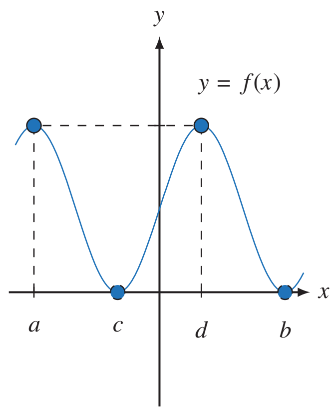
10

Linear Programming

CHAPTER CONTENTS

Optimization is a term used to describe any step-by-step procedure created to find the best solution to a problem with more than one solution. Ever since its inception in the 1940s, initially in the context of military planning, linear optimization (originally and still known as linear programming (LP for short)) has found extensive application across various industries and disciplines.

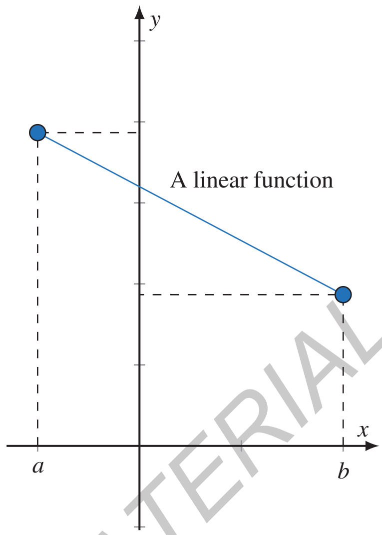
Let $f : [a, b] \to \mathbb{R}$ be a nonlinear continuous function. In Calculus, if we want to minimize/-maximize the function $f(x)$, we must take the derivative, and then find the critical points. We also check the endpoints, if there are any. We can justify our maxima or minima either by the first derivative test, or the second derivative test. In the graph shown to the right, a and b are end-points of the function $f(x)$, and c and d are their critical numbers ($f'(x) = 0$ when $x = c, d$). The function f has maximum values at $x = a, d$, and has minimum values at $x = c, b$.



Combinatorial and Algorithmic Mathematics: From Foundation to Optimization, First Edition. Baha Alzalg.

Companion website: www.wiley.com/go/alzalg

Now, instead of optimizing a nonlinear function on $[a, b]$, consider a linear function on $[a, b]$. In this case, we need to check only the endpoints as in the figure shown to the right. Linear optimization studies generalizations of this easy (linear) case to higher dimensions. More specifically, instead of optimizing a linear function of only one variable, say cx, on the closed interval $[a, b]$, we optimize a linear function of a finite number of variables, say $\boldsymbol{c}^{\mathsf{T}}x = c_1x_1 + c_2x_2 + \cdots + c_nx_n$, on polytopes, which are generalizations of polygons from $\mathbb{R}^2$ to $\mathbb{R}^n$, where the set $\mathbb{R}^n$ consists of all n-tuples of real numbers, $\mathbb{R}$. This study is "easy" to understand because of linearity, but it is "difficult" to carry out because of high dimensionality.



In this chapter, we introduce linear programming, the graphical method, and study the LP duality and geometry. We also study the most common linear programming algorithm, the simplex method. Over and above that, we study an interior-point method as one of the non-simplex methods. The references Bertsimas and Tsitsiklis (1997) and Nemhauser and Wolsey (1988), for example, is a good source for information relative to this topic.

10.1 Linear Programming Formulation and Examples

In this section, we will see that applications of LP touch a vast range of real-world areas. First, we present the general form of an LP problem.

10.1.1 General Form Linear Programs

An LP problem is the problem of minimizing a linear cost function subject to linear equality and inequality constraints. We have the following example.

Example 10.1 The following is an LP problem.

$$\begin{array}{lrl}
\text{minimize} & 4x_1 - x_2 + 3x_3 & \\
\text{subject to} & x_1 + x_2 \qquad + x_4 & \le 7, \\
& 2x_2 - x_3 \qquad & = 6, \\
& x_3 + x_4 & \ge 4, \\
& x_1 \qquad\qquad & \ge 0, \\
& x_3 \qquad & \le 0.
\end{array}$$

Here, x_1, x_2, x_3, and x_4 are the decision variables whose values are to be chosen to minimize the linear cost function $4x_1 - x_2 + 3x_3$ subject to linear equality and inequality constraints. ■

