Limits

The idea of a limit underlies the various branches of calculus.

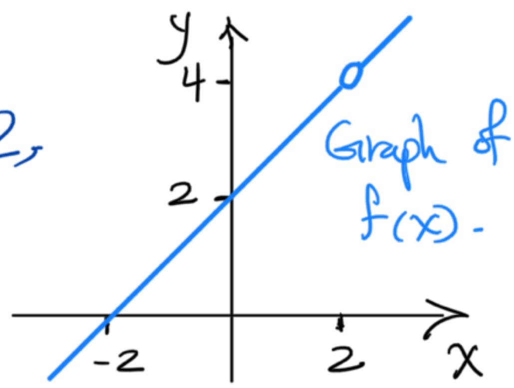
The limit of a function.

Let us investigate the behavior of the function $f(x) = (x^2-4)/(x-2)$ for values of x near 2.

Note that

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2,$$

provided that $x \neq 2$.



x	$f(x)$
1.9	3.9
1.99	3.99
1.999	3.99

This table gives values of $f(x)$ for values of x close to 2 but not equal to 2.

From the table and the graph of f, we see that the closer x is to 2, the closer $f(x)$ is to 4.

We express this by saying "the limit of the func. $f(x)=\frac{x^2-4}{x-2}$ as x approaches 2 is equal to 4".
The notation for this is $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$.

<u>Notation</u>: In general, we write $\lim_{x\to a} f(x) = L$ to mean that the limit of $f(x)$, as x approaches a, equals L.

<u>Notations of one-sided limits</u>.

- We write $\lim_{x\to a^-} f(x) = L$ to mean that the left-hand limit of $f(x)$ as x approaches a is equal to L, or the limit of $f(x)$ as x goes to a from the left equals L.
- We write $\lim_{x\to a^+} f(x) = L$ to mean that the right-hand limit of $f(x)$ as x approaches a is equal to L, or the limit of $f(x)$ as x goes to a from the right equals L.

<u>Fact</u>: $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$.

Re-visting the func. $f(x) = \frac{x^2-4}{x-2} = x+2 ; (x \neq 2)$.

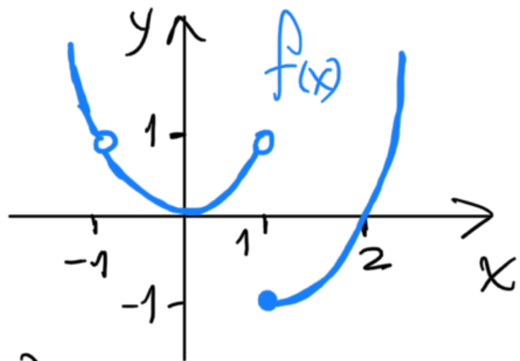
x	$f(x)$
1.9	3.9
1.99	3.99
1.999	3.99
2.1	4.1
2.01	4.01
2.001	4.001

$\therefore \lim_{x \to 2^-} f(x) = 4$. ——— (1)

$\therefore \lim_{x \to 2^+} f(x) = 4$. ——— (2)

Thus, from (1) and (2), $\lim_{x \to 2} f(x) = 4$.

Ex. Use the graph of $f(x)$ to find the limits.



$\lim_{x \to 1^-} f(x) = 1$.

$\lim_{x \to 1^+} f(x) = -1$.

$\lim_{x \to 1} f(x)$ DNE (does not exist).

(as $\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$)

$\lim_{x \to -1^-} f(x) = 1$.

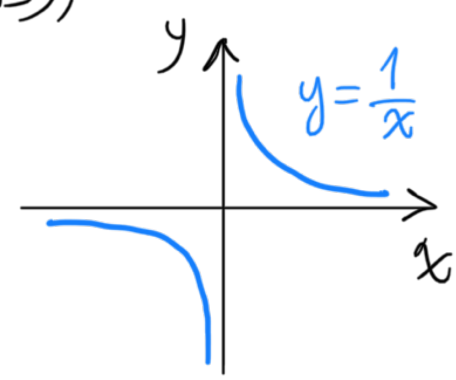
$\lim_{x \to -1^+} f(x) = 1$.

