Increasing/decreasing functions and extrema.

Increasing and decreasing functions.

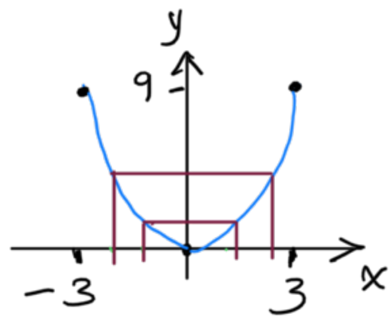
Def. A func. f is called:

(1) increasing on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ on } I.$$

(2) decreasing on I if

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ on } I.$$



Ex. Consider $f(x) = x^2$ on $[-3, 3]$.
Note that the graph of f falls from -3 to zero and rises from zero to 3.
Then the func. f is decreasing on the interval $[-3, 0]$ and is increasing on $[0, 3]$.

By Def. $0 < a < b < 3 \Longrightarrow a^2 < b^2 \Longrightarrow$ incr. on $[0,3]$.

$-3 < a < b < 0 \Longrightarrow a^2 > b^2 \Longrightarrow$ decr. on $[-3,0]$.

Increasing/decreasing test:

(A) If $f'(x) > 0$ on an interval I, then f is increasing on that interval.

(B) If $f'(x) < 0$ on an interval I, then f is decreasing on that interval.

Ex. Find the interval(s) on which f increases and the interval(s) on which f decreases.

(1) $f(x) = x^2$.

Soln. $Dom(f) = \mathbb{R}$, $f'(x) = 2x \begin{cases} +ve & \text{when } x > 0, \\ -ve & \text{when } x < 0. \end{cases}$

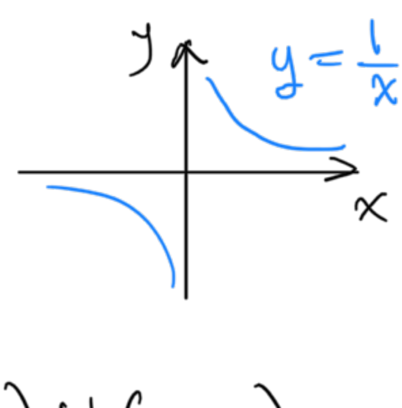
f is $\nearrow$ on $[0, \infty)$ and f is $\searrow$ on $(-\infty, 0]$.

f is increasing. f is decreasing.

(2) $f(x) = \frac{1}{x}$.

Soln. $Dom(f) = \mathbb{R} - \{0\}$.

$f'(x) = \frac{-1}{x^2}$, $x \neq 0$.



Then $f'(x) < 0$, $\forall x \neq 0$.

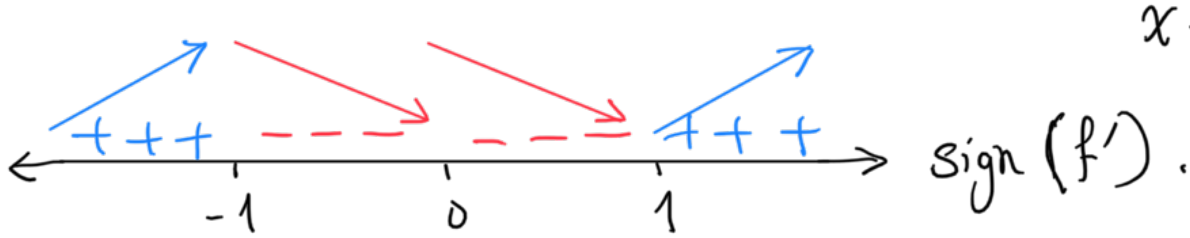
$\therefore f$ is decreasing on $(-\infty, 0) \cup (0, \infty)$.

(3) $f(x) = x + \frac{1}{x}$.

Soln. $Dom(f) = \mathbb{R} - \{0\}$.

$f'(x) = 1 - \frac{1}{x^2} = 0$, then $1 = \frac{1}{x^2}$, so $x^2 = 1$.

$x = \pm 1$.



