Differentials and linear approximations.

Differentials.

Def. Let f be a diff. func. and $h \neq 0$, then

(1) The increment of f from x to $x+h$ is defined as $\boxed{\Delta f = f(x+h) - f(x)}$.

(2) The differential of f at x with increment h is defined as $\boxed{df = f'(x)\, h}$.

Fact: $\Delta f \simeq df$.

Ex. Use a differential to estimate the change in $f(x) = x^{2/5}$ if:

(A) x is increased from 32 to 34.

(B) x is decreased from 1 to 0.9.

Soln. $f(x) = x^{2/5}$, then $f'(x) = \dfrac{2}{5x^{3/5}}$,

and hence $df = f'(x)\, h = \dfrac{2h}{5x^{3/5}}$.

(A) $h = 34 - 32 = 2$ and $x = 32$. Then

$$df = \frac{2(2)}{5(32)^{3/5}} = \frac{4}{5(8)} = \frac{4}{40} = 0.1.$$

Thus, a change in x from 32 to 34 increases the value of f by approximately 0.1.

Note that $df = 0.1$

$$\cong \Delta f = f(34) - f(32) \cong 0.0982.$$

(B) $h = 0.9 - 1 = -0.1$ and $x = 1$. Then

$$df = \frac{2(-0.1)}{5\,(1)^{3/5}} = \frac{-.2}{5} = -0.04.$$

Thus, a change of x from 1 to 0.9 decreases the value of f by approximately 0.04.

Estimation.

Note that

$$f(x+h) = f(x) + \Delta f \cong f(x) + \Delta f = f(x) + h f'(x).$$

We have the Fact: $\boxed{f(x+h) \cong f(x) + h f'(x)}$.

Ex. Use a differential to estimate:

(A) $\sqrt{104}$. (B) $\cos 40^\circ$.

Soln. (A) Let $x = 100$ and $h = 4$.

Since $\sqrt{104} = \sqrt{100+4}$, we have

$$\sqrt{104} = \sqrt{x+h} \cong \sqrt{x} + h(\sqrt{x})' = \sqrt{x} + \frac{h}{2\sqrt{x}}.$$

Then $\sqrt{104} \cong \sqrt{100} + \frac{\cancel{4}^{\,2}}{\cancel{2}\sqrt{100}} = 10 + \frac{2}{10} = 10.2$.

Thus $\sqrt{104} \cong 10.2$.

(B) Let $x = 45 = \frac{\pi}{4}$ in Radian,

and $h = -5 = -5\left(\frac{\pi}{180}\right) = -\frac{\pi}{36}$ in Rad.

Then
$$\begin{aligned}
\cos(40) &= \cos(45+(-5)) \\
&= \cos(x+h) \\
&\cong \cos x + h(\cos x)' \\
&= \cos x - h \sin x \\
&= \cos\frac{\pi}{4} - \left(-\frac{\pi}{36}\right)\sin\frac{\pi}{4} \\
&= \frac{1}{\sqrt{2}} + \frac{\pi}{36}\frac{1}{\sqrt{2}} \\
&\cong 0.7688.
\end{aligned}$$

Linear approximation.

Recall that $f(t+h) \cong f(t) + h f'(t)$.

Letting $t = a$ and $h = x - a$, we get

$$f(x) \cong f(a) + (x-a) f'(a).$$

Fact: The linear func. $\boxed{L(x) = f(a) + (x-a) f'(a)}$

is called the linearization of f at a.

Ex. Find the linearization of $f(x) = \sqrt{x+3}$ at $a = 1$.

Solu. $f(1) = 2$, $f'(x) = \dfrac{1}{2\sqrt{x+3}}$, $f'(1) = \dfrac{1}{4}$.

The linearization is

$$L(x) = f(1) + f'(1)(x-1)$$

$$= 2 + \frac{1}{4}(x-1)$$

$$\therefore\ L(x) = \frac{7}{4} + \frac{x}{4}.$$

For instance,

$$\sqrt{3.98} = f(3+0.98) \cong \frac{7}{4} + \frac{0.98}{4} = 1.995.$$

Searching keywords:

- Differential, estimation, linearization, linear approximation
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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