The fundamental theorem of calculus.

The fundamental theorem of calculus (FTC) is very important because it establishes a connection between the two branches of calculus: differential calculus and integral calculus.

The fundamental theorem of calculus, part 1 (FTC1).

If f is cts on $[a,b]$, then the func. F defined by $F(x) = \int_a^x f(t)\,dt$, $a \le x \le b$, is cts on $[a,b]$ and diff. on (a,b), and $F'(x) = f(x)$.

Examples:

1) let $F(x) = \int_{-1}^{x} (3t + t^2)\,dt$, for $-1 \le x \le 5$,

   then $F'(x) = 3x + x^2$ on $[-1, 5]$.

2) Let $F(x) = \int_0^x \tan(\pi t)\,dt$, for $-\frac{1}{3} \le t \le \frac{1}{3}$,

   then $F'(x) = \tan(\pi x)$, hence $F'(\frac{1}{4}) = \tan\frac{\pi}{4} = 1$.

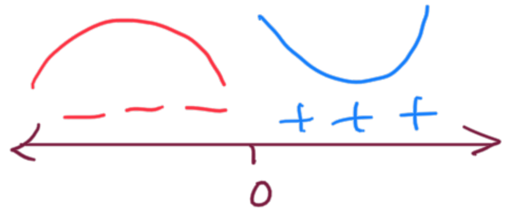
3) let $F(x) = \int_x^1 \sqrt{t+1}\,dt$, for $t \ge 0$.

   Then $F(x) = -\int_1^x \sqrt{t+1}\,dt$, so $F'(x) = -\sqrt{x+1}$.

__Ex.__ Find the interval(s) on which $F(x)=\int_0^x \frac{dt}{1+t^2}$, $\forall x \in \mathbb{R}$, is increasing or decreasing, then find the interval(s) of concavity and the inflection point(s).

__Soln.__ $F'(x)=\frac{1}{1+x^2}>0$, $\forall x \in \mathbb{R}$. So F is increasing on $\mathbb{R}$.

$F''(x)=\frac{-2x}{(1+x^2)^2}$ which equals zero when $x=0$.



$\therefore$ F is concave up on $(-\infty,0)$,

F is concave down on $(0,\infty)$,

and $(0,0)$ is an inflection point.

__Remark 1.__ If $F(x)=\int_a^{g(x)} f(t)\,dt$, then by the chain rule,

$F'(x)=f(g(x))\,g'(x)$.

__Ex.__ Let $F(x)=\int_2^{x^2} \cos(t^3+e^3)\,dt$, then

$F'(x)=\cos((x^2)^3+e^3)\frac{d}{dx}(x^2)=2x\cos(x^6+e^3)$.

Remark 2. If $F(x)=\int_{h(x)}^{g(x)} f(t)\,dt$, then

$$F(x)=\int_{h(x)}^{0} f(t)\,dt+\int_{0}^{g(x)} f(t)\,dt=-\int_{0}^{h(x)} f(t)\,dt+\int_{0}^{g(x)} f(t)\,dt.$$

Using the chain rule, $F'(x)=-f(h(x))\,h'(x)+f(g(x))\,g'(x)$.

Ex. Find the equation of the tangent line of the func. $F(x)=\int_{2x}^{x^2}\sqrt{t^2+1}\,dt$, at $x=0$.

Soln. $F(x)=-\int_{0}^{2x}\sqrt{t^2+1}\,dt+\int_{0}^{x^2}\sqrt{t^2+1}\,dt$.

Then $F'(x)=-\sqrt{(2x)^2+1}\,\frac{d}{dx}(2x)+\sqrt{(x^2)^2+1}\,\frac{d}{dx}(x^2)$

$$=-2\sqrt{4x^2+1}+2x\sqrt{x^4+1}.$$

Now, $F(0)=\int_{0}^{0}\sqrt{t^2+1}\,dt=0$ and $F'(0)=-2$.

