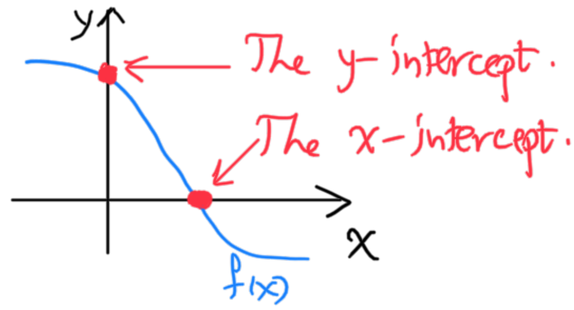
Curve sketching.

Guidelines:-

All information you need to sketch the curve $y = f(x)$:

① **Domain:** Determine $Dom(f)$.

② **Intercepts:**



- The y-intercept is $f(0)$.
- The x-intercept: Set $y=0$ and solve for x.

③ **Symmetry:**

- Symmetry about the y-axis:

If $f(-x) = f(x)$ $\forall x \in Dom(f)$, then f is an even func.

- Symmetry about the origin:

If $f(-x) = -f(x)$ $\forall x \in Dom(f)$, then f is an odd func.

④ **Periodicity:**

If $f(x+p) = f(x)$ $\forall x \in Dom(f)$ ($p > 0$ is constant), then f is called a <u>periodic func</u>.

The smallest such number p is called the <u>period</u>.

Examples: $y=\sin x$ has period 2π.
$y=\tan x$ has period π.

⑤ Asymptotes:

- Horizontal asymptotes:
  $y=L$ is an H.A. if $\lim_{x\to\pm\infty} f(x) = L$.
- Vertical asymptotes:
  $x=c$ is a V.A. if $\lim_{x\to c^\pm} f(x) = \pm\infty$.
- Slant asymptotes: To be disscussed later on.

⑥ What does f' say about the graph of f?

- Intervals of increase or decrease.
- Extreme values.

⑦ What does f'' say about the graph of f?

- Intervals of concavity.
- Points of inflection.

Use information in items 1–7 to sketch the curve.

Ex. Sketch the graph of $f(x)$.

1) $f(x) = x^5 - 5x$.

Soln. $\mathrm{Dom}(f) = \mathbb{R}$.

y-intercept: $f(0) = 0$.

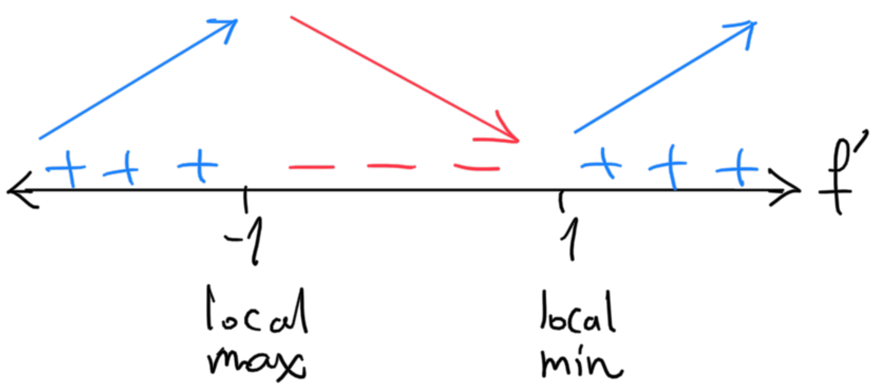
x-intercept: $f(x) = x^5 - 5x = x(x^4 - 5) = 0$.

So $x = 0$ and $x = \pm\sqrt[4]{5} \cong \pm 1.5$.

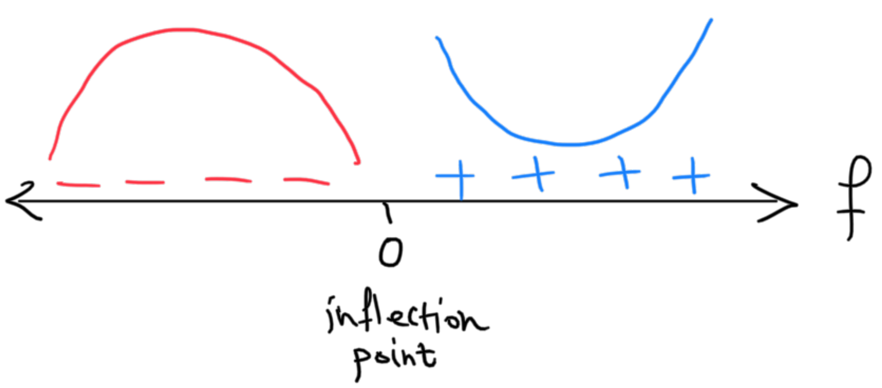
$f(x)$ is a poly., so f has no vertical asy. nor horizontal asy.