Generally speaking, assume that we are given a cost vector $\boldsymbol{c} = (c_1, c_2, \dots, c_n)^\mathsf{T}$ and we minimize a linear cost function $\boldsymbol{c}^\mathsf{T}\boldsymbol{x} = \sum_{i=1}^{n} c_i x_i$ over all nth-dimensional vectors $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^\mathsf{T}$ subject to linear equality and inequality constraints. Then we are interested in a problem of the form:

$$
\begin{array}{lll}
\min & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} & \\
\text{s.t.} & \boldsymbol{a}_i^\mathsf{T}\boldsymbol{x} \geq b_i, & i = 1, 2, \dots, m_1, \\
& \boldsymbol{a}_j^\mathsf{T}\boldsymbol{x} \leq b_j, & j = 1, 2, \dots, m_2, \\
& \boldsymbol{a}_k^\mathsf{T}\boldsymbol{x} = b_k, & k = 1, 2, \dots, m_3, \\
& x_p \geq 0, & p = 1, 2, \dots, m_4, \\
& x_q \leq 0, & q = 1, 2, \dots, m_5.
\end{array}
\tag{10.1}
$$

Problem (10.1) is said to be the general form LP. We have the following definition.

Definition 10.1 Consider the minimization problem (10.1). Then:

(a) The variables $x_1, x_2, \dots, x_n$ are called decision variables;
(b) A vector $\boldsymbol{x}$ satisfying all of the constraints is called a feasible solution;
(c) The set of all feasible solutions is called the feasible set or feasible region;
(d) If $x_i \geq 0$ or $x_i \leq 0$, then x_i is called a restricted variable; otherwise, it is called a free or unrestricted variable (urs);
(e) The function $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ is called the objective function or cost function;
(f) A feasible solution $\boldsymbol{x}^\star$ that minimizes the objective function (i.e., $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}^\star \leq \boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ for any feasible solution $\boldsymbol{x}$) is called an optimal solution;
(g) The value of $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}^\star$, corresponding to an optimal solution $\boldsymbol{x}^\star$, is called the optimal cost or optimal value;
(h) If the optimal cost is $-\infty$, we say that the minimization problem is unbounded.

Example 10.2 Consider the following nonlinear minimization problem.

$$
\begin{array}{ll}
\min & 2x_1 + |x_2| \\
\text{s.t.} & 5x_1 + 7x_2 \leq 3, \\
& |x_1| + x_2 \leq 4, \\
& x_1, x_2 \text{ urs.}
\end{array}
$$

Using the fact that $|t| = \max\{t, -t\}$ for $t \in \mathbb{R}$, this problem can be expressed as

$$
\begin{array}{ll}
\min & 2x_1 + y \\
\text{s.t.} & 5x_1 + 7x_2 \leq 3, \\
& x_1 + x_2 \leq 4, \\
& -x_1 + x_2 \leq 4, \\
& y - x_2 \geq 0, \\
& y + x_2 \geq 0, \\
& x_1, x_2 \text{ urs,}
\end{array}
$$

which is an LP problem. ■

Note that there is no need to study maximization problems separately because maximizing $\boldsymbol{c}^\top \boldsymbol{x}$ subject to some constraints is equivalent to minimizing $(-\boldsymbol{c})^\top \boldsymbol{x}$ subject to the same constraints (why?).

10.1.2 Examples of Linear Programming Problems

This part presents some examples of LP problems and allows the reader gain to some familiarity with the art of constructing mathematical optimization models.

The procedure given in the following workflow, followed by some examples, will teach us how to formulate linear optimization models.

Workflow 10.1 There are three steps involved in the formation of an LP problem:

(i) Identify the decision variables of interest to the decision-maker and express them as $x_1, x_2, x_3, \dots$.
(ii) Ascertain the objective function in terms of the decision variables. This would be a cost in case of minimization problem or a profit in case of maximization problem.
(iii) Ascertain the constraints representing the maximum availability or minimum commitment.

Example 10.3 **(Maximizing profit in product manufacturing)**
A company is involved in the production of two items, denoted as P_1 and P_2. The manufacturing process for each unit of product P_1 necessitates 2 kg of raw material and 4 labor hours for processing, while each unit of product P_2 requires 5 kg of raw material and 3 labor hours of the same type. On a weekly basis, the company has access to 45 kg of raw material and 55 labor hours. For the financial aspect, the company gains a profit of JD 25 for every unit of product P_1 sold and JD 35 for every unit of product P_2 sold. Formulate this problem as an LP problem that maximizes the total profit.

Solution
The given data can be summarized in the following table.

Product	Row material	Labour hours	Profit
P_1	2 kg	4 h	JD 25
P_2	5 kg	3 h	JD 35
Restrictions	45 kg	55 h	

- The first step is to identify the decision variables. Let x_i denote the number of units that should be produced from product P_i per week, $i = 1, 2$.
- The second step is to determine the objective function. The objective is to maximize the total profit. So, our objective function is $z = 25x_1 + 35x_2$.
- The third step is to formulate the constraints. In this example, the constraints are:
  - A restriction on the row material. This can be formulated as $2x_1 + 5x_2 \leq 45$.
  - A restriction on the labor hours. This can be formulated as $4x_1 + 3x_2 \leq 55$.
  - Nonnegativity constraints. This can be formulated as $x_1 \geq 0$ and $x_2 \geq 0$.