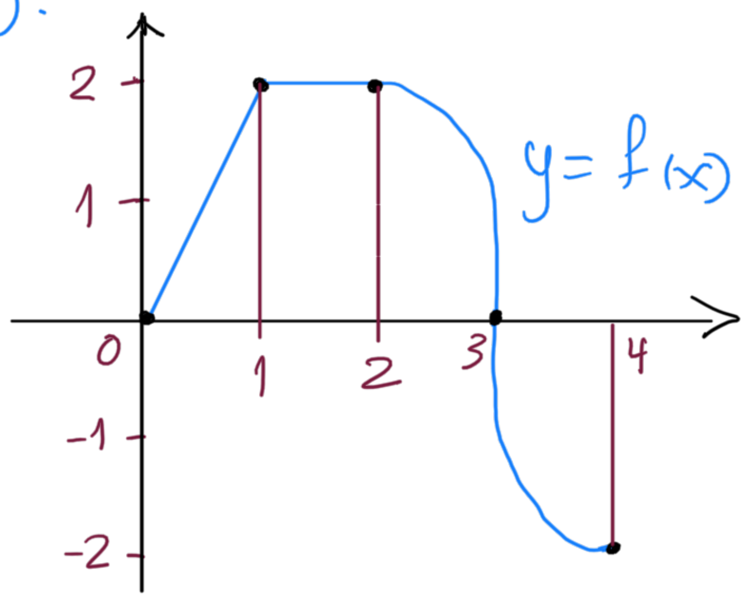
Thus, the equation of the tangent line is

$$y-F(0)=F'(0)\,(x-0)$$

$$y-0=-2\,(x-0)$$

$$\therefore\quad y=-2x.$$

Ex. If f is the func. whose graph is shown below and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$ and $g(4)$.



Soln. $g(0) = \int_0^0 f(t)\,dt = 1$

$$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2}(1 \cdot 2) = \frac{1}{2}.$$

$$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$$

$$= 1 + (1 \cdot 2) = 3.$$

Let A^* be the area under f from 2 to 3.

Then $g(4) = \int_0^4 f(t)\,dt$

$$= \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt$$

$$= 3 + A^* - A^* = 3.$$

Def. Let f be a cts func. on $[a,b]$. A func. G is called antiderivative for f on $[a,b]$ if G is cts on $[a,b]$ and $G'(x) = f(x)$, $\forall x \in (a,b)$.

Ex. For any constant c, $\left(\frac{1}{3}x^3 + c\right)' = x^2$.
Then $\left(\frac{1}{3}x^3 + c\right)$ is an antiderivative of x^2.

The fundamental theorem of calculus, part 2 (FTC2).
Let f be a cts func. on $[a,b]$. If G is any antiderivative for f on $[a,b]$, then

$$\int_a^b f(t)\,dt = G(x)\Big|_a^b = G(b) - G(a).$$

Ex. Because $(x^4)' = 4x^3$, we have

$$\int_a^b 4x^3\,dx = x^4\Big|_a^b = b^4 - a^4.$$

Ex. Find the area under the curve of $y = e^x$ from 1 to 3.

Soln. The func. e^x is cts everywhere and we know that an antiderivative is $G(x) = e^x$, so by FTC2

$$\text{Area} = \int_1^3 e^x \, dx = F(3) - F(1) = e^3 - e.$$

Exc. Evaluate $\int_3^6 \frac{dx}{x}$.

Exc. Find the area under the cosine curve from 0 to $\pi/2$.

Ex. What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2} \, dx = \left.\frac{-1}{x}\right|_{-1}^{3} = -\frac{1}{3} - 1 = \frac{-4}{3}.$$

Soln. We know that $\int_a^b f(x)\,dx \geq 0$ when $f \geq 0$. Note that $\frac{1}{x^2} \geq 0$, but $\frac{-4}{3} \ngeq 0$, so this calculation must be wrong. In fact the FTC2 applies to cts funcs. So, it cannot be applied here because $f(x) = 1/x^2$ is not cts on $[-1,3]$.

f has an infinite discontinuity at $x=0$, so

$$\int_{-1}^{3} \frac{1}{x^2} \, dx \quad \text{DNE}.$$

Net change theorem.

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Some integration formulas.