$f'(x) = 5x^4 - 5 = 5(x^4 - 1)$

$f'(x) = 0$ when $x = \pm 1$



$f''(x) = 20x^3$, $f''(x) = 0$ when $x = 0$



Information summary:

- **Points**:-

y-intercept: $(0,0)$.

x-intercepts: $(0,0)$, $(\sqrt[4]{5}, 0)$, $(-\sqrt[4]{5}, 0)$.

≈ 1.5 ≈ -1.5

local max: $(-1, 4)$.

local min: $(1, -4)$.

Inflection point: $(0,0)$.

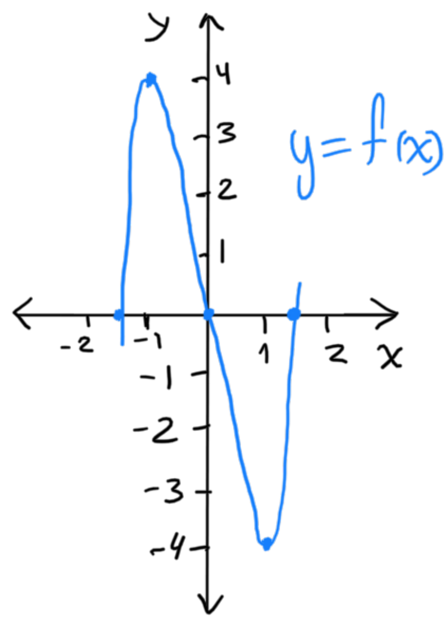
- **Intervals**:-

increasing: $(-\infty, -1) \cup (1, \infty)$.

decreasing: $(-1, 1)$.

concave down: $(-\infty, 0)$.

concave up: $(0, \infty)$.



(2) $f(x) = \dfrac{\sin x}{1 - \sin x}$, $x \in [-\pi, \pi]$.

Soln. $f(x)$ is undefined when $1 - \sin x = 0$, which is true when $x = \pi/2$.

$\text{Dom}(f) = [-\pi, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

y-intercept: $f(0)=0$.

x-intercepts: $f(x)=0$ when $\sin x=0$, which is true when $x=-\pi, 0, \pi$.

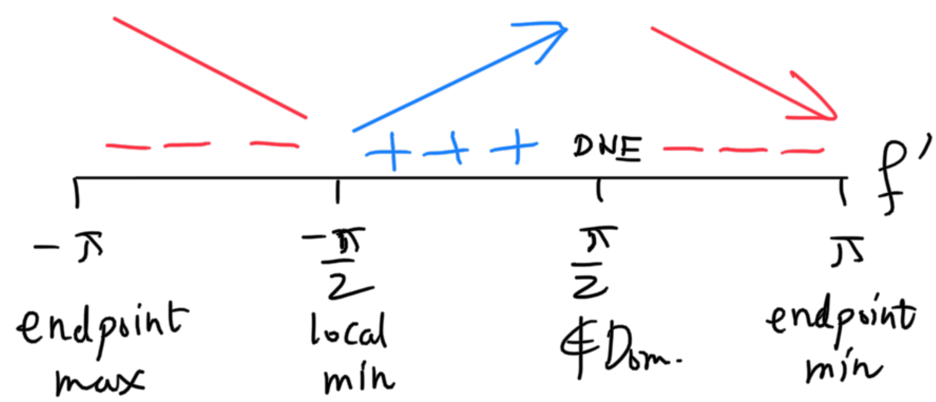
Vertical asym.: $\displaystyle\lim_{x\to\frac{\pi}{2}^+} \frac{\sin x}{1-\sin x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{1-\sin x} = \infty$

$\therefore f$ has a V.A. at $x=\pi/2$.

Note that f has no H.A.