As a result, this problem can be formulated as the following LP model.

$$\begin{aligned} \max \quad & 25x_1 + 35x_2 \\ \text{s.t.} \quad & 2x_1 + 5x_2 \leq 45, \\ & 4x_1 + 3x_2 \leq 55, \\ & x_1, x_2 \geq 0. \end{aligned}$$

■

Example 10.4 (Maximizing profit in corn chip production)
A corn chip company operates with two distinct departments, each responsible for producing two types of corn chips: "extra larges" and "really smalls." The company earns a profit of 225 per kilobag of extra larges and 175 per kilobag of really smalls (where a kilobag contains 1000 bags). Each department adheres to specific production regulations per day. The company's primary objective is to maximize its profit while complying with these regulations.

(a) Identify the decision variables.
(b) Write the objective function z in terms of the decision variables.
(c) Write inequalities expressing the following constraints:
  (i) The production of extra larges should not exceed 20 kilobags per day, and the production of really smalls should not exceed 30 kilobags per day.
  (ii) No more than a total of 45 kilobags can be produced each day.
  (iii) The number of extra larges produced daily must be at least 2/3 of the number of really smalls produced.
  (iv) The company must utilize more than 250 hours of labor each day to satisfy union requirements. Making one kilobag of extra larges consumes 10 hours and making one kilobag of really smalls consumes 15 hours.

Solution

(a) The decision variables are:

x: The number of extra large corn chips produced per day;

y: The number of really small corn chips produced per day.

(b) $z = 225x + 175y$.

(c) (i) $x \leq 20, \ y \leq 30$.
  (ii) $x + y \leq 45$.
  (iii) $x \geq \frac{2}{3}y$.
  (iv) $10x + 15y > 250$. ■

Example 10.5 (Minimizing nutritional costs)
Two different food items, denoted as F_1 and F_2, contain vitamins A and B. Food F_1 provides 2 units of vitamin A and 5 units of vitamin B per unit, while food F_2 offers 4 units of vitamin A and 2 units of vitamin B per unit. The cost of one unit of food F_1 is JD 10, and for food F_2, it is JD 12.50. The objective is to meet or exceed the minimum daily nutritional requirements for vitamins A and B, which are 40 and 50 units, respectively, at the lowest possible cost. Formulate this problem as an LP problem.

Solution

The given data can be summarized in the following table.

Food/vitamin	A	B	Cost
F_1	2 units	5 units	JD 10
F_2	4 units	2 units	JD 12.5
Restrictions	40 units	50 units	

Let x_i be the number of units that should be daily produced from food F_i for a person, $i = 1, 2$. This problem can be formulated as the following LP model.

$$\begin{aligned} \min \quad & 10x_1 + 12.5x_2 \\ \text{s.t.} \quad & 2x_1 + 4x_2 \geq 40, \\ & 5x_1 + 2x_2 \geq 50, \\ & x_1, x_2 \geq 0. \end{aligned}$$

■

Example 10.6 (Maximizing advertising audience)

A marketing manager has an annual advertising budget of JD 25,000, which he intends to allocate to two advertising media, A and B. Media A, a monthly magazine, costs JD 1000 per message, and media B costs JD 1500 per message. The following conditions apply: For media A, not more than one insertion is desired in the issue. For media B, at least five messages should be placed. The expected effective audience for one message in media A is 40,000 people, while for media B, it is 50,000 people. Formulate this problem as an LP problem to maximize the total audience reached through advertising while staying within the budget constraints.

Solution

The given data can be summarized in the following table.

Media	Media A	Media B	Restrictions
Audience	40,000 people	50,000 people	
One message cost	JD 1000	JD 1500	JD 25,000
Number of messages	At most 1	At least 5	

Let x_1 and x_2 be the number of messages that should appear in media A and B, respectively. This problem can be formulated as the following LP model.

$$\begin{aligned} \max \quad & 40,000x_1 + 50,000x_2 \\ \text{s.t.} \quad & 1000x_1 + 1500x_2 \leq 25,000, \\ & x_1 \leq 1, \\ & x_2 \geq 5, \\ & x_1, x_2 \geq 0. \end{aligned}$$

■

Example 10.7 (Minimizing cost in sheep nutrition)
A farmer is actively involved in breeding sheep, and the sheep's diet primarily consists of various products grown on the farm. To ensure that the sheep receive the required nutrient constituents, the farmer must consider purchasing additional products, which we will refer to as Product A and Product B. The essential nutrient constituents (vitamins and protein) contained in each of these products are detailed in the table below:

Nutrient Constituents	Nutrient in product A	Nutrient in product B	Minimum requirement of nutrient constituents
X	36	6	108
Y	3	12	36
Z	20	10	100

Product A is priced at JD 20 per unit, while Product B is priced at JD 40 per unit. Formulate an LP problem that can minimize the total cost and satisfy the requirements.

