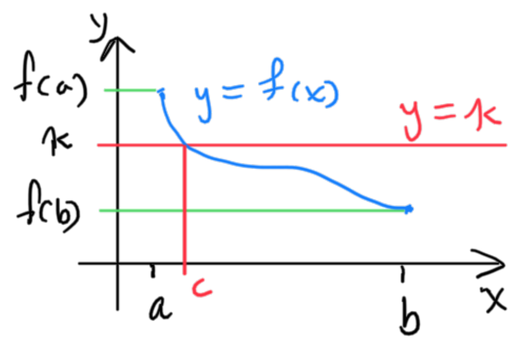
I.V.T., V.A. & H.A.

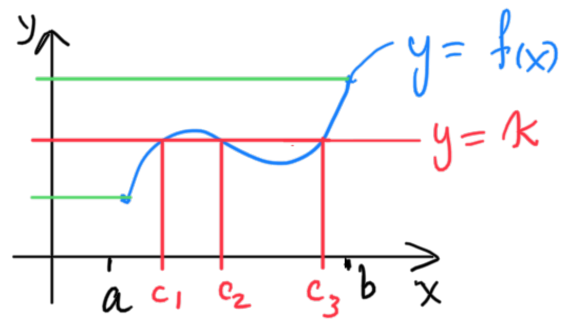
The Intermediate Value Theorem (I.V.T.).

Let f be a func. defined on the closed interval $[a,b]$, where $f(a) \neq f(b)$. Suppose that

- f is cts on $[a,b]$.
- N is a number between $f(a)$ and $f(b)$.

Then there exists a number $c \in (a,b)$ such that $f(c) = N$.





Ex. Show that the func. has a root in the indicated interval.

1) $f(x) = 2x^2 + x - 3$, $x \in [-2, 0]$.

Proof: f is cts on $[-2,0]$.

Also, $f(-2) \cdot f(0) = (3)(-3) = -9 < 0$.

Using I.V.T. $\Longrightarrow$ there exists $c \in (-2,0)$ such that $f(c) = 0$. That is, c is a root of f between -2 and 0.

In fact, $f(c) = 2c^2 + c - 3 = 0$.

So, $(2c+3)(c-1) = 0$. Then either

$c = -\frac{3}{2} \in (0,2)$ or $c = 1 \in (-2,0)$.

Thus, $c = -\frac{3}{2}$.

2) $f(x) = 2\cos x - x^2, \quad x \in [0, \frac{\pi}{2}]$.

Proof. f is cts on $[0, \frac{\pi}{2}]$.

Also, $f(0)\, f(\frac{\pi}{2}) = (2)\left(-\frac{\pi^2}{4}\right) = -\frac{\pi^2}{2} < 0$.

Using I.V.T. $\Longrightarrow$ there exists $c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$. That is, c is a root of f between 0 and $\frac{\pi}{2}$.

$2\cos c - c^2 = 0$.

(3) $f(x) = \cos(\frac{\pi}{2} x) - x^2, \quad x \in [0,1]$. Exc

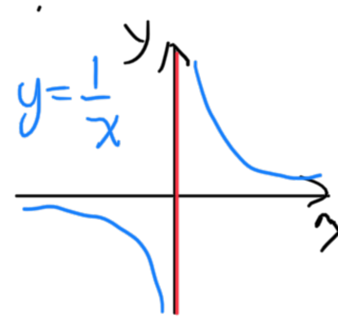
Vertical asymptotes.

Def- The vertical line $x=a$ is called a vertical asymptote (V.A.) of the curve $y=f(x)$ if at least one of the following statements is true:

$\lim_{x\to a} f(x)=\infty$	$\lim_{x\to a^-} f(x)=\infty$	$\lim_{x\to a^+} f(x)=\infty$
$\lim_{x\to a} f(x)=-\infty$	$\lim_{x\to a^-} f(x)=-\infty$	$\lim_{x\to a^+} f(x)=-\infty$

Ex: Find the vertical asymptotes of:

(1) $f(x)=\frac{1}{x}$.

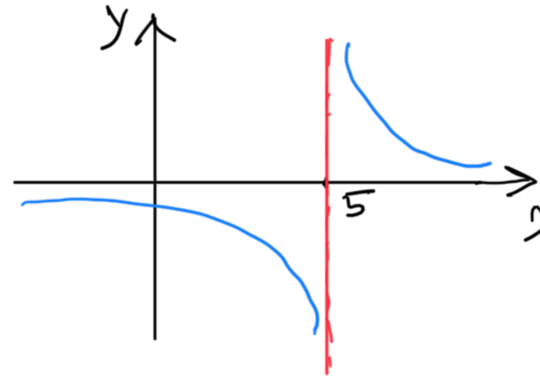


Soln. $\lim_{x\to 0^+} \frac{1}{x}=\infty$,

and $\lim_{x\to 0^-} \frac{1}{x}=-\infty$.