$f'(x) = \dfrac{\cos x}{(1-\sin x)^2}$, $f'(x)=0$ when $\cos x = 0$.

So $x=\pm\pi/2$.

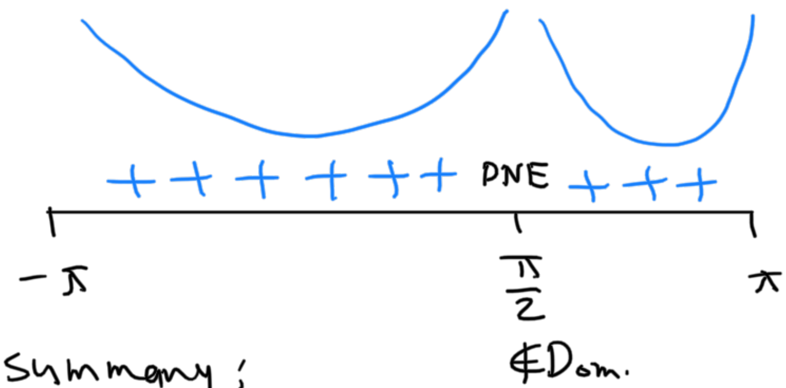


$f''(x) = \dfrac{\sin^2 x + \sin x - 2}{(\sin x - 1)^2}$,

$f''(x)=0$ when $\sin^2 x + \sin x - 2 = 0$

or $(\sin x + 2)(\sin x - 1) = 0$.

Then $\sin x = 2$ (impossible!) or $\sin x = 1$, so $x=\frac{\pi}{2}$.



Information summary:

• **Points:-**

y-intercept: $(0,0)$.

x-intercepts: $(-\pi,0)$, $(0,0)$, $(\pi,0)$.

endpt. max: $(-\pi,0)$.

endpt. min: $(\pi,0)$.

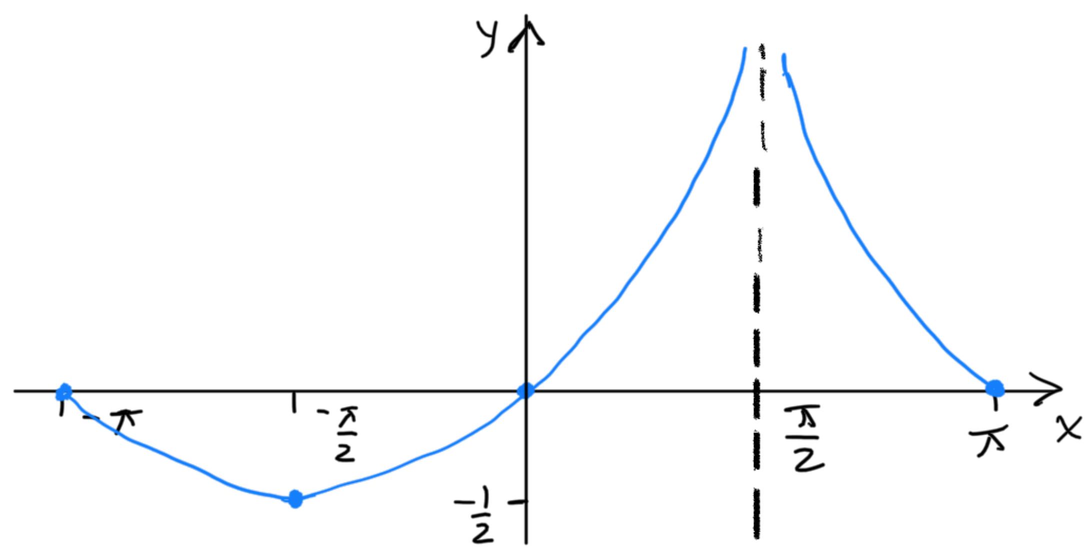
local min: $\left(-\frac{\pi}{2}, -\frac{1}{2}\right)$ ⟵ absolute min.

• **Intervals:-**

increasing: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

decreasing: $\left(-\pi, -\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$.