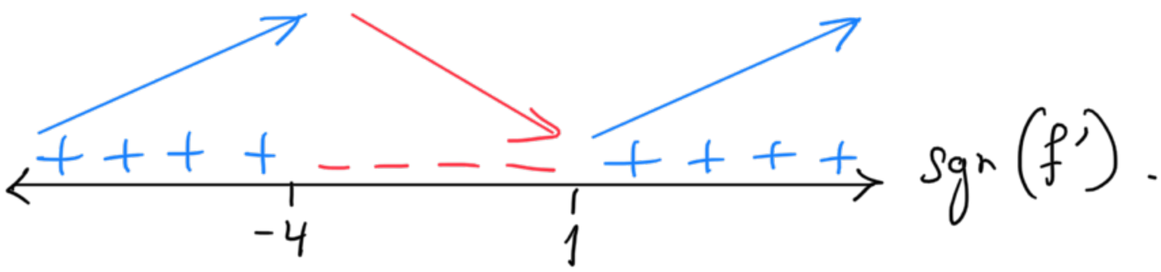
$\therefore$ f is increasing on $(-\infty, -1] \cup [1, \infty)$

f is decreasing on $[-1, 0) \cup (0, 1]$.

(4) $f(x) = 2x^3 + 9x^2 - 24x - 10$.

Soln. $Dom(f) = \mathbb{R}$.

$f'(x) = 6x^2 + 18x - 24 = 6(x^2 + 3x - 4) = 6(x-1)(x+4)$



$\therefore$ f is $\nearrow$ on $(-\infty, -4] \cup [1, \infty)$.

and f is $\searrow$ on $[-4, 1]$.

(5) $f(x) = \cos x - x$.

$-1 \le \sin x \le 1$,

$1 \ge -\sin x \ge -1$,

$0 \ge -\sin x - 1 \ge -2$.

<u>Soln.</u> $\mathrm{Dom}(f) = \mathbb{R}$.

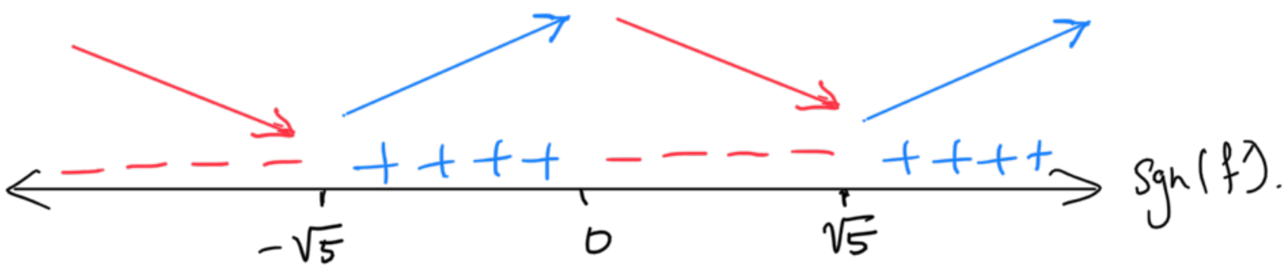
$f'(x) = -\sin x - 1 \le 0$ for all x.

$\therefore$ f is $\searrow$ on $\mathbb{R}$.

(6) $f(x) = |x^2 - 5|$.

<u>Soln.</u> $f(x) = \begin{cases} x^2 - 5 & \text{if } x \le -\sqrt{5} \text{ or } x \ge \sqrt{5}, \\ 5 - x^2 & \text{if } -\sqrt{5} < x < \sqrt{5}. \end{cases}$

Then $f'(x) = \begin{cases} 2x & \text{if } x < -\sqrt{5} \text{ or } x > \sqrt{5}, \\ -2x & \text{if } -\sqrt{5} < x < \sqrt{5}, \\ \text{DNE} & \text{if } x = \pm\sqrt{5}. \end{cases}$

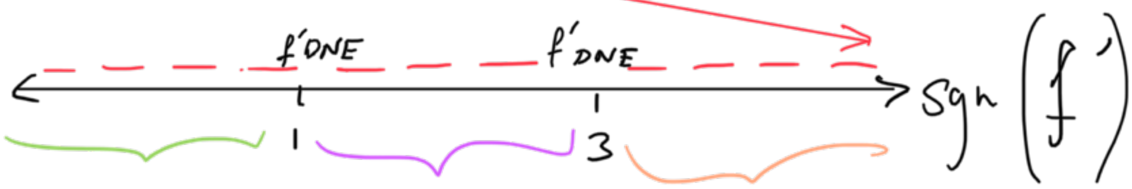


$\therefore$ f is $\nearrow$ on $[-\sqrt{5}, 0] \cup [\sqrt{5}, \infty)$,

and f is $\searrow$ on $(-\infty, -\sqrt{5}] \cup [0, \sqrt{5}]$.

