Series.

If we add the terms of an infinite seq. $\{a_n\}_{n=1}^{\infty}$ we get $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ which is called an infinite series (or just a series) and is denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

Ex. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n+1}$ is an infinite series.

Def. $\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} \sum_{n=1}^{k} a_n$

Exists $(=L)$: then the series is called convergent (conv.) to L.

DNE: then the series is called divergent (div.)

Two questions arise when dealing with series

Q$_1$: How could we find the shme of a series if it is convergent?

There are 2 cases:

- 1. Geometric Series.
- 2. Apply the definition and use Telescoping series.

To be discussed today!

Q$_2$: How can we a series for convergence?

There are 7 tests

- 1. The divergence test
- 2. The P- test
- 3. The integral test
- 4. The comparison test
- 5. The limit comparison test
- 6. The ratio test
- 7. The root test

The geometric series (G.S):

$$\sum_{k=a}^{\infty} x^k = \begin{cases} \dfrac{x^a}{1-x} & \text{if } |x|<1 \quad (\text{Conv.}) \\ \text{DNE} & \text{if } |x| \geq 1 \quad (\text{Div.}) \end{cases}$$

Ex. Find the sum if it exists

(1) $$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{\cancel{3}}}{\frac{1}{\cancel{3}}} = 2.$$

In the words, the series $\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k$ is conv. to 2.

(2) $$\sum_{k=1}^{\infty} 3^{2k} = \sum_{k=1}^{\infty} 9^k \quad \text{DNE}$$

In the words, the series $\sum_{k=1}^{\infty} 3^{2k}$ is div.

(3) $$\sum_{k=1}^{\infty}\left(\frac{5}{4}\right)^{3-k} = \sum_{k=3}^{\infty}\left(\frac{5}{4}\right)^3\left(\frac{4}{5}\right)^k = \cancel{\left(\frac{5}{4}\right)^3}\cdot\frac{\overset{1}{\cancel{\left(\frac{4}{5}\right)^3}}}{1-\frac{4}{5}}$$

$$= \frac{1}{\frac{1}{5}} = 5.$$

(4) $$\sum_{k=2}^{\infty}\frac{2}{5^{k-1}} = \sum_{k=2}^{\infty}\frac{2}{5^{-1}}\left(\frac{1}{5}\right)^k = 10\sum_{k=2}^{\infty}\left(\frac{1}{5}\right)^k$$

$$= 10\cdot\frac{\frac{1}{25}}{1-\frac{1}{5}} = \overset{2}{\cancel{10}}\cdot\frac{\frac{1}{\cancel{25}\,\cancel{5}}}{\frac{4}{\cancel{5}}} = \frac{1}{2}.$$

(5) $$\sum_{k=1}^{\infty}(-4)^{2k} = \sum_{k=1}^{\infty}\left((-4)^2\right)^k = \sum_{k=1}^{\infty}16^k \quad \text{DNE.}$$

So the series is div.

Remark: If $|x|<1$, then $$\sum_{k=a}^{\infty}x^k = \frac{x^a}{1-x} \text{ ——— } (*).$$

(1) Differentiating both sides of $(*)$, we get

$$\sum_{k=a}^{\infty}k\,x^{k-1} = \frac{d}{dx}\left(\frac{x^a}{1-x}\right),$$ provided that $|x|<1$.

(2) Integrating both sides of $(*)$, we obtain

$$\sum_{k=a}^{\infty} \frac{x^{k+1}}{k+1} = \int \frac{x^a}{1-x}\,dx, \text{ provided that } |x|<1.$$

<u>Ex</u>: Find the sum if it exists.

(1) $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$.

<u>Soln.</u> For $|x|<1$, we have $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$.

Diff. both sides, we get

$$\sum_{k=1}^{\infty} k x^{k-1} = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{(x-1)(1)-x(1)}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

Letting $x = \frac{2}{3} < 1$, then

$$\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^{k-1} = \frac{1}{\left(1-\frac{2}{3}\right)^2} = \frac{1}{\frac{1}{9}} = 9.$$

So $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k = \frac{2}{3}\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^{k-1} = \frac{2}{3}(9) = 6.$

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(2) $\sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)^k}{k+1}$.

<u>Soln.</u> For $|x|<1$, we have $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$.

Integ. both sides we get

$$\sum_{k=1}^{\infty} \frac{x^{k+1}}{k+1} = \int \frac{x}{1-x}\,dx = \int \left(-1+\frac{1}{1-x}\right) dx = -x - \ln|1-x| + C.$$

Take $x = 1/2 < 1$, we have

$$\sum_{k=1}^{\infty} \frac{(1/2)^{k+1}}{k+1} = -\frac{1}{2} - \ln\left(1-\frac{1}{2}\right) = -\frac{1}{2} - \ln\left(\frac{1}{2}\right).$$

Then $$\sum_{k=1}^{\infty} \frac{(1/2)^{k}}{k+1} = \frac{-\frac{1}{2} - \ln\left(\frac{1}{2}\right)}{\frac{1}{2}} = -1 - 2\ln\left(\tfrac{1}{2}\right) = (\ln 4) - 1.$$