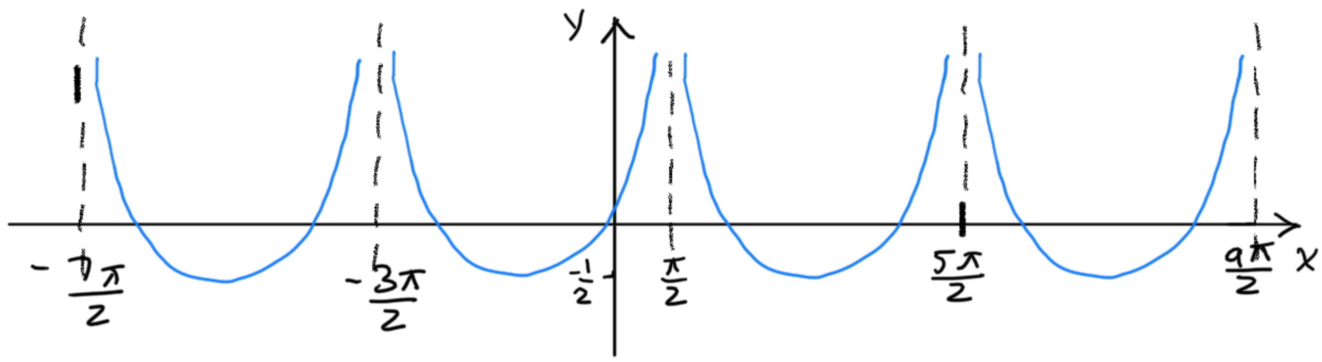
concave up: $\left(-\pi, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$.



(3) $f(x) = \dfrac{\sin x}{1-\sin x}$, $x \in \mathbb{R}$.

Soln. $Dom(f) = \{x \in \mathbb{R} : x \neq \frac{\pi}{2} \pm 2n\pi, n \in \mathbb{N}\}$

$f(x)$ is periodic with period 2π. Use Ex. (2).



(4) $f(x) = \dfrac{\cos x}{2+\sin x}$. Exc.

(5) $f(x) = \dfrac{x^2}{x^2-4}$.

Soln. $Dom(f) = \mathbb{R} - \{\pm 2\}$.

The x- and y-intercepts are both 0.

Asymptotes:

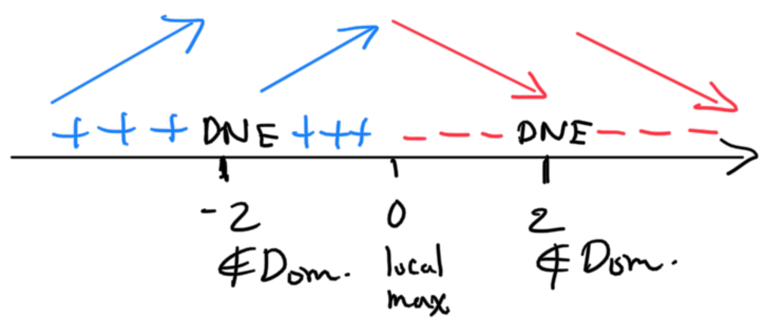
$$\lim_{x\to 2^+} \frac{x^2}{x^2-4} = \infty, \quad \lim_{x \to -2^-} \frac{x^2}{x^2-4} = \infty.$$

$\therefore f$ has V.A.s at $x = \pm 2$,

$$\lim_{x\to\infty} \frac{x^2}{x^2-4} = 1.$$

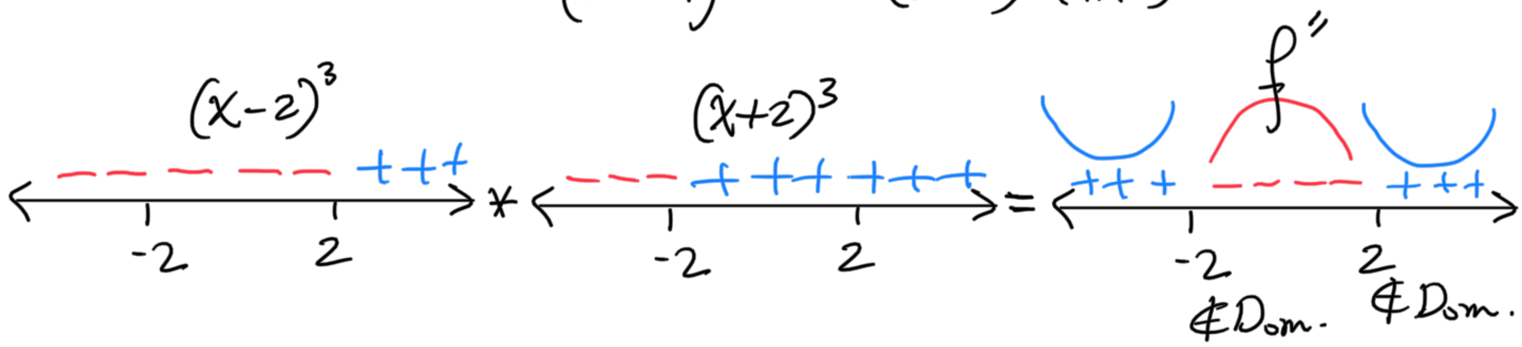
$\therefore$ f has an H.A. at $y = 1$.