(7) $f(x) = \begin{cases} (x-1)^2 & \text{if } x < 1, \\ 5 - x & \text{if } 1 \le x < 3, \\ 7 - 2x & \text{if } 3 \le x. \end{cases}$

Soln. $f'(x) = \begin{cases} 2(x-1) & \text{if } x<1, \\ -1 & \text{if } 1<x<3, \\ -2 & \text{if } 3<x. \\ \text{DNE} & \text{if } x=1,3. \end{cases}$



$\therefore$ f is decr. on $\mathbb{R}$.

Critical numbers

Def. Given a func. f. A number $c \in \text{Dom}(f)$ is called a critical number of f if

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ is undefined.}$$

Ex. Find all the critical numbers of $f(x) = \frac{2x^2}{x+2}$.

Soln. $\text{Dom}(f) = \mathbb{R} - \{2\}$.

$$f'(x) = \frac{(x+2)(4x) - (2x^2)(1)}{(x+2)^2} = \frac{2x(x+4)}{(x+2)^2}.$$

Now, $f'(x) = 0$ for $x = 0, -4 \in Dom(f)$, and $f'(x)$ is undefined for $x = -2 \notin Dom(f)$.
Thus, the only critical numbers of f are

$$x = 0 \text{ and } x = -4.$$

Local extreme values.

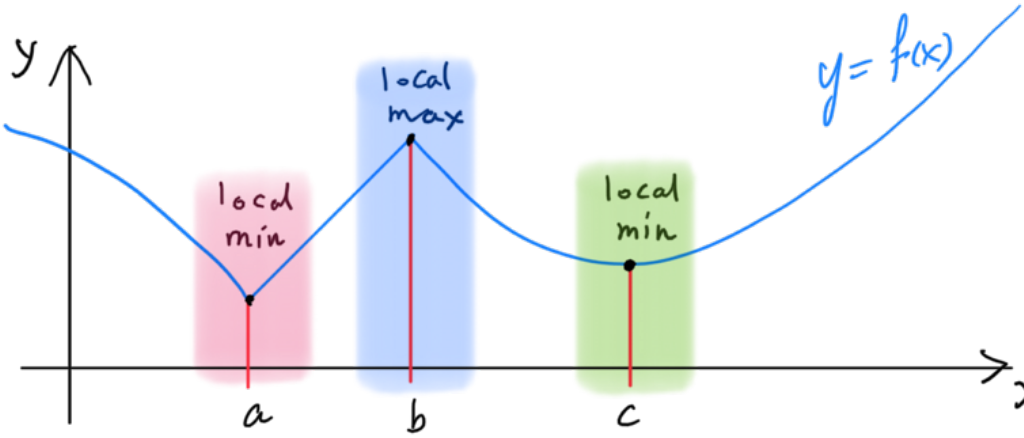
Def. Let f be a func. and $c \in Dom(f)$. The number $f(c)$ is called a:

(1) local maximum value of f if $f(c) \geq f(x)$ when x is near c.

(2) local minimum value of f if $f(c) \leq f(x)$ when x is near c.

(3) local extremum of f if it is local max or local min.

When we say that something is true near c, we mean that it is true on some open interval containing c.

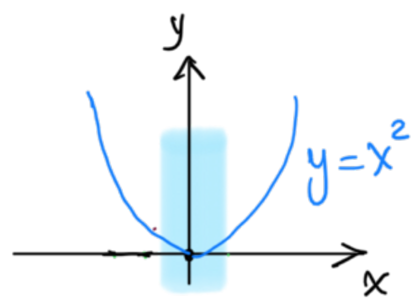


- $f(a)$ is a local min. As $f'(a)$ is undefined, a is a critical number.
- $f(b)$ is a local max. As $f'(b)$ is undefined, b is a critical number.
- $f(c)$ is a local min. As $f'(c) = 0$, c is a critical number.

<u>Fermat's theorem</u>.

If $f(c)$ is a local extremum (max or min), then c must be a critical number.