This shows that the line $x=0$ is a V.A. of f.

(2) $f(x) = \dfrac{1}{(x-5)^3}$.



Soln. $\lim\limits_{x \to 5^+} \dfrac{1}{(x-5)^3} = \infty$,

and $\lim\limits_{x \to 5^-} \dfrac{1}{(x-5)^3} = -\infty$.

This shows that the line $x=5$ is a V.A. of f.

(3) $f(x) = \dfrac{x-4}{x^2-4x+4} = \dfrac{x-4}{(x-2)^2}$.

Soln. $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x) = -\infty$.

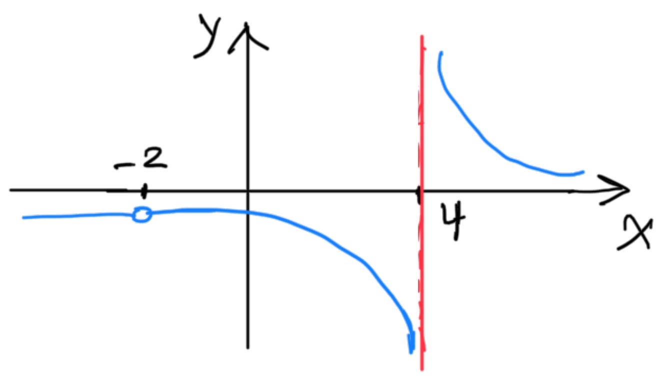
This shows that the line $x=2$ is a V.A. of f.

(4) $f(x) = \dfrac{3x+6}{x^2-2x-8} = \dfrac{3(x+2)}{(x+2)(x-4)} = \dfrac{3}{x-4}$ with $x \neq -2$

Soln. Note that f is not cts at $x=-2, 4$.

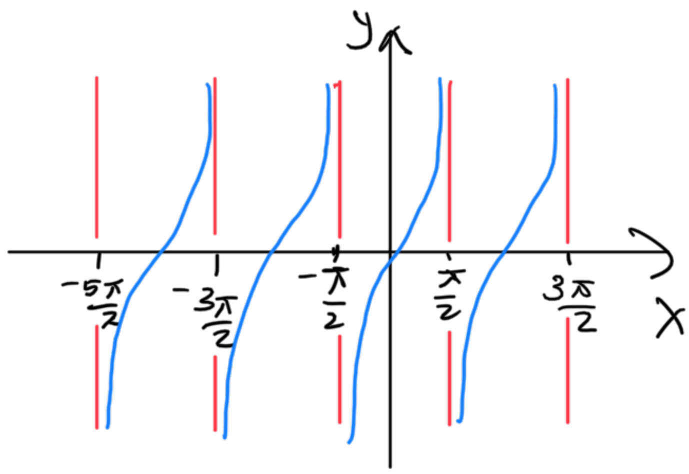
$\lim\limits_{x \to 4^+} f(x) = \infty$,

and $\lim\limits_{x \to 4^-} f(x) = \infty$.



This shows that the line $x=4$ is a V.A. of f,
but f has no V.A. at the line $x=2$.

(5) $f(x) = \tan x = \dfrac{\sin x}{\cos x}$.



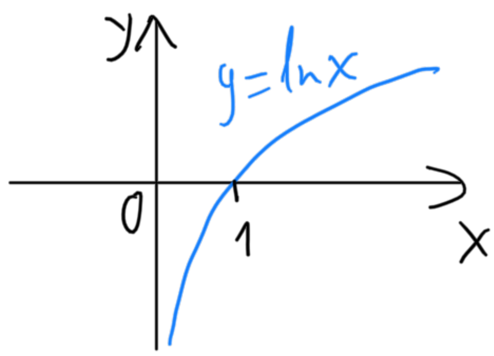
Soln.

$$\lim_{x \to \frac{\pi}{2}^-} \tan x = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \infty,$$

$$\lim_{x \to \frac{\pi}{2}^+} \tan x = \lim_{x \to \frac{\pi}{2}^+} \frac{\sin x}{\cos x} = -\infty.$$

This shows that the line $x = \frac{\pi}{2}$ is a V.A. of f.

(6) $f(x) = \ln x$. Exc.



Horizontal Asymptotes.