$$f'(x) = \frac{2x(x^2-4) - x^2(2x)}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2},$$

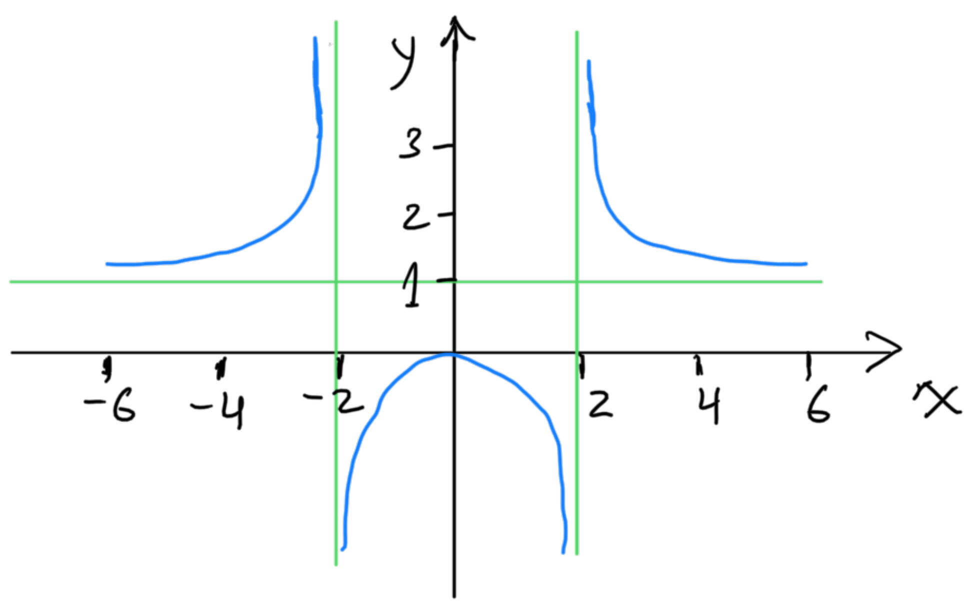


f has no local min.

$$f''(x) = \cdots = \frac{8(3x^2+4)}{(x^2-4)^3} = \frac{8(3x^2+4)}{(x-2)^3(x+2)^3}.$$



$2, -2 \notin \mathrm{Dom}(f)$, so f has no inflection points.



(6) $f(x) = \dfrac{2x^2}{x^2-1}$. Exc.

(7) $f(x) = xe^x$.

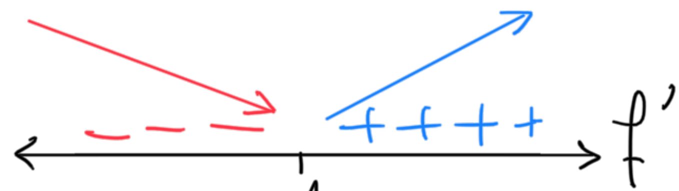
Soln. $\text{Dom}(f) = \mathbb{R}$.

The x- and y-intercepts are both 0.

$$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = 0.$$

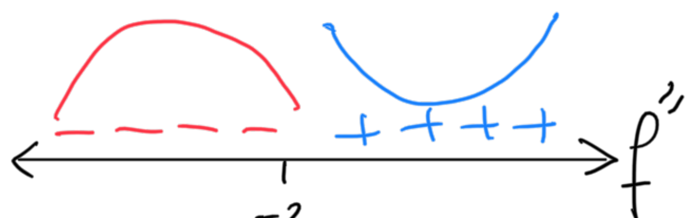
$\therefore$ The line $y=0$ (the x-axis) is an H.A.