<u>Ex</u>. Find the critical number(s) and local extrema of $f(x) = x^2$.



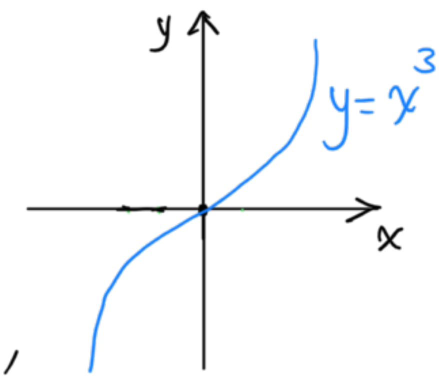
<u>Soln</u>. $f'(x) = 2x$ exists everywhere.

$f'(x) = 0$ when $x = 0$.

So, $x=0$ is the only critical number.
$f(0)=0$ is a local min.

Remark: The converse of Fermat's thm is not true in general.

Ex. Take $f(x)=x^3$.



$f'(x)=3x^2$ exists everywhere.
$f'(x)=0$ when $x=0$.
So, $x=0$ is a critical number,
but $f(0)=0$ is not a local extremum.

The first derivative test (FDT).

Assume that c is a critical number of a cts func. f.

(A) If sign(f') changes from positive to negative at c, then f has a local maximum at c.

(B) If sign(f') changes from negative to positive at c, then f has a local minimum at c.

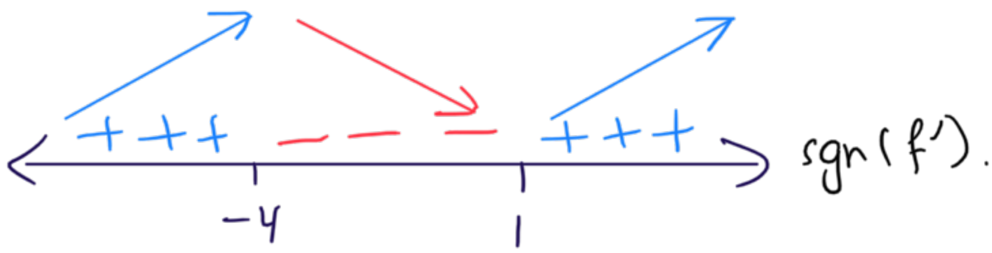
(C) If f' does not change its sign at c, then f has no local extremum at c.

Ex. Find the local extrema of the following funcs.

(1) $f(x) = 2x^3 + 9x^2 - 24x + 10$.

Soln. $f'(x) = 6x^2 + 18x - 24 = 6(x-1)(x+4)$.

$f'(x) = 0$ when $x = -4, 1$.



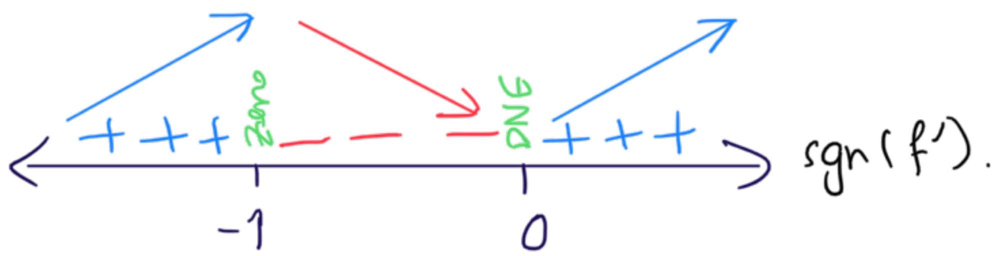
By FDT, f has a local max at $x = -4$,

and a local min at $x = 1$.

(2) $f(x) = 2x^{5/3} + 5x^{2/3}$.

Soln. $f'(x) = \frac{10}{3} x^{2/3} + \frac{10}{3} x^{-1/3} = \frac{10}{3} x^{-1/3}(x+1)$.

Then $f'(-1) = 0$ and $f'(0)$ DNE.



By FDT, $f(-1) = 3$ is a local max,

and $f(0) = 0$ is a local min.

