Implicit differentiation

Consider the equation $y = x^2$, then $y' = 2x$.

Consider the eq. $y = (x^2+1)^3$, then $y' = 3(x^2+1)^2 (2x)$.

But how about the eq. $x^2+y^2=1$ or eq. $\cos(x+y)=1$?
How can we find y' in this case?

<u>Def.</u> Given the equation $F(x,y)=0$. The process of differentiating both sides of this equation with respect to x and then solving for $y'(x)$ is called <u>implicit differentiation</u>.

<u>Remark:</u>

$\frac{d}{dx}(x) = 1$.	$\frac{d}{dx}(y) = y'$.
$\frac{d}{dx}(x^2) = 2x$.	$\frac{d}{dx}(y^2) = 2yy'$.
$\frac{d}{dx}(x^3) = 3x^2$.	$\frac{d}{dx}(y^3) = 3y^2y'$.

$$\frac{d}{dx}(x^2y^2) = x^2\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x^2) = x^2(2yy') + y^2(2x).$$

$$\frac{d}{dx}\cos(x+y) = -\sin(x+y)\frac{d}{dx}(x+y) = -(\sin(x+y))(1+y').$$

Ex. Use the implicit differentiation to find dy/dx for

(1) $x^2+y^2=4$.

Soln. $\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(4)$, then $2x+2yy'=0$,

and hence $\cancel{2}yy'=-\cancel{2}x$.

Thus $y'=-x/y$.

(2) $x^4-4x^3y+y^4=1$.

Soln. $\frac{d}{dx}(x^4-4x^3y+y^4)=\frac{d}{dx}(1)$,

then $4x^3-\frac{d}{dx}(4x^3y)+4y^3y'=0$.

Now, $\frac{d}{dx}(4x^3y)=4x^3y'+y(12x^3)$.

Then $4x^3-4x^3y'-12x^2y+4y^3y'=0$.

Thus, $y'=\dfrac{12x^2y-4x^3}{4y^3-4x^3}=\dfrac{3x^2y-x^3}{y^3-x^3}$.

For instance, the slope at the point $(1,0)$ is

$$dy/dx\Big|_{\substack{x=1\\y=0}}=\frac{0-1}{0-1}=1.$$

(2) $\sin(x+y) = xy$.

Soln. $\frac{d}{dx}\sin(x+y) = \frac{d}{dx}(xy)$,

then $\cos(x+y)\,[1+y'] = xy' + y$.

This implies that $\cos(x+y)\,y' - xy' = y - \cos(x+y)$.

Thus, $y' = \dfrac{y-\cos(x+y)}{\cos(x+y)-x}$.

(4) $\sin(x-y) = (2x+1)^3 y$. Exc.

Ex. Find the equation of the tangent line at the indicated point.

(1) $x^2 + xy + 2y^2 = 28$ at $(-2,-3)$.

Soln. The eq. of the tangent line at $(-2,-3)$ is $y - y(-2) = y'(-2)\,(x-(-2))$.

Now $y(-2) = -3$, then $y-(-3) = y'(-2)\,(x+2)$.

Want: $y'(-2)$.

Diff. both sides of the original eq. w.r.t. x to get $2x + xy' + y + 4yy' = 0$.

Then $y' = -\dfrac{2x+y}{x+4y}$.

Thus $y'(-2) = -\dfrac{2(-2)+(-3)}{(-2)+4(-3)} = \dfrac{-7}{14} = -\dfrac{1}{2}$.

Therefore the eq. of the tangent line is

$$y+3 = -\frac{1}{2}(x+2).$$

(2) $x^2y^2 - 2x = 4 - 4y$ at $(2,-2)$.

FINAL ANS. $y+2 = \frac{7}{6}(x-2)$.

(3) $\tan(xy) = x$ at $(1, \pi/4)$.

Exc.

Ex. Use the implicit differentiation to express d^2y/dx^2 in terms of x and y, for

(1) $y^3 - x^2 = 4$.

Soln. $\frac{d}{dx}(y^3 - x^2) = \frac{d}{dx}(4)$, then $3y^2y' - 2x = 0$ —(*)

This implies that $y' = \dfrac{2x}{3y^2}$.

To find y'', we diff. both sides of Eq. (*) w.r.t. x to obtain $\frac{d}{dx}\left(3y^2y'-2x\right)=\frac{d}{dx}(0)$.

Then $\frac{d}{dx}(3y^2y')-2=0$.

Now $\frac{d}{dx}(3y^2y')=3y^2y''+y'\,6y\,y'=3y^2y''+6y(y')^2$.

Thus, $3y^2y''+6y(y')^2-2=0$.

It follows that $y''=\dfrac{2-6y(y')^2}{3y^2}$. But $y'=\dfrac{2x}{3y^2}$.

Then $y''=\dfrac{2-6y(2x/3y^2)^2}{3y^2}=\dfrac{\left(2-6y\left(\frac{4x^2}{9y^4}\right)\right)}{(3y^2)}*\dfrac{3y^2}{3y^2}$

$$=\frac{6y^3-8x^2}{9y^5}.$$

Or, as $y'=2x/3y^2$, we have $y''=\dfrac{3y^2(2)-2x(6yy')}{(3y^2)^2}$.

Then $y''=\dfrac{6y^2-12xyy'}{9y^4}=\dfrac{6y^2-\cancel{12}^{\,4}xy\left(\frac{2x}{\cancel{3}y^2}\right)}{9y^4}$

$$=\frac{6y^2-\left(\frac{8x^2}{y}\right)}{9y^4}=\frac{6y^3-8x^2}{9y^5}.$$

(2) $x^2y^2 - 2x = 4 - 4y$. Exc.

Ex. Find the point(s) on the curve $x^2+2y^2=1$ where the tangent line has slope 1.

Soln. $x^2+2y^2=1$, then $2x+4y\frac{dy}{dx}=0$.

Thus, $\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{4}_{2}y} = \frac{-x}{2y}$.

We are looking for the point(s) (x,y) where the tangent line has slope 1, that is, when $\frac{dy}{dx}=\frac{-x}{2y}=1$.

Then $x=-2y$. But the curve is $x^2+2y^2=1$.

It follows that $(-2y)^2+2y^2=1$,

$$4y^2+2y^2=1,$$

$$6y^2=1,$$

$$y=\pm\frac{1}{\sqrt{6}},$$

$$x=\mp\frac{2}{\sqrt{6}}.$$

Thus, the points are $\left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$.

Searching keywords:

- Use implicit differentiation استخدام الاشتقاق الضمني
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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