$\lim_{x \to -1} f(x) = 1$.

Notation of infinite limits.

- We write $\lim_{x \to a} f(x) = \infty$ to mean that $f(x)$ becomes infinite as x approaches a.
- We write $\lim_{x \to a} f(x) = -\infty$ to mean that $f(x)$ becomes negative infinite as x approaches a.

Ex. To find $\lim_{x \to 0} \frac{1}{x}$ (if it exists), note that $\lim_{x \to 0^+} \frac{1}{x} = \infty$, and $\lim_{x \to 0^-} \frac{1}{x} = -\infty$. Thus $\lim_{x \to 0} \frac{1}{x}$ DNE.

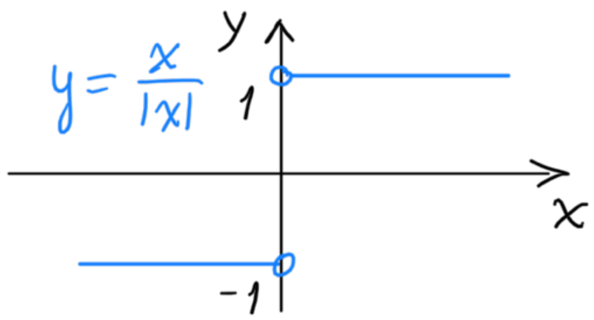


Ex. Evaluate $\lim_{x \to 0} \frac{x}{|x|}$, if exists.

Soln. Note that $|x| = \begin{cases} x & , \text{ if } x \leq 0, \\ -x & , \text{ if } x > 0. \end{cases}$

Then $\frac{1}{|x|} = \begin{cases} 1/x & , \text{ if } x > 0, \\ \text{undefined} & , \text{ if } x = 0, \\ -1/x & , \text{ if } x < 0, \end{cases}$

hence $\frac{x}{|x|} = \begin{cases} 1 & , \text{ if } x > 0, \\ \text{undefined} & , \text{ if } x = 0, \\ -1 & , \text{ if } x < 0. \end{cases}$



Thus $\lim_{x\to 0^+} \frac{x}{|x|} = 1$,

and $\lim_{x\to 0^-} \frac{x}{|x|} = -1$.

Therefore $\lim_{x\to 0} \frac{x}{|x|}$ DNE.

Limit laws.

<u>Fact</u>: For any constant c, $\lim_{x\to a} c = c$.

<u>Ex</u>: $\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$.

$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} = \lim_{x\to 0} -1 = -1$.

<u>Fact</u>: Let n be a positive integer, then

1) $\lim_{x\to a} x^n = a^n$.

2) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, where, if n is even, we assume that $a > 0$.

<u>Fact</u>: Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ <u>exist</u>. Then

1) $\lim_{x\to a} [f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$. (Sum law).

2) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$. (Difference law).

3) $\lim_{x\to a}[c\,f(x)] = c\lim_{x\to a} f(x)$. (Constant multiple law)

4) $\lim_{x\to a}[f(x)\,g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$. (Product law).

5) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$. (Quotient law).

Ex. $\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)}$

$$= \frac{\lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - 3\lim_{x\to -2} x}$$

$$= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)}$$

$$= \frac{-1}{11}.$$

Fact: If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided that the limits exist.

Ex. Note that $\frac{x^2-4}{x-2} = x+2$, when $x \neq 2$.

So $\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2} x+2 = 2+2 = 4.$

Direct substitution property: If f is a polynomial or a rational function and $a \in \mathrm{Dom}(f)$, then

$$\lim_{x \to a} f(x) = f(a).$$

Ex. $\displaystyle \lim_{x \to -2} \frac{x^2-4}{x-2} = \frac{(-2)^2-4}{(-2)-2} = \frac{0}{-4} = 0.$

Fact: If $\displaystyle \lim_{x \to a} f(x) = L \neq 0$ and $\displaystyle \lim_{x \to a} g(x) = 0$, then $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)}$ DNE.