(3) $f(x) = \sin^2 x, \quad 0 < x < 2\pi.$

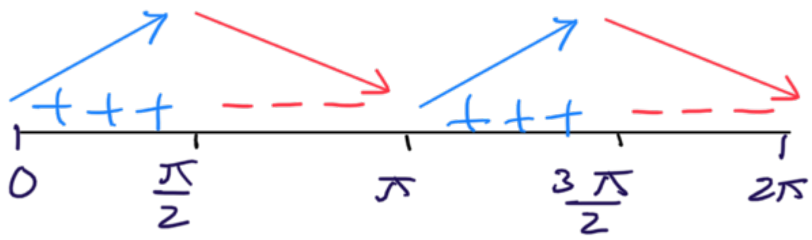
Soln. $f'(x) = 2\sin x \cos x = \sin 2x.$

$f'(x) = 0$ when $2x = 0, \pi, 2\pi, 3\pi, 4\pi.$

Then $x = 0, \underbrace{\frac{\pi}{2}, \pi, \frac{3\pi}{2}}_{\in (0, 2\pi)}, 2\pi.$

$0 \notin (0, 2\pi)$ $\quad$ $2\pi \notin (0, 2\pi)$

So, the critical numbers are $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ only.



sign (f').

By FDT, $f(\frac{\pi}{2}) = f(\frac{3\pi}{2}) = 1$ is a local max,

and $f(\pi) = 0$ is a local min.

(4) $f(x) = x + 2\sin x, \quad 0 \le x \le 2\pi.$ Exc.

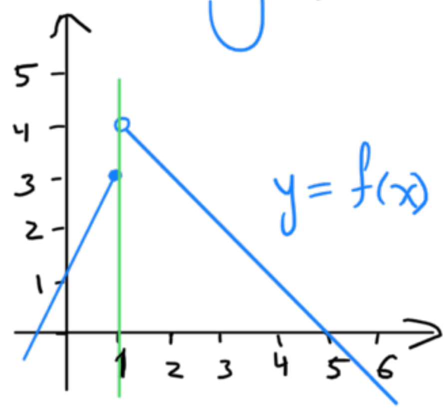
FINAL ANS. $f(2\pi/3)$ is a local max. value

$f(4\pi/3)$ is a local min. value.

Remark: The requirement that f be cts at c in the hypothesis of the FDT is necessary.

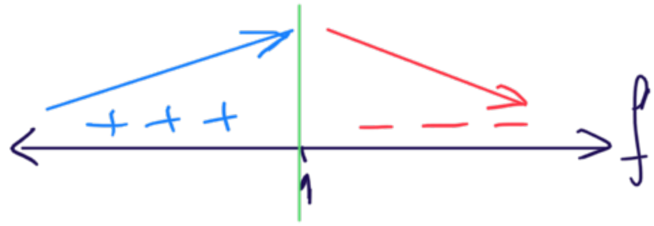
Ex. Find the local extrema of the following func.

$$f(x) = \begin{cases} 1+2x, & x \le 1, \\ 5-x, & x > 1. \end{cases}$$



Note that f is not cts at $x = 1$.

Soln. $f'(x) = \begin{cases} 2 & , x < 1 \\ -1 & , x > 1 \\ \text{DNE} & , x = 1 \end{cases}$



$f(1)$ is defined and $x = 1$ is a critical number, but f has no local extremum at $x = 1$.

The second derivative test (SDT).

Assume that $f'(c) = 0$ and that $f''(c)$ exists.

(A) If $f''(c) > 0$, then f has a local min. at c.

(B) If $f''(c) < 0$, then f has a local max. at c.

Ex. Use the SDT to find the local extrema of the following funcs.

(1) $f(x) = x^4 - 8x^2 + 10$.

Soln. $f'(x) = 4x^3 - 16x = 4x(x^2-4)$.

$f'(x) = 0$ when $x = -2, 0, 2$.

The critical numbers are $-2, 0$ and 2.

$f''(x) = 12x^2 - 16$.

$f''(0) = -16 < 0 \xrightarrow{\text{SDT}} f(0) = 10$ is a local max.

$f''(\pm 2) = 32 > 0 \xrightarrow{\text{SDT}} f(\pm 2) = -6$ is a local min.

(2) $f(x) = 2x^3 - 3x^2 - 12x + 5$. Exc.

Note: The SDT is inconclusive when $f''(c)=0$ or $f''(c)$ is undefined.