Solution
Let x_1 and x_2 be the number of units that must be purchased from products A and B, respectively. This problem can be formulated as the following LP model.

$$\begin{aligned} \min \quad & 20x_1 + 40x_2 \\ \text{s.t.} \quad & 36x_1 + 6x_2 \geq 108, \\ & 3x_1 + 12x_2 \geq 36, \\ & 20x_1 + 10x_2 \geq 100, \\ & x_1, x_2 \geq 0. \end{aligned}$$

■

Example 10.8 (Nurse scheduling at a university hospital)
A university hospital is seeking your assistance in scheduling nurses for their intensive care unit. In this scenario, it is assumed that the same daily schedule repeats, and the nurse requirements remain constant. Each workday is divided into four shifts: 12AM–6AM, 6AM–12PM, 12PM–6PM, and 6PM–12AM. Every day, each nurse is assigned to work two of these shifts. Nurses working two consecutive shifts are compensated at a rate of \$20 per hour, while those working a "split schedule" (e.g., 12AM–6AM and 12PM–6PM) receive \$25 per hour. (It is important to note that the shifts 6PM–12AM and 12AM–6AM are considered consecutive.) The following table indicates the daily nurse requirements for each shift:

Shift	Number required
12AM–6AM	5
6AM–12PM	12
12PM–6PM	7
6PM–12AM	10

Formulate an LP that can assist this hospital in determining the optimal nurse scheduling to meet daily requirements and minimize the total nurse compensation cost.

Solution
The decision variables are:

x_1: The number of nurses that work from 12AM to 12PM;

x_2: The number of nurses that work from 6AM to 6PM;

x_3: The number of nurses that work from 12PM to 12AM;

x_4: The number of nurses that work from 6PM to 6AM;

x_5: The number of nurses that work from 12AM to 6AM and 12PM to 6PM;

x_6: The number of nurses that work from 6AM to 12PM and 6PM to 12AM.

Minimizing the total cost, we obtain the following LP problem.

$$
\begin{array}{lllllll}
\min & 20x_1 + 20x_2 + 20x_3 + 20x_4 + 25x_5 + 25x_6 \\
\text{s.t.} & x_1 + & & & x_4 + & x_5 & \geq 5, \\
& x_1 + & x_2 + & & & & x_6 \geq 12, \\
& & x_2 + & x_3 + & & x_5 & \geq 7, \\
& & & x_3 + & x_4 + & & x_6 \geq 10, \\
& x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \geq 0.
\end{array}
$$

■

10.2 The Graphical Method

In this section, we discuss the graphical method for linear optimization problems of two variables. We will also visually demonstrate different LP cases which may result in different types of solutions. We start by presenting the following workflow of six steps to find the extremum (maximum or minimum) solution graphically.

Workflow 10.2 The following steps involved in solving two-dimensional LP problems graphically:

(i) Graph constraint equations on a rectangular coordinate plane.
(ii) Determine the valid side of each constraint equation.
(iii) Isolate and identify the feasible region.
(iv) Determine the direction of improvement.
(v) Locate the extreme corner.
(vi) Find the optimum solution and the corresponding optimal value.

As a direct application of the above steps, we have the following examples.

Example 10.9 Use the graphical method to solve the following LP problem.

$$
\begin{array}{ll}
\min & z = 2x + 5y \\
\text{s.t.} & 3x + 2y \leq 6, \\
& -x + 2y \leq 4, \\
& x + y \geq 1, \\
& x, y \geq 0.
\end{array}
$$

Solution
Following the steps in Workflow 10.2, we obtain the graphical solution visualized in Figure 10.1. Note that the given objective function $z = 2x + 5y$ is perpendicular to the vector $\boldsymbol{c} = (2, 5)^{\mathsf{T}}$ for any given scalar z. For simplicity, we represent this using the vector $\boldsymbol{c}$ in Figure 10.1. Furthermore, decreasing z corresponds to moving the line $z = 2x + 5y$ in the direction of $-\boldsymbol{c}$. Therefore, to minimize z, we move the line $2x + 5y = z$ as much as possible in the direction of $-\boldsymbol{c}$, as long as we do not leave the feasible region. From Figure 10.1, we find that the unique optimal solution is $\boldsymbol{x} = (1, 0)^{\mathsf{T}}$ and the optimal value is $z = 2 \times 1 + 5 \times 0 = 2$. ∎

For a system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$, we have three possibilities: The system has a unique solution, it has infinitely many solutions, or it is inconsistent. For an LP, we have the corresponding three possibilities, but we have one more possibility in addition. An LP problem may have:

- A unique/finite optimal solution;
- No bounded solution (so the LP is unbounded);
- No feasible solution (so the LP is infeasible);
- Alternative (multiple or infinite number of) optimal solutions.