Ex. Find the limit if it exists (show your solution in red!).

1) $\displaystyle \lim_{x \to 4} \left(\frac{1}{x} - \frac{1}{4}\right)\left(\frac{1}{x-4}\right) = \lim_{x \to 1} \frac{\cancel{4-x}}{4x}\left(\frac{1}{\underset{-1}{\cancel{x-4}}}\right)$

$$= \lim_{x \to 1} \frac{-1}{4x}$$

$$= \frac{-1}{16}.$$

2) $\displaystyle \lim_{x \to 4} \left(\frac{1}{x} - \frac{1}{4}\right)\left(\frac{1}{x-4}\right)^2 = \lim_{x \to 4} \left(\frac{\overset{-1}{\cancel{4-x}}}{4x}\right)\left(\frac{1}{x-4}\right)^{\cancel{2}}$

$$= \lim_{x \to 4} \frac{-1}{4x(x-4)} \text{ DNE.}$$

3) $\lim_{t \to 0} \dfrac{1+\frac{1}{t}}{1+\frac{1}{t^2}} = \lim_{t \to 0} \dfrac{\frac{t+1}{t}}{\frac{t^2+1}{t^2}} = \lim_{t \to 0} \dfrac{t(t+1)}{t^2+1} = 0.$

4) $\lim_{x \to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x \to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x} \cdot \dfrac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

$= \lim_{x \to 0} \dfrac{^{1}(\cancel{x}+\cancel{2})-\cancel{2}}{\cancel{x}(\sqrt{x+2}+\sqrt{2})}$

$= \lim_{x \to 0} \dfrac{1}{\sqrt{x+2}+\sqrt{2}}$

$= \dfrac{1}{2\sqrt{2}}.$

5) $\lim_{x \to 1} \dfrac{\sqrt{2-x}-1}{2-\sqrt{x+3}} = \lim_{x \to 1} \left(\dfrac{\sqrt{2-x}-1}{2-\sqrt{x+3}} \cdot \dfrac{\sqrt{2-x}+1}{\sqrt{2-x}+1} \cdot \dfrac{2+\sqrt{x+3}}{2+\sqrt{x+3}} \right)$

$= \lim_{x \to 1} \dfrac{\cancel{[(2-x)-1]}(2+\sqrt{x+3})}{\cancel{[4-(x+3)]}(\sqrt{2-x}+1)}$

$= \lim_{x \to 1} \dfrac{2+\sqrt{x+3}}{\sqrt{2-x}+1}$

$= \dfrac{2+\sqrt{4}}{\sqrt{1}+1}$

$= 2.$

6) $\lim_{x\to 3} \frac{x^2+x-12}{(x-3)^2} = \lim_{x\to 3} \frac{(x+4)\cancel{(x-3)}}{(x-3)^{\cancel{2}}}$

$= \lim_{x\to 3} \frac{x+4}{x-3}$ DNE.

7) $\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0} \frac{(\cancel{9}+6\cancel{h}+h^{\cancel{2}})-\cancel{9}}{\underset{1}{\cancel{h}}}$

$= \lim_{h\to 0} 6+h$

$= 6.$

Ex. Let $f(x) = \begin{cases} \sqrt{x-4} & \text{if } x>4, \\ 8-2x & \text{if } x<4. \end{cases}$

Determine whether $\lim_{x\to 4} f(x)$ exists.

Soln: $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0.$

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (8-2x) = 8-2(4) = 0.$

Thus, the limit exists and $\lim_{x\to 4} f(x) = 0.$

Ex. Let $f(x) = \begin{cases} x^3 & , \text{ if } x<3, \\ 7 & , \text{ if } x=3, \\ 2x+3 & , \text{ if } x>3. \end{cases}$

Determine whether $\lim_{x \to 3} f(x)$ exists.

