Integration by substitution

To find $\int f(g(x))\, g'(x)\, dx$, we let $u = g(x)$, then $du = g'(x)\, dx$.

So, if F is an antiderivative of f, then

$$\underbrace{\int f(g(x))\, g'(x)\, dx}_{\text{"Not nice" integral.}} = \underbrace{\int f(u)\, du}_{\text{Very nice!}} = F(u) + C = F(g(x)) + C.$$

Ex. Evaluate the given integral.

1) $I = \int (x^3+5)^{100} (3x^2)\, dx$.

Soln. Let $u = x^3 + 5$, then $du = 3x^2\, dx$, so

$$I = \int u^{100} (\cancel{3x^2}) \frac{du}{\cancel{3x^2}} = \int u^{100}\, du = \frac{u^{101}}{101} + C$$

Thus, $I = \frac{1}{101}(x^3+5)^{101} + C$.

2) $I = \int x \cos x^2\, dx$.

Soln. Let $u = x^2$, then $du = 2x\, dx$, so

$$I = \int \cancel{x} \cos u \frac{du}{2\cancel{x}} = \frac{1}{2}\int \cos u\, du = \frac{1}{2}\sin u + C.$$

Thus, $I = \frac{1}{2}\sin(x^2) + C$.

3) $I = \int (3\sin x + 4)^5 \cos x \, dx.$

Soln. Let $u = 3\sin x + 4$, then $du = 3\cos x \, dx.$

It follows that $I = \int u^5 \cancel{\cos x} \dfrac{du}{3\cancel{\cos x}}$

$$= \frac{1}{3}\int u^5 \, du$$

$$= \frac{1}{3}\left(\frac{1}{6}u^6\right) + C$$

$$= \frac{1}{8}(3\sin x + 4)^6 + C.$$

4) $\int \dfrac{\sinh \sqrt{x}}{\sqrt{x}} \, dx.$

Soln. let $u = \sqrt{x}$, then $du = \dfrac{dx}{2\sqrt{x}} = \dfrac{dx}{2u}$

or $dx = 2u \, du.$

So $\int \dfrac{\sinh\sqrt{x}}{\sqrt{x}} \, dx = \int \dfrac{\sin u}{\cancel{u}} \, 2\cancel{u} \, du$

$$= 2\int \sin u \, du$$

$$= -2\cos u + C$$

$$= -2\cos\sqrt{x} + C.$$

5) $\int x\sqrt{2-x}\,dx$.

Soln. Let $u = 2-x$, then $du = -dx$.

So $\int x\sqrt{2-x}\,dx = \int (2-u)\sqrt{u}\,(-du)$

$$= -\int \left(2u^{1/2} - u^{3/2}\right) du$$

$$= -2\left(\tfrac{2}{3}\right)u^{3/2} - \tfrac{2}{5}u^{5/2} + C$$

$$= -\tfrac{4}{3}(2-x)^{3/2} - \tfrac{2}{5}(2-x)^{5/2} + C.$$

6) $\int \sec^3 x \tan x\,dx$.

Soln. Let $u = \sec x$, then $du = \sec x \tan x\,dx$.

Note that $\sec^3 x \tan x\,dx = \sec^2 x\,(\sec x \tan x\,dx)$

$$= u^2 du.$$

So $\int \sec^3 x \tan x\,dx = \int u^2\,du$

$$= \tfrac{1}{3}u^3 + C$$

$$= \tfrac{1}{3}\sec^3 x + C.$$

7) $\int e^{\pi x}\,dx$.

Soln. Let $u = \pi x$, then $du = \pi\,dx$ or $dx = \frac{1}{\pi}\,du$.

So $\int e^{\pi x}\,dx = \int e^{u}\left(\frac{1}{\pi}\right)du = \frac{1}{\pi}e^{u} + C = \frac{1}{\pi}e^{\pi x} + C$.

8) $\int \sqrt{1+x^2}\,x^5\,dx$.

Soln. Let $u = 1+x^2$, then $du = 2x\,dx$ or $x\,dx = \frac{1}{2}du$.

Note that $x^5\,dx = (x^2)^2 x\,dx = (u-1)^2\left(\frac{1}{2}du\right)$.

So $\int \sqrt{1+x^2}\,x^5\,dx = \int \sqrt{u}\,(u-1)^2\left(\frac{1}{2}\right)du$

$$= \frac{1}{2}\int \sqrt{u}\,(u^2-2u+1)\,du$$

$$= \frac{1}{2}\int \left[u^{5/2} - 2u^{3/2} + u^{1/2}\right]du$$

$$= \frac{1}{2}\left[\frac{2}{7}u^{7/2} - 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right] + C$$

$$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C.$$

9) $\int x^3\cos(x^4+2)\,dx$. Exc. FINAL ANS: $\frac{1}{4}\sin(x^4+2) + C$.

10) $\int \sqrt{2x+1}\,dx$. Exc. FINAL ANS: $\frac{1}{3}(2x+1)^{3/2} + C$.

11) $\int \frac{x}{\sqrt{1-4x^2}}\,dx$. Exc. FINAL ANS: $-\frac{1}{4}\sqrt{1-4x^2} + C$.

Note: We can also use the substitution rule to prove the formula $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$.

To see this let $u = f(x)$, then $du = f'(x)\,dx$.

So $\int \frac{f'(x)}{f(x)}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|f(x)| + C$.

The substitution rule for definite integrals

$$\int_a^b f(g(x))\, g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Ex: Evaluate the given integral.

1) $\int_0^4 \sqrt{2x+1}\,dx$.

Soln. Let $u = 2x+1$, then $dx = \frac{1}{2}du$.

When $x=0$, $u=1$; when $x=4$, $u=9$.

Then $\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big]_1^9$

$= \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right) = 26/3$.

2) $\int_1^e \frac{\ln x}{x}\,dx.$

Soln. Let $u = \ln x$, then $du = dx/x$.

When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$.

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}.$$

3) $I = \int_0^1 x^3\sqrt{x^4+5}\,dx.$

Soln. Let $u = x^4+5$, then $du = 4x^3\,dx$.

When $x=0$, $u=5$; when $x=1$, $u=6$.

$$\text{So, } I = \int_5^6 \cancel{x^3}\sqrt{u}\,\frac{du}{4\cancel{x^3}} = \frac{1}{4}\int_5^6 u^{1/2}\,du$$

$$= \frac{1}{4}\left[\frac{u^{3/2}}{3/2}\right]_5^6 = \frac{1}{6}u^{3/2}\Big|_5^6$$

$$= \frac{1}{6}\left[6^{3/2} - 5^{3/2}\right].$$

4) $\int_{1}^{2} \frac{dx}{(3-5x)^2}$. Exc. FINAL ANS. $1/14$.

Searching keywords:

- Integration by substitution التكامل بالتعويض
- Evaluate the integral
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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