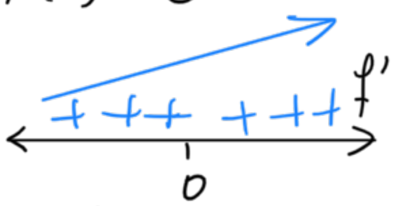
Examples:

- $f(x)=x^3$

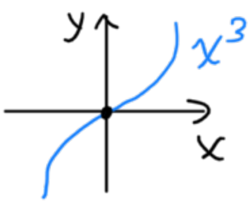
  $f'(x)=3x^2$

  $f''(x)=6x$

  $f''(0)=0$



$\therefore f(0)=0$ is not a local extremum.

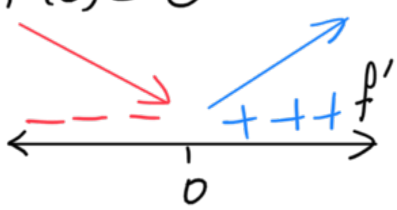


- $f(x)=x^4$

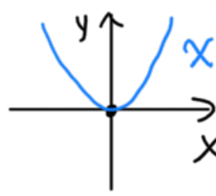
  $f'(x)=4x^3$

  $f''(x)=12x^2$

  $f''(0)=0$



$\therefore f(0)=0$ is a local min

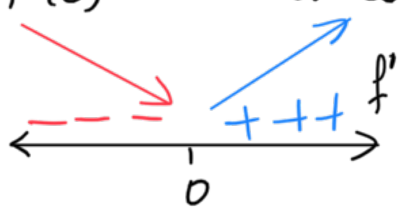


- $f(x)=x^{4/3}$

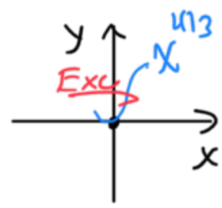
  $f'(x)=\frac{4}{3}x^{1/3}$

  $f''(x)=\frac{4}{9}x^{-2/3}$

  $f''(0)$ is undefined



$\therefore f(0)=0$ is a local min.



Conclusion: The FDT is more general than the SDT, because, if $f''(c)=0$ or $f''(c)$ is undefined, we cannot use the SDT, but we must use the FDT.

Absolute extreme values.

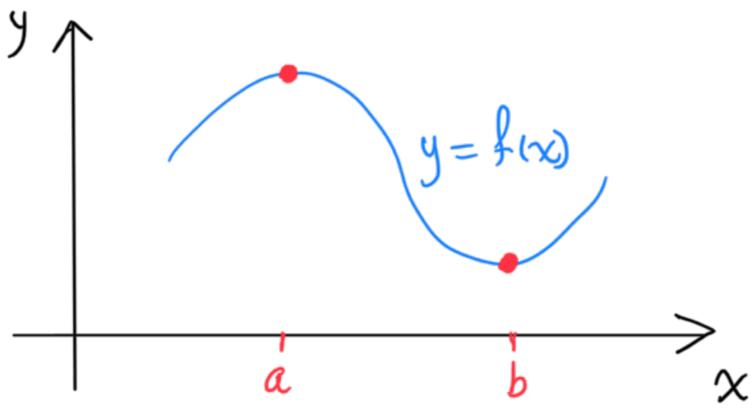
Def. Let $c \in \mathrm{Dom}(f)$. Then $f(c)$ is the

(1) <u>absolute maximum</u> value of f if

$$f(c) \geq f(x) \text{ for all } x \in \mathrm{Dom}(f).$$

(2) <u>absolute minimum</u> value of f if

$$f(c) \leq f(x) \text{ for all } x \in \mathrm{Dom}(f).$$



- $f(a)$ is the absolute max.
- $f(b)$ is the absolute min.