Soln. $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^3 = 27$,

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x+3) = 9$.

$27 \neq 9$

Thus $\lim_{x \to 3} f(x)$ DNE.

Trigonometric limits.

Facts: 1) $\lim_{x \to a} \sin x = \sin a$.

2) $\lim_{x \to a} \cos x = \cos a$.

3) $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

4) $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$.

Recall that

$\sin^2 x + \cos^2 x = 1$.

$\tan^2 x + 1 = \sec^2 x$.

$\cot^2 x + 1 = \csc^2 x$.

Proof of item (4):

$$\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{1+\cos x}\right)\right]$$

$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{1+\cos x}$$

$$= (1)(0/2)$$

$$= 0.$$

Ex. Find the limit if it exists (show your solution in red!).

1) $\lim_{x\to 0}(x\cot x) = \lim_{x\to 0} x\left(\frac{\cos x}{\sin x}\right)$

$$= \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{x}$$

$$= \lim_{x\to 0}\cos x \Big/ \lim_{x\to 0}\left(\frac{\sin x}{x}\right)$$

$$= 1/1$$

$$= 1.$$

2) $\lim_{x\to 0}\frac{\sin 4x}{4x} = \lim_{y\to 0}\frac{\sin y}{y} = 1$ (By setting $y = 4x$, $y\to 0$ as $x\to 0$).

3) $\lim_{x\to 0} \frac{\sin 4x}{5x} = \lim_{x\to 0} \frac{4}{5} \frac{\sin 4x}{4x}$

$= \frac{4}{5} \lim_{x\to 0} \frac{\sin 4x}{4x}$

$= \left(\frac{4}{5}\right)(1) = 4/5$.

4) $\lim_{x\to 0} \frac{\sin 4x}{\sin 5x} = \lim_{x\to 0} \frac{\frac{\sin 4x}{4x}}{\frac{\sin 5x}{4x}} = \frac{1}{\left(\frac{5}{4}\right)} = \frac{4}{5}$.

5) $\lim_{x\to \frac{\pi}{3}} \frac{\sin\left(x-\frac{\pi}{3}\right)}{\left(x-\frac{\pi}{3}\right)^2} = \lim_{y\to 0} \frac{\sin y}{y^2}$ $\left(\begin{array}{l}\text{By setting} \\ y = x-\frac{\pi}{3}\end{array}\right)$

$= \lim_{y\to 0}\left(\frac{\sin y}{y}\right) \cdot \frac{1}{\lim_{y\to 0} y}$ DNE.

6) $\lim_{x\to 0} \frac{1-\sec^2(2x)}{x^2} = -\lim_{x\to 0} \frac{\tan^2 2x}{x^2}$

$= -\lim_{x\to 0} \frac{\sin^2 2x}{x^2} \cdot \frac{1}{\cos^2 2x}$

$= -\lim_{x\to 0} \frac{\sin^2 2x}{(2x)^2} \cdot (4)\left(\frac{1}{\cos^2 2x}\right)$

$= -4\left(\lim_{x\to 0} \frac{\sin 2x}{2x}\right)^2 \left(\lim_{x\to 0} \frac{1}{\cos 2x}\right)^2$

$= -4(1)(1) = -4$.

$$7)\ \lim_{x\to 0} \frac{x^2}{\sec x - 1} = \lim_{x\to 0} \frac{x^2}{\sec x - 1} \cdot \frac{\sec x + 1}{\sec x + 1}$$

$$= \lim_{x\to 0} \frac{x^2(\sec x + 1)}{\sec^2 x - 1}$$

$$= \lim_{x\to 0} \frac{x^2(\sec x + 1)}{\tan^2 x}$$

$$= \lim_{x\to 0} \frac{x^2(\sec x + 1)}{\left(\frac{\sin x}{\cos x}\right)^2}$$

$$= \lim_{x\to 0} \frac{x^2(\sec x + 1)\cos^2 x}{\sin^2 x}$$

$$= \lim_{x\to 0} \left(\frac{x}{\sin x}\right)^2 \lim_{x\to 0} \left[(\sec x + 1)\cos^2 x\right]$$