$$f'(x) = xe^x + e^x$$
$$= (x+1)e^x.$$

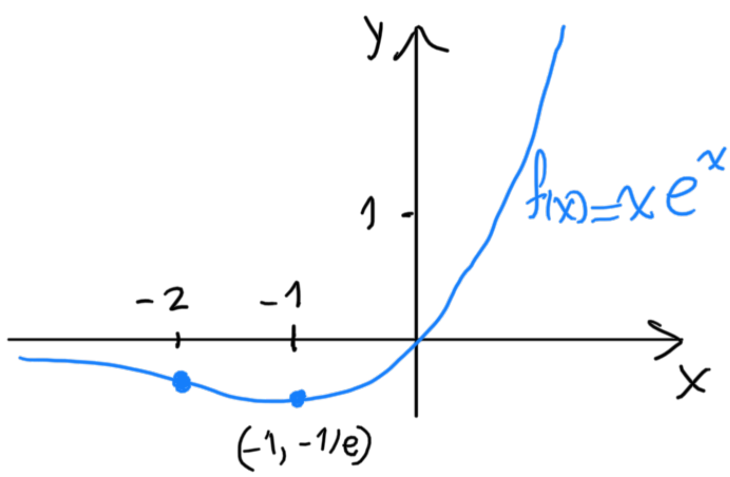


local (and absolute) min $(1, -1/e)$

$$f''(x) = (x+1)e^x + e^x$$
$$= (x+2)e^x$$



inflection pt.
$(-2, -2e^{-2}) = (-2, -0.27)$



(8) $f(x) = \ln(4-x^2)$. Exc.

Slant asymptotes.

Some curves have asymptotes that are slant (oblique), that is, neither horizontal or vertical.

Def. If $\lim_{x\to\infty}\left[f(x)-(mx+b)\right]=0$, where $m\neq 0$, then the line $y=mx+b$ is called a slant asymptote.

Ex. Since $f(x)=\dfrac{x^3}{x^2+1}=x+\dfrac{x}{x^2+1}$,

we have $\lim_{x\to\infty}\left(f(x)-x\right)=\lim_{x\to\infty}\dfrac{x}{x^2+1}=0$.

So, the line $y=x$ is a slant asym. of $f(x)$.

Ex. Since $f(x)=\dfrac{3x^2+2}{x+4}=(3x-12)+\dfrac{50}{x+4}$,

we have $\lim_{x\to\infty}\left[f(x)-(3x-12)\right]=0$.

Thus, $y=3x-12$ the slant asym. of $f(x)$.

Ex. Sketch the graph of $f(x) = \dfrac{x^3}{x^2+1}$.

Soln. $\text{Dom}(f) = \mathbb{R}$.

The x- and y-intercepts are both 0.

Since $f(-x) = -f(x)$, f is odd and its graph is symmetric about the origin.

$\lim\limits_{x \to \infty} f(x) = -\infty$; there is no H.A.

x^2+1 is never zero; there is no V.A.

From a previous example, $y = x$ is a slant asym.

$$f'(x) = \frac{x^2(x^2+3)}{(x^2+1)^2} > 0 \text{ for all } x.$$

No local max. or min.