The closed interval method:

To find the absolute max. and min. values of a cts func. f on a closed interval $[a,b]$;

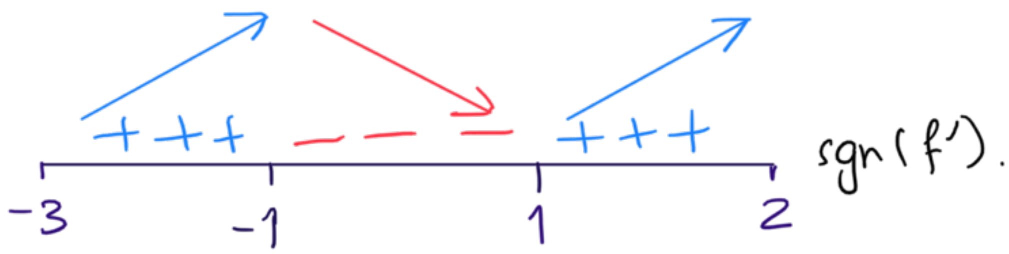
1. Find the values of f at the critical numbers of f in (a,b).
2. Find $f(a)$ and $f(b)$; the values of f at the endpoints.
3. The largest of the values from steps 1 and 2 is the absolute max value; the smallest of these values is the absolute min. value.

Ex. Find and classify all the extreme values.

(1) $f(x) = x^3 - 3x + 1$, $x \in [-3, 2]$.

Soln. $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$.

$f'(x) = 0$ when $x = \pm 1$.



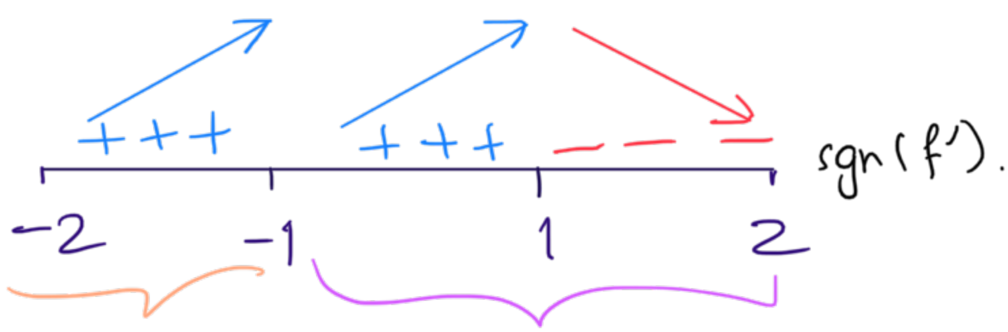
- $f(-3) = -17$ is an endpoint min.
- $f(-1) = 3$ is a local max.
- $f(1) = -1$ is a local min.
- $f(2) = 3$ is an endpoint max.

Thus, $f(-1) = f(2) = 3$ is the absolute max.

and $f(-3) = -17$ is the absolute min.

(2) $f(x) = \begin{cases} x^2 + 1 & , -2 \le x < -1, \\ -x^2 + 2x + 5 & , -1 \le x \le 3. \end{cases}$

Soln. $f'(x) = \begin{cases} 2x & , -2 < x < -1, \\ -2x + 2 & , -1 < x < 3, \\ \text{DNE} & , x = -1. \end{cases}$



- $f(-2)=5$ is an endpoint max.
- $f(-1)=2$ is a local min.
- $f(1)=6$ is a local max.
- $f(3)=2$ is an endpoint min.

Thus, $f(-1)=f(3)=2$ is the absolute min., and $f(1)=6$ is the absolute max.

(3) $f(x)=2\cos^3 x+3\cos x,\ x\in[0,2\pi]$. Exc.

<u>Remark</u>: Let a be a real number, and $Dom(f)$ be $(-\infty,a)$, (a,∞) or $(-\infty,\infty)$, then

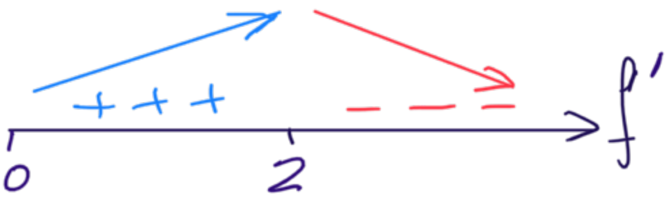
(1) If $\lim\limits_{x\to\pm\infty} f(x)=\infty$, then f cannot have an absolute max. value.

(2) If $\lim\limits_{x\to\pm\infty} f(x)=-\infty$, then f cannot have an absolute min. value.

<u>Ex</u>. Classify all the extreme values of $f(x)=6\sqrt{x}-x\sqrt{x}$.

Soln. $\text{Dom}(f) = [0, \infty)$.

$$f'(x) = 3x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{3(2-x)}{2\sqrt{x}}.$$



- $f(0) = 0$ is an endpoint min.
- $f(2) = 4\sqrt{2}$ is a local max. value and the absolute max. value.