$$= \frac{\lim_{x\to 0} (\sec x + 1) \lim_{x\to 0} \cos^2 x}{\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2}$$

$$= \frac{(1+1)(1)}{1}$$

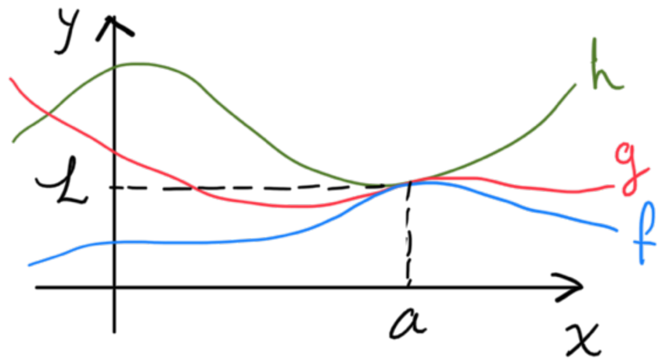
$$= 2.$$

The squeeze theorem.

If $f(x) \le g(x) \le h(x)$ when x is near a and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then $\lim_{x\to a} g(x) = L$.



Ex. Show that (a) $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$.

(b) $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

(c) $\lim_{x\to 0} x^2 \cos^2\left(\frac{1}{x}\right) = 0$.

Proof: First note that we cannot use

$$\lim_{x\to 0} x \sin\frac{1}{x} = \lim_{x\to 0} x \cdot \lim_{x\to 0} \sin\frac{1}{x}$$

because $\lim_{x\to 0} \sin\frac{1}{x}$ DNE.

Instead we apply the squeeze thm here!

(a) Note that $-1 \le \sin\frac{1}{x} \le 1$,

Multiplying each side by x^2, we get $-x^2 \le x^2 \sin\frac{1}{x} \le x^2$.

Now, as $x \to 0$, we have $\quad -x^2 \to 0, \quad x^2\sin\frac{1}{x} \to 0, \quad x^2 \to 0$

$\therefore \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

(b) $-1 \le \sin\frac{1}{x} \le 1$.

For $x \geq 0$, we have $-x \le x\sin\frac{1}{x} \le x$.

As $x \to 0$, $\quad -x \to 0, \quad x\sin\frac{1}{x} \to 0, \quad x \to 0$

$\therefore \lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$.

(c) $-1 \le \cos\frac{1}{x} \le 1$. Then $0 \le \cos^2\frac{1}{x} \le 1$.

Hence $0 \le x^2\cos^2\frac{1}{x} \le x^2$.

As $x \to 0$, $\quad 0 \to 0, \quad x^2\cos^2\frac{1}{x} \to 0, \quad x^2 \to 0$

Thus, $\lim\limits_{x\to 0} x^2\cos^2\frac{1}{x} = 0$.

Precise definition of a limit.

Def Let f be a func. defined on some open interval that contains the number a, except possibly at a itself. Then $\lim\limits_{x\to a} f(x) = L$ if for every number $\epsilon > 0$, there is a number $\delta > 0$ such that if $|x-a| < \delta$, then $|f(x) - L| < \epsilon$.

Ex: Show that $\lim_{x\to 1} 3x+5 = 8$ using the definition.

Proof. Let $\epsilon > 0$, we look for $\delta > 0$ such that if $|x-\underset{c}{1}| < \delta$, then $|\underbrace{(3x+5)}_{f(x)}-\underset{L}{8}| < \epsilon$.

Now $|3x+5-8| = |3x-3| = 3|x-1| < \epsilon$.

Then $|x-1| < \epsilon/3$. So we take $\delta < \epsilon/3$.

Searching keywords:

- Limits. النهايات
- Find the limit, trigonometric limits
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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