In the context of graphical method, it is easy to visualize these four different cases, as will be evident from the following examples.

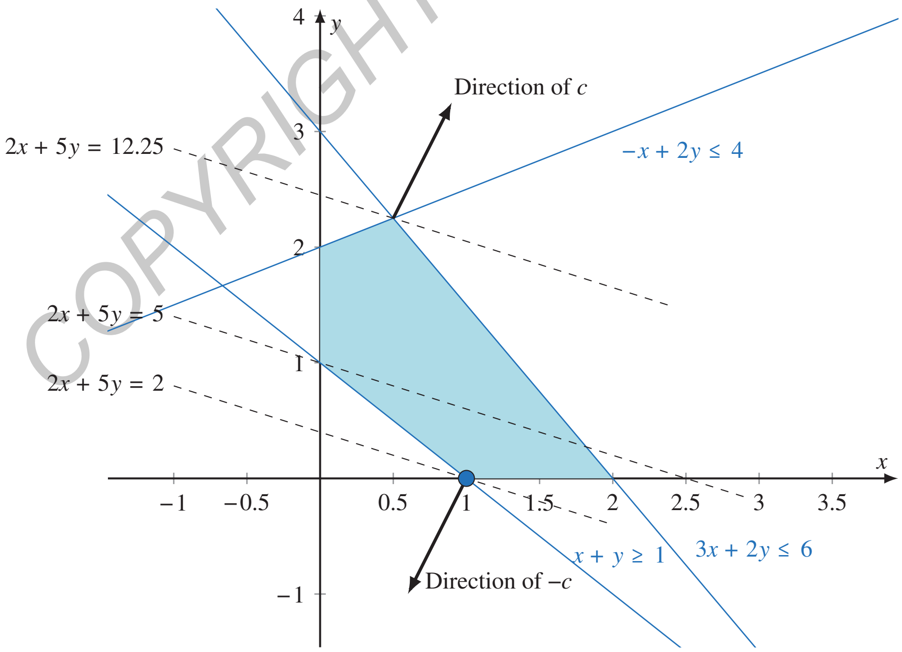


Figure 10.1 Graphical solution of the LP problem in Example 10.9

Example 10.10 Use the graphical method to solve the following LP problems.

(a)
$$\begin{aligned} \max\quad & z = 13x_1 + 23x_2 \\ \text{s.t.}\quad & x_1 + 3x_2 \le 96, \\ & x_1 + x_2 \le 40, \\ & 7x_1 + 4x_2 \le 238, \\ & x_1, \quad x_2 \ge 0. \end{aligned}$$

(b)
$$\begin{aligned} \max\quad & z = x_1 + x_2 \\ \text{s.t.}\quad & x_1 + 3x_2 \le 96, \\ & x_1 + x_2 \le 40, \\ & 7x_1 + 4x_2 \le 238, \\ & x_1, \quad x_2 \ge 0. \end{aligned}$$

Solution

(a) The graphical representation of the given LP problem is shown in Figure 10.2, with the feasible region shaded in cyan. From the graph, we find that the maximum value for z is 800 at $\boldsymbol{x} = (12, 28)^{\mathsf{T}}$. So, this LP problem has a unique optimal solution.

(b) The graphical representation of the given LP problem is shown in Figure 10.3, with the feasible region shaded in cyan. Note that the z-line hits the entire line segment between the points (12, 28) and (26, 14). From the graph, we find that the maximum value for z is 40, and that every point in the line segment between (12, 28) and (26, 14) is an optimal solution. So, this LP problem has alternative optimal solutions. ■

Example 10.11 Use the graphical method to solve the following LP problems.

(a)
$$\begin{aligned} \min\quad & z = 3x_1 + x_2 \\ \text{s.t.}\quad & 5x_1 + x_2 \ge 42, \\ & 2x_1 + x_2 \ge 30, \\ & x_1, \quad x_2 \ge 0. \end{aligned}$$

(b)
$$\begin{aligned} \max\quad & z = 3x_1 + x_2 \\ \text{s.t.}\quad & 5x_1 + x_2 \ge 42, \\ & 2x_1 + x_2 \ge 30, \\ & x_1, \quad x_2 \ge 0. \end{aligned}$$