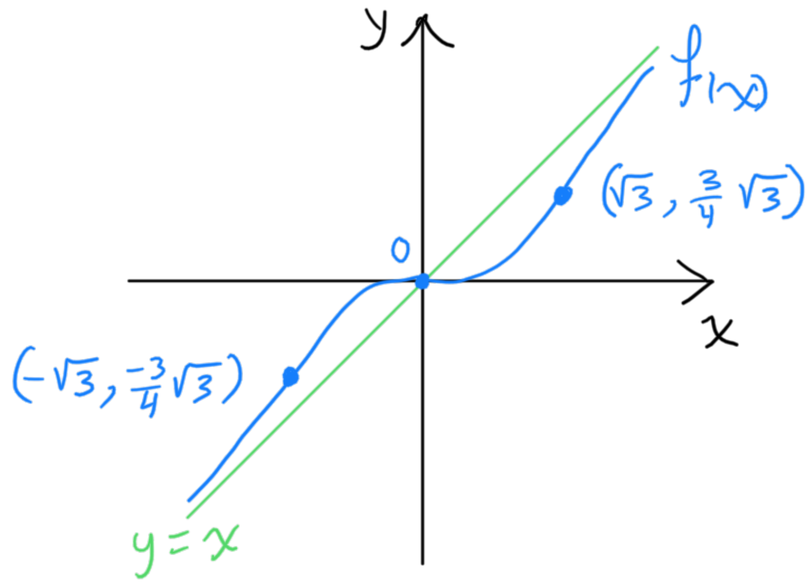
$+ \; + \; + \; +$ $\quad f'$

$$f''(x) = \frac{2x(3-x^2)}{(x^2+1)^3}$$

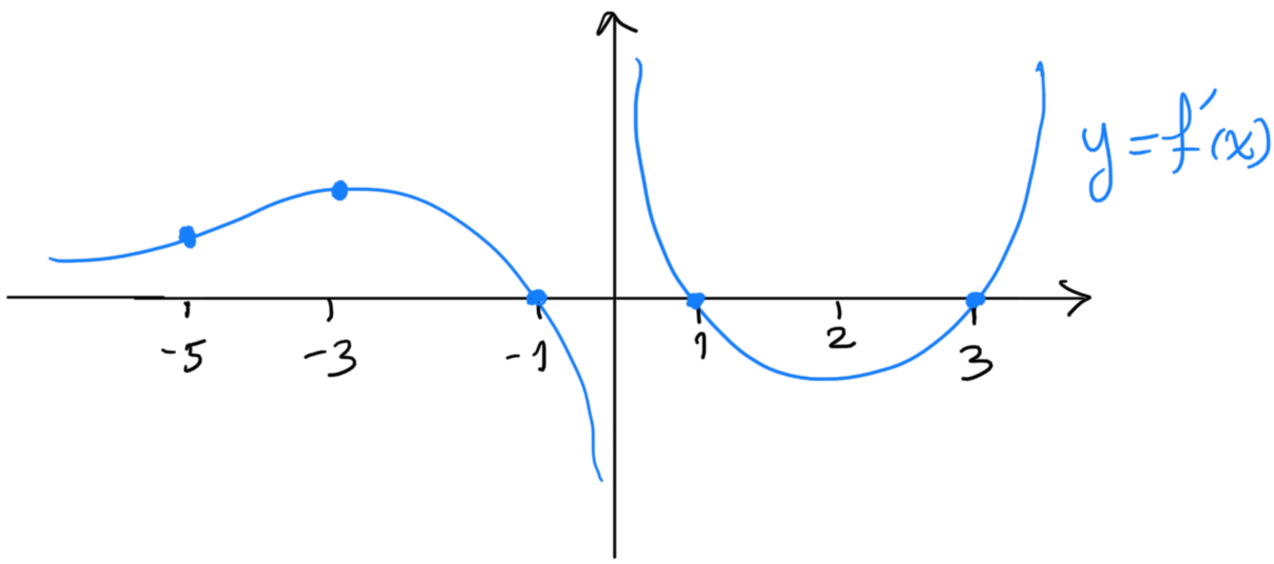
$+ + + \quad - - - \quad + + + \quad - - -$ $\quad f''$

$-\sqrt{3} \qquad 0 \qquad \sqrt{3}$

The points of inflection are $\left(-\sqrt{3}, \frac{-3}{4}\sqrt{3}\right)$, $(0,0)$ and $\left(\sqrt{3}, \frac{3}{4}\sqrt{3}\right)$.



Ex. A func. f is cts on $(-\infty, \infty)$, diff. for all $x \neq 0$. The graph of f' is shown below



(A) Determine the interval(s) on which f increases and on which f decreases, and find the critical numbers of f.

(B) Determine when the graph of f is concave up and where it is concave down.

Soln- (A) • f increases when $f' \geq 0$, which occurs when $x \in (-\infty, -1] \cup (0, 1] \cup [3, \infty)$.

• f decreases when $f' \leq 0$, which occurs when $x \in [-1, 0) \cup [1, 3]$.

• The critical numbers are $x = -1, 0, 1, 3$.

(B) • f is concave up when f' increases, which occurs when $x \in (-\infty, -3) \cup (2, \infty)$.

• f is concave down when f' decreases, which occurs when $x \in (-3, 0) \cup (0, 2)$.

Searching keywords:

- Sketch the curve, curve sketching, رسم المنحنيات
- Slant asymptote
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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