Def. The line $y = L$ is called a horizontal asymptote (H.A.) of the curve $y = f(x)$ if either

$$\boxed{\lim_{x\to\infty} f(x) = L} \quad \text{or} \quad \boxed{\lim_{x\to-\infty} f(x) = L.}$$

Fact: If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \begin{cases} \text{zero} & , \deg(p) < \deg(q) \\ \infty & , \deg(p) > \deg(q) \\ \dfrac{\text{Cofn. of term of largest degree of } f(x)}{\text{Cofn. of term of largest degree of } g(x)} & , \deg(p) = \deg(q) \end{cases}$$

Ex: Find the horizontal asymptotes of:

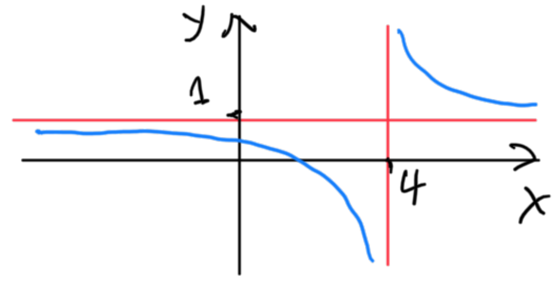
(1) $f(x) = \frac{1}{x}$.

Soln. $\lim_{x\to\infty} \frac{1}{x} = 0$,

and $\lim_{x\to-\infty} \frac{1}{x} = 0$.

This shows that the line $y = 0$ is an H. A. of f.

(2) $f(x) = \dfrac{x}{x-4}$.



Soln. $\displaystyle\lim_{x\to\pm\infty} \frac{x}{x-4} = 1$.

This shows that the line $y=1$ is an H.A. of f.

(3) $f(x) = \dfrac{9-4x^2}{1-2x^2}$.

Soln. $\displaystyle\lim_{x\to\infty} \frac{9-4x^2}{1-2x^2} = \frac{-4}{-2} = 2$.

This shows that the line $y=2$ is an H.A. of f.

(4) $f(x) = \dfrac{x}{x^2+3}$.

Soln. $\displaystyle\lim_{x\to\infty} \frac{x}{x^2+3} = 0$. $\therefore$ $y=2$ is an H.A. of f.

(5) $f(x) = \sqrt{x^2+x} - x$.

Soln. $\displaystyle\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right)\frac{\left(\sqrt{x^2+x}+x\right)}{\left(\sqrt{x^2+x}+x\right)}$

$\displaystyle= \lim_{x\to\infty} \frac{\cancel{x^2} + x - \cancel{x^2}}{\sqrt{x^2+x}+x}$

$$= \lim_{x\to\infty} \frac{1x}{\sqrt{1x^2+x}+1x} \qquad \frac{1}{1+1}$$

$$= \frac{1}{2}.$$

This shows that the line $y=\frac{1}{2}$ is an H.A. of f.

(6) $f(x) = \sqrt{4x^2-2x+1} - 2x.$ <u>Exc.</u>

(7) $f(x) = \dfrac{\cos^3 x}{x+1}.$

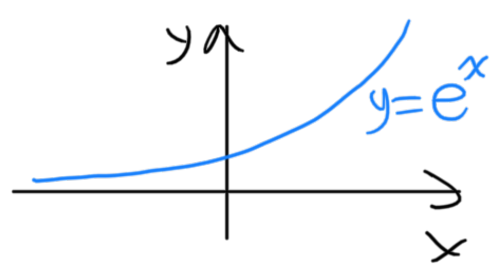
<u>Soln.</u> $-1 \le \cos^3 x \le 1$, so $\dfrac{-1}{x^2+1} \le \dfrac{\cos^3 x}{x^2+1} \le \dfrac{1}{x^2+1}.$

as $x \longrightarrow \infty$, $\quad \downarrow 0 \qquad \downarrow 0 \qquad \downarrow 0$

Then $\displaystyle\lim_{x\to\infty} \frac{\cos^3 x}{x^2+1} = 0.$

This shows that the line $y=0$ is an H.A. of f.

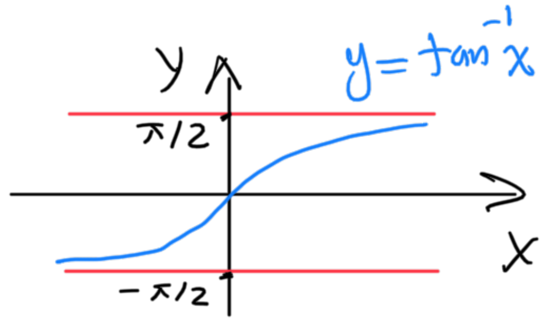
(8) $f(x) = e^x.$



<u>Soln.</u> $\displaystyle\lim_{x\to-\infty} e^x = 0.$

$\therefore$ $y=0$ is an H.A of f.

(8) $f(x) = \tan^{-1} x$. Exc.



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- Intermediate value theorem.
- Vertical asymptotes, horizontal asymptotes.
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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