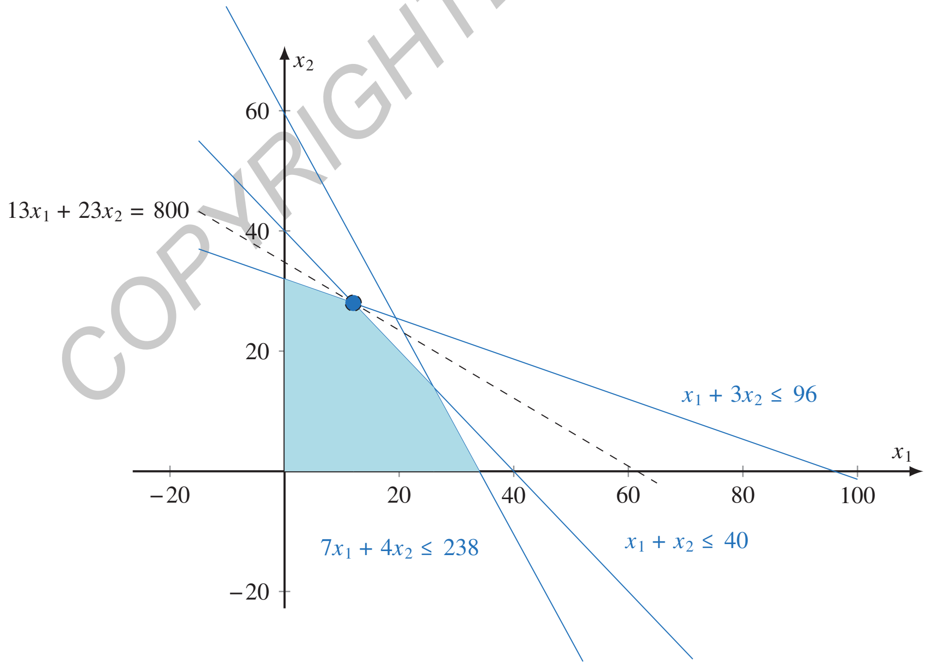


Figure 10.2 Graphical solution of the optimization problem in Example 10.10 (*a*).

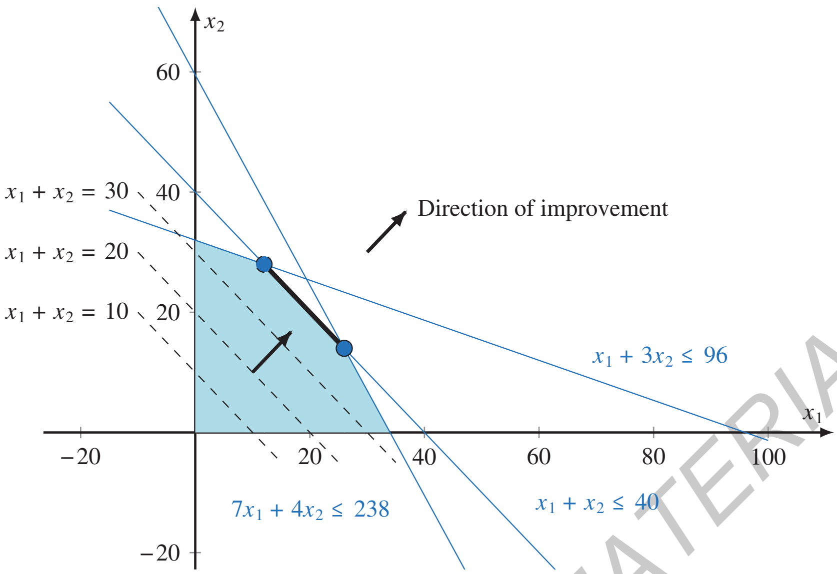


Figure 10.3 Graphical solution of the optimization problem in Example 10.10 (*b*).

Solution

(a) In Figure 10.4, we have provided a graphical representation of the LP problem at hand. The feasible region is distinctly shaded in a cyan color for clarity. Upon inspecting the graph, we can readily deduce that the lowest attainable value for the objective function z occurs at 34. This minimal value of z is achieved when the decision variables are set to $\boldsymbol{x} = (4, 22)^{\mathsf{T}}$.

(b) We provide a visual depiction of the given LP problem in Figure 10.5. Within this graph, the feasible region is distinctly highlighted in cyan. One can observe that the z-line, which represents the objective function's values, can be continuously extended toward the upper-right corner of the feasible region without any bound or limit. This observation implies that there is no finite or optimal value of z that can be achieved within the problem's constraints. Consequently, we can conclude that this LP problem is unbounded, emphasizing the open-ended nature of this particular problem. ■

Example 10.12 Use the graphical method to solve the following LP problems.