1) $\int_a^b k\,dx = kx\Big|_a^b = k(b-a).$

2) $\int_a^b x^r\,dx = \frac{x^{r+1}}{r+1}\Big|_a^b, \quad (r \neq -1).$

3) $\int_a^b \frac{1}{x}\,dx = \ln|x|\Big|_a^b.$

4) $\int e^x\,dx = e^x\Big|_a^b.$

5) $\int b^x\,dx = \frac{b^x}{\ln b}\Big|_a^b.$

6) $\int \sin x\,dx = -\cos x\Big|_a^b.$

7) $\int \cos x\,dx = \sin x\Big|_a^b.$

8) $\int \sec^2 x\,dx = \tan x\Big|_a^b.$

9) $\int \csc^2 x \, dx = -\cot x \Big|_a^b$.

10) $\int \sec x \tan x \, dx = \sec x \Big|_a^b$.

11) $\int \csc x \cot x \, dx = -\csc x \Big|_a^b$.

12) $\int \frac{dx}{1+x^2} = \tan^{-1} x \Big|_a^b$.

13) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_a^b$.

14) $\int \sinh x \, dx = \cosh x \Big|_a^b$.

15) $\int \cosh x \, dx = \sinh x \Big|_a^b$.

<u>Examples;</u>

1) $\int_0^1 \frac{dx}{\sqrt[5]{x}} = \int_0^1 x^{-\frac{1}{5}} dx = \frac{x^{4/5}}{4/5} \Big|_0^1 = \frac{1}{4/5} = 5/4$.

2) $\int_0^1 (3x^2 - 7x^6 + 1) \, dx = \left(x^3 - x^7 + x \right|_0^1 = 1$.

3) $\int_1^2 (x+1)^2 \, dx = \frac{(x+1)^3}{3} \Big|_1^2 = \frac{1}{3} \left(3^2 - 2^3\right) = \frac{19}{3}$.

4) $\int_0^{\pi/4} \sec x\left(3\tan x - 4\sec x\right) dx = 3\int_0^{\pi/4} \tan x \sec x \, dx - 4\int_0^{\pi/4} \sec^2 x \, dx$

$= 3\sec x \Big|_0^{\pi/4} - 4\tan x \Big|_0^{\pi/4} = 3(\sqrt{2}-1) - 4(1-0) = 3\sqrt{2}-7.$

5) $\int_1^2 \frac{x^4+1}{x^2} dx = \int_1^2 \left(x^2 + x^{-2}\right) dx = \frac{1}{3}x^3 + \frac{x^{-1}}{-1}\Big|_1^2 = \frac{17}{6}.$

6) $\int_0^1 \left(3-\sqrt{x+1}\right)^2 dx = \int_0^1 \left[9 - 6\sqrt{x+1} + (x+1)\right] dx$

$= \int_0^1 \left(x - 6\sqrt{x+1} + 10\right) dx$

$= \frac{x^2}{2} - 6\,\frac{(x+1)^{3/2}}{3/2} + 10x \Big|_0^1$

$= \frac{1}{2} - 4\left[2^{3/2} - 1^{3/2}\right] + 10$

7) $\int_0^{\pi/4} -2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) dx = \int_0^{\pi/4} -\sin x \, dx = \cos x \Big|_0^{\pi/4}$

$= \frac{1}{\sqrt{2}} - 1.$

8) $\int_2^3 \frac{dx}{(x+1)^2} = \int_2^3 -(x+1)^2 dx = (x+1)^{-1}\Big|_2^3 = \frac{1}{4} - \frac{1}{3} = \frac{-1}{12}.$

9) $\int_{-1}^2 |x|\, dx = \int_{-1}^0 |x|\, dx + \int_0^2 |x|\, dx$

$= \int_{-1}^0 (-x)\, dx + \int_0^2 x\, dx$

$= -\frac{x^2}{2}\Big|_{-1}^0 + \frac{x^2}{2}\Big|_0^2 = \frac{1}{2} + 2 = \frac{5}{2}.$

Searching keywords:

- The definite integral التكامل المحدود
- Fundamental theorem of calculus النظرية الأساسية للتكامل
- Evaluate the integral
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

د. بهاء محمود الزالق
The University of Jordan
Dr. Baha Alzalg
baha2math@gmail.com