Note that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sqrt{x}(6-x) = -\infty$.

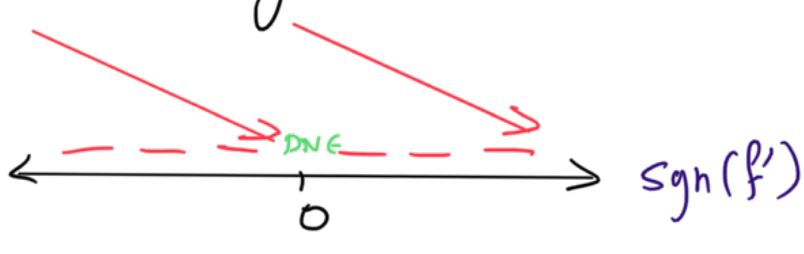
Thus, f has no absolute min. value.

Remark: If $\lim_{x\to\pm\infty} f(x)$ is finite, this does not mean that f has an absolute (max. or min.) value.

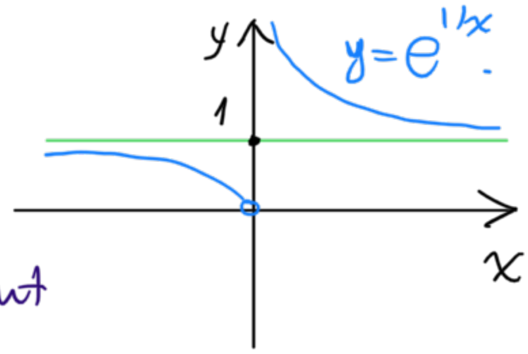
Ex. Consider $f(x) = e^{1/x}$. $\text{Dom}(f) = \mathbb{R} - \{0\}$.

$f'(x) = -\frac{1}{x^2}e^{1/x} < 0$ for all $x \neq 0$ as $x^2 > 0$ and $e^{1/x} > 0$ for all $x \neq 0$.

Thus, f is decreasing on $(-\infty, 0)$ and $(0, \infty)$.



So, there is no critical number, so the func. has no local max. or min. Thus, f has no absolute max. or min.



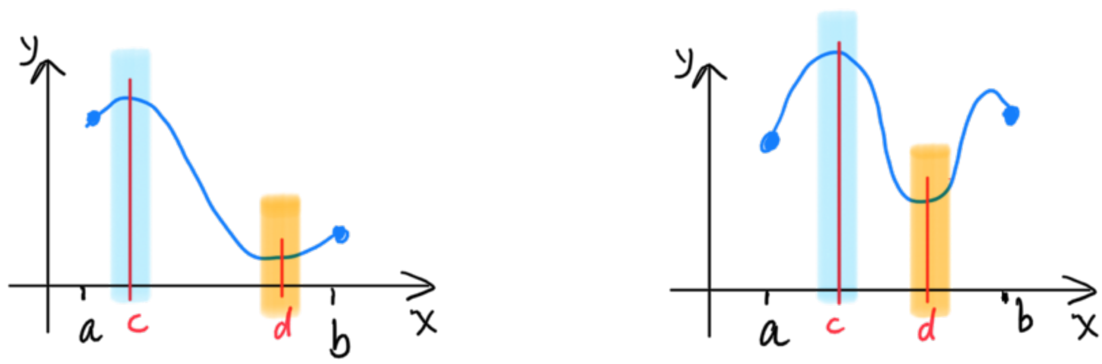
Note that

$$\lim_{x\to\pm\infty} e^{1/x} = e^0 = 1 \text{ , but}$$

$$\lim_{x\to 0^+} e^{1/x} = \lim_{t\to\infty} e^t = \infty.$$

The extreme value theorem.

If f is cts on a closed interval [a,b], then f attains an absolute max. value f(c) and an absolute min. value f(d) at some c and d in [a,b].



Searching keywords:

- Find where the function is increasing and where it is decreasing التزايد، التناقص
- Find the critical numbers of f القيم الحرجة
- Find the local maximum and minimum values القيم القصوى المحلية، القيم الصغرى، القيم العظمى
- Absolute maximum and minimum values, extremum values القيم القصوى المطلقة
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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