(a)
$$\begin{array}{lrl} \max & z = & 13x_1 + 23x_2 \\ \text{s.t.} & x_1 + 3x_2 & \leq 96, \\ & x_1 + x_2 & \geq 30, \\ & 7x_1 + 4x_2 & \leq 238, \\ & x_1, \quad x_2 & \geq 0. \end{array}$$

(b)
$$\begin{array}{lrl} \max & z = & 13x_1 + 23x_2 \\ \text{s.t.} & x_1 + 3x_2 & \geq 96, \\ & x_1 + x_2 & \leq 30, \\ & 7x_1 + 4x_2 & \geq 238, \\ & x_1, \quad x_2 & \geq 0. \end{array}$$

Solution

(a) The graphical representation of the given LP problem is shown in Figure 10.6, with the feasible region shaded in cyan. From the graph, we find that the maximum value for z is 839.52 at $\boldsymbol{x} = (19.41, 25.53)^{\mathsf{T}}$. So, this LP problem has a unique optimal solution.

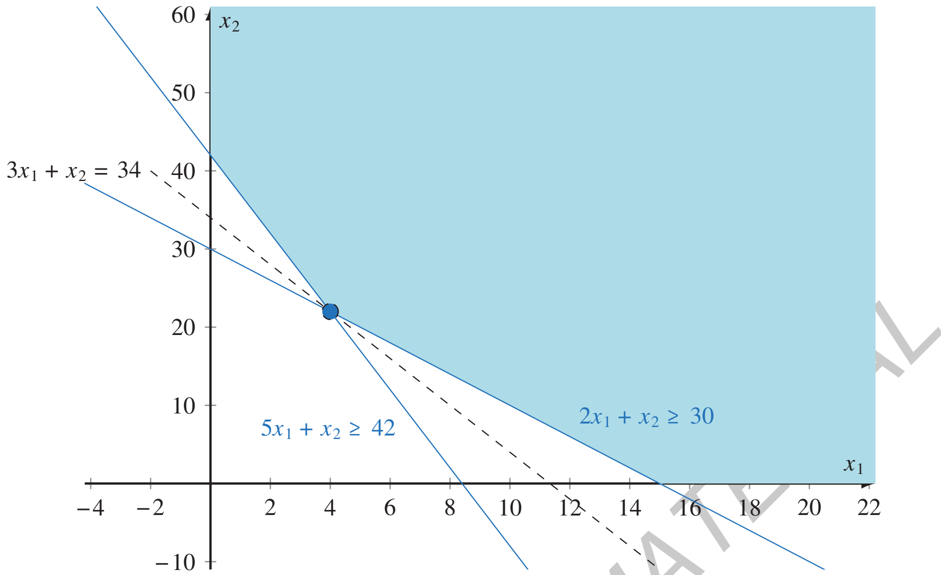


Figure 10.4 Graphical solution of the optimization problem in Example 10.11 (*a*).

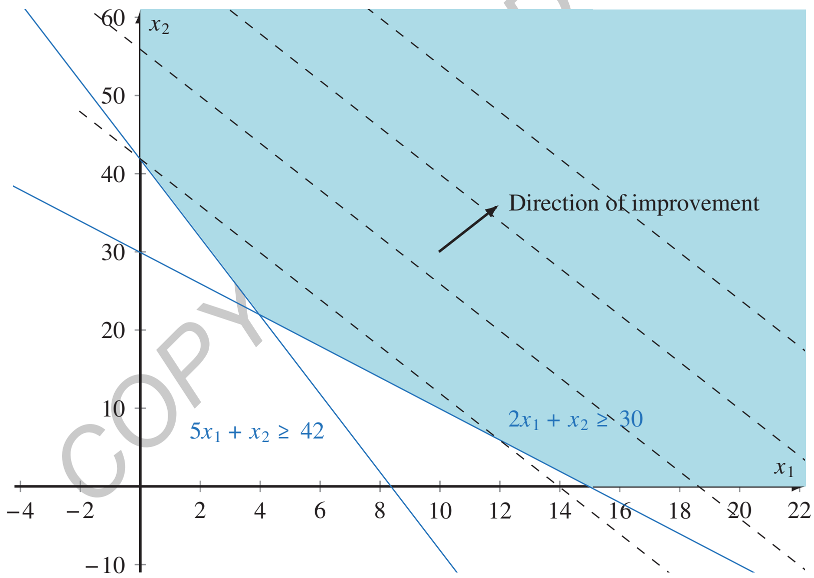


Figure 10.5 Graphical solution of the optimization problem in Example 10.11 (*b*).

