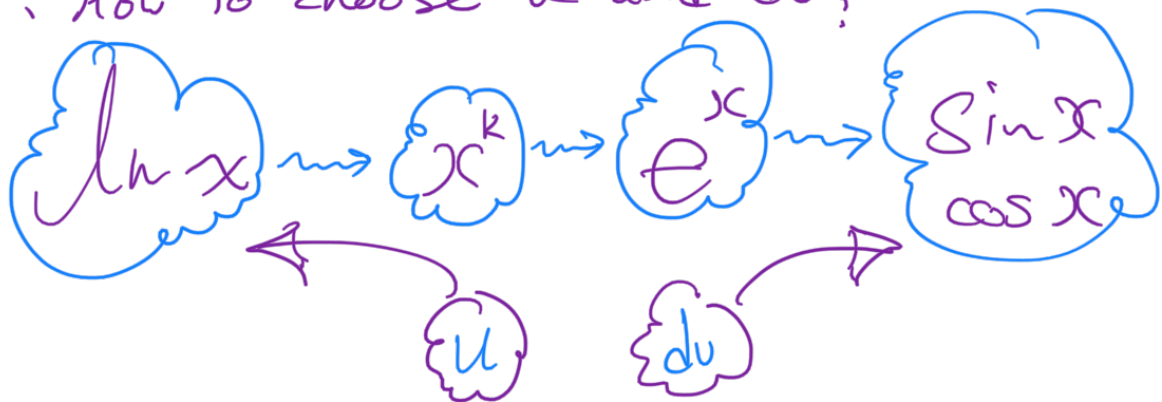
This is I !!

$I = -e^x \cos x + e^x \sin x - I$ . Thus

$$I = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

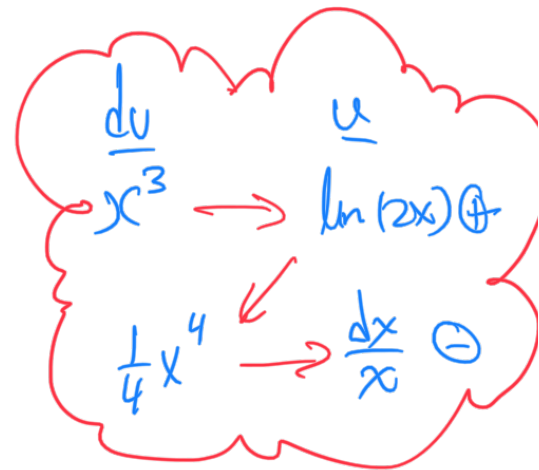
Question: How to choose u and du?

Ans:



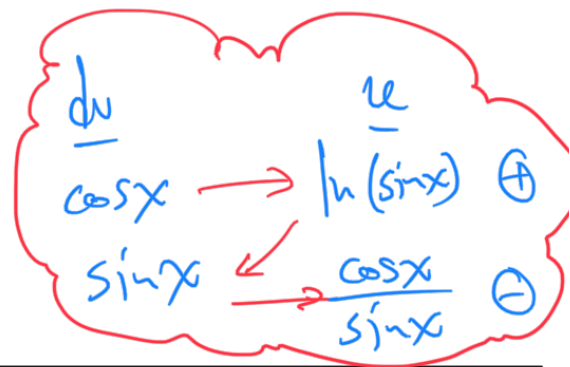
$$(9) I = \int x^3 \ln(2x) dx.$$

$$\begin{aligned} \underline{\text{Soln.}} \quad I &= \frac{1}{4} x^4 \ln(2x) - \int \frac{x^3}{4} dx \\ &= \frac{1}{4} x^4 \ln(2x) - \frac{1}{16} x^4 + C. \end{aligned}$$



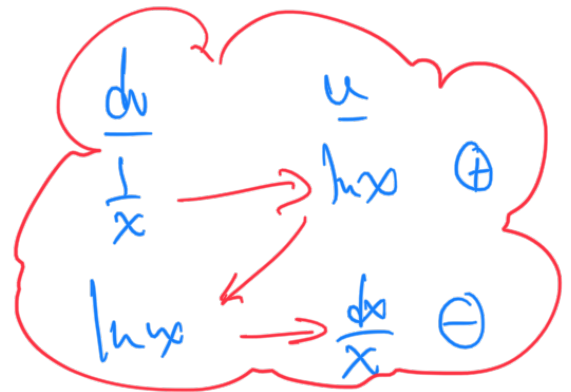
$$(10) I = \int \cos x \ln(\sin x) dx.$$

$$\begin{aligned} \underline{\text{Soln.}} \quad I &= \sin x \ln(\sin x) - \int \cos x dx \\ &= \sin x \ln(\sin x) - \sin x + C. \end{aligned}$$