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(3) $$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{3}\right)^{k}}{k+1}.$$

$|x|<1$

Soln. $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \xRightarrow{\text{Integ.}} \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = -\ln(1-x).$$

Let $x = \frac{1}{3} < 1$, then $$\sum_{k=0}^{\infty} \frac{(1/3)^{k+1}}{k+1} = -\ln\left(1-\frac{1}{3}\right).$$

Thus, $$\sum_{k=0}^{\infty} \frac{(1/3)^{k}}{k+1} = \frac{-\ln(1-1/3)}{1/3} = -3\ln\left(\frac{2}{3}\right).$$

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(4) $$\sum_{k=2}^{\infty} (-1)^k \left(\frac{2}{5}\right)^{k-2}.$$

Soln. $$\sum_{k=2}^{\infty} x^k = \frac{x^2}{1-x},$$ provided that $|x|<1$.

Replace x with $-x$, we get

$$\sum_{k=2}^{\infty} (-1)^k x^k = \frac{(-x)^2}{1+x} = \frac{x^2}{1+x}.$$

let $x = \frac{2}{5} < 1$, then $\left[\sum_{k=2}^{\infty} (-1)^k \left(\frac{2}{5}\right)^k = \frac{\left(\frac{2}{5}\right)^2}{1+\frac{2}{5}}\right] * \left(\frac{2}{5}\right)^{-2}$

Then $\sum_{k=2}^{\infty} (-1)^k \left(\frac{2}{5}\right)^k = \frac{\cancel{\left(\frac{2}{5}\right)^2}}{1+\frac{2}{5}} * \cancel{\left(\frac{2}{5}\right)^{-2}} = \frac{1}{\frac{7}{5}} = \frac{5}{7}.$

Apply the definition and use telescoping series

Apply the definition: $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$

S_n (let)

<u>Telescoping series</u>:

$$\sum_{k=p}^{n} \left[f(k) - f(k+1)\right] = f(p) - f(n+1), \text{ and}$$

$$\sum_{k=p}^{n} \left[f(k) - f(k-1)\right] = f(n) - f(p-1).$$

We use this method if we have a rational func.
So, recall the idea of partial fractions.

Ex. Find the sum if it exists.

(1) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$. $\longrightarrow a_k \longrightarrow$

Soln. Note that $\displaystyle\frac{1}{k(k+1)} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.

Let $\displaystyle S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cancel{\cdots} + \left(\cancel{\tfrac{1}{n}} - \tfrac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}.$$

Then $\displaystyle\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \overbrace{\sum_{k=1}^{n} a_k}^{S_n} = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.$

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(2) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k}\,\sqrt{k+1}}$.

Soln. $\displaystyle\frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k}\,\sqrt{k+1}} = \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right)$$

$$= \left(1 - \cancel{\tfrac{1}{\sqrt{2}}}\right) + \left(\cancel{\tfrac{1}{\sqrt{2}}} - \cancel{\tfrac{1}{\sqrt{3}}}\right) + \cdots + \left(\cancel{\tfrac{1}{\sqrt{n}}} - \tfrac{1}{\sqrt{n+1}}\right)]$$

$$= 1 - \frac{1}{\sqrt{n+1}}$$

So $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k}\,\sqrt{k+1}} = \lim_{n\to\infty}\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1.$

(3) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k+1)(2k-1)}$ ← a_k

Soln. $$\frac{1}{(2k+1)(2k-1)} = \frac{1}{2}\,\frac{2}{(2k+1)(2k-1)}$$

$$= \frac{1}{2}\,\frac{(2k+1)-(2k-1)}{(2k+1)(2k-1)}$$

$$= \frac{1}{2}\left[\frac{1}{2k-1} - \frac{1}{2k+1}\right] \leftarrow a_k.$$

let $$S_n = \sum_{k=1}^{n} a_k$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

$$= \frac{1}{2}\left[\left(1 - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2n+1}\right].$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)(2k-1)} = \lim_{n\to\infty} S_n = \frac{1}{2}\lim_{n\to\infty}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2}.$$

Thm. If $\sum a_k$ and $\sum b_k$ are conv. series, then so are the series $\sum c\,a_k$ (where c is a constant) and $\sum (a_k \pm b_k)$. In this case,

$$\sum (c\,a_k \pm b_k) = c\sum a_k \pm \sum b_k.$$

Ex. Find the sum of $\sum_{k=1}^{\infty}\left(\frac{2^{k+2}}{3^k}+\frac{4}{k(k+1)}\right)$.

Soln. From previous example, we have

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = 2 \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{1}{k(k+1)} = 1.$$

It follows that

$$\sum_{k=1}^{\infty}\left(\frac{2^{k+2}}{3^k}+\frac{4}{k(k+1)}\right) = 4\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k + 4\sum\frac{1}{k(k+1)}$$

$$= 4(2) + 4(1) = 12.$$

Today, we have answered Q1.
Next time, we will answer Q2.

This lecture: Infinite series - Introduction.
Next lecture: Series convergence tests.

Searching keywords:
- Series, find the sum, geometric series, sequence of partial sums, المتتاليات
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

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