(b) The graphical representation of the given LP problem is shown in Figure 10.7. Note that there are no feasible solutions, that is, there are no points satisfying all constraints. Therefore, the feasible region is empty, and the LP problem is infeasible. ■

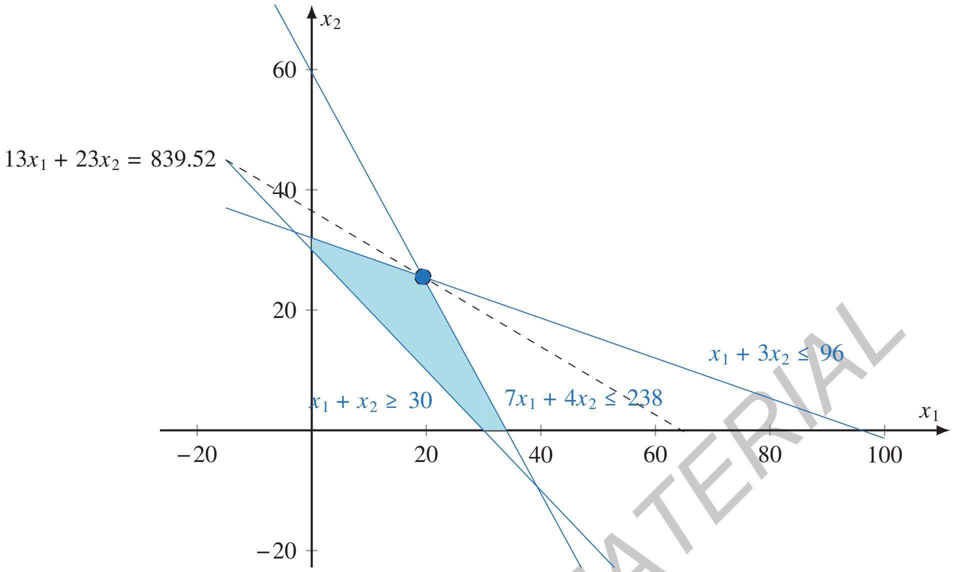


Figure 10.6 Graphical solution of the optimization problem in Example 10.12 (*a*).

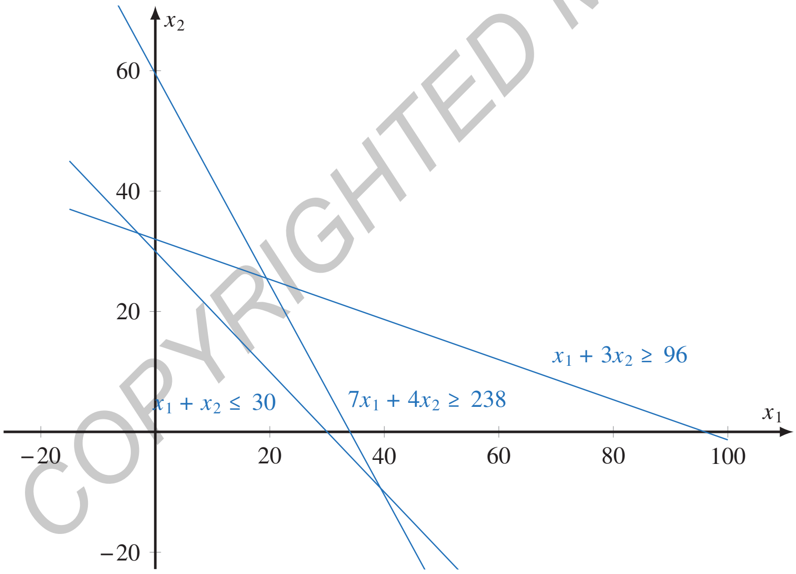


Figure 10.7 Graphical solution of the optimization problem in Example 10.12 (*b*).

Example 10.13 For the LP problems given in Examples 10.10–10.12, indicate which case the LP belongs to (i.e., if the LP has a unique optimal solution, has many optimal solutions, is unbounded, or is infeasible), and which type the feasible region is found (i.e., if the feasible region is bounded, unbounded, or empty).

Solution

The answer is given in Table 10.1. ■

Table 10.1 The answer of Example 10.13

LP problem	LP case/type	Feasible region type
The LP in Example 10.10 (a)	Unique optimal solution	Bounded
The LP in Example 10.10 (b)	Alternative optimal solutions	Bounded
The LP in Example 10.11 (a)	Unique optimal solution	Unbounded
The LP in Example 10.11 (b)	Unbounded LP	Unbounded
The LP in Example 10.12 (a)	Unique optimal solution	Bounded
The LP in Example 10.12 (b)	Infeasible LP	Empty

Example 10.14 Consider the following LP problem.

$$\begin{array}{ll} \max & y \\ \text{s.t.} & -x + \ \ y \le 1, \\ & 3x + 2y \le 12, \\ & 2x + 3y \ge 12, \\ & x, \quad\ \ y \ge 0. \end{array}$$

(a) Sketch the feasible region of this LP and solve it using the graphical method.
(b) Generally speaking, if (some of) the variables are restricted to be integer-valued, then the underlying optimization problem is called an integer (a mixed-integer) program.