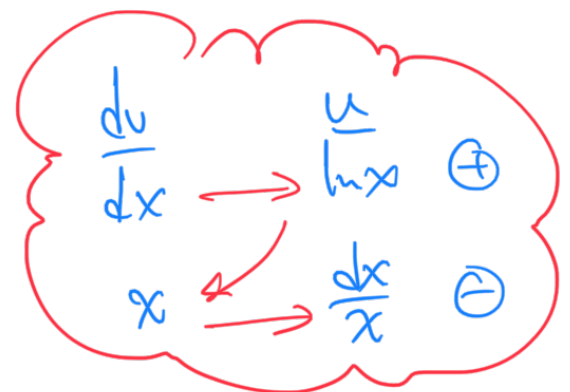
$$(11) I = \int \frac{\ln x}{x} dx.$$

$$\begin{aligned} \underline{\text{Soln.}} \quad I &= (\ln x)^2 - \int \frac{\ln x}{x} dx \leftarrow I \\ \text{Thus, } I &= \frac{1}{2} (\ln x)^2 + C. \end{aligned}$$



$$(12) I = \int \ln x dx.$$

$$\begin{aligned} \underline{\text{Soln.}} \quad I &= x \ln x - \int \left(\frac{1}{x}\right) (x) dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + C. \end{aligned}$$



$$(13) \int \ln \sqrt{x} dx. \quad \underline{\text{Exc.}}$$

$$(14) I = \int e^x \left( \frac{1}{x} - \frac{1}{x^2} \right) dx.$$

Soln.  $I = \int \frac{1}{x} e^x dx - \int \frac{1}{x^2} e^x dx.$

let  $II \rightarrow$

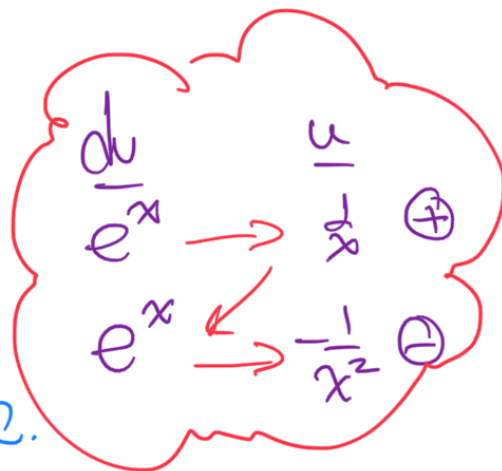
$III \rightarrow$

Now,  $II = \frac{1}{x} e^x + \int \frac{1}{x^2} e^x dx.$

Then

This is  $III$  again!

$I = \frac{1}{x} e^x + \cancel{III} - \cancel{III} + C = \frac{1}{x} e^x + C.$



(15)  $I = \int \sin^{-1} x dx.$

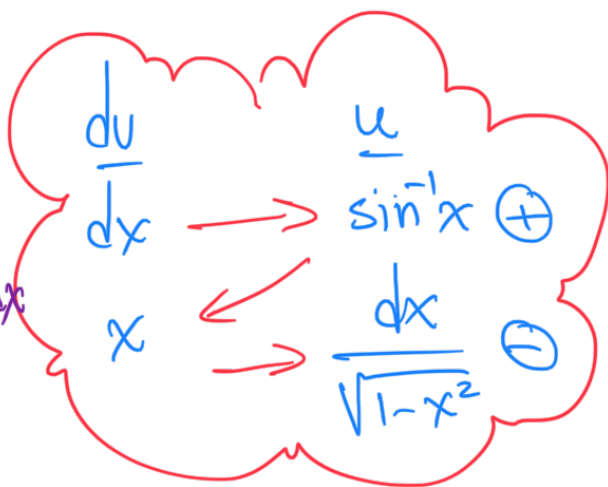
Soln.  $I = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$

let  $w = 1 - x^2$ , then  $dw = -2x dx$

Then  $I = x \sin^{-1} x + \frac{1}{2} \int \frac{dw}{\sqrt{w}}$

$= x \sin^{-1} x + \sqrt{w} + C$

$= x \sin^{-1} x + \sqrt{1-x^2} + C.$



Facts:

(1)  $\int \ln x dx = x \ln x - x + C.$

(2)  $\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + C.$

(3)  $\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C.$

(4)  $\int \sec^{-1} x dx = x \sec^{-1} x - \ln|x + \sqrt{x^2-1}| + C.$

This lecture: Integration by parts.

Next lecture: Integration of trigonometric functions.



Searching keywords:

- Evaluate the integral احسب التكامل
- Integration by parts التكامل بالأجزاء
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

<http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx>

For any comments or concerns, please use my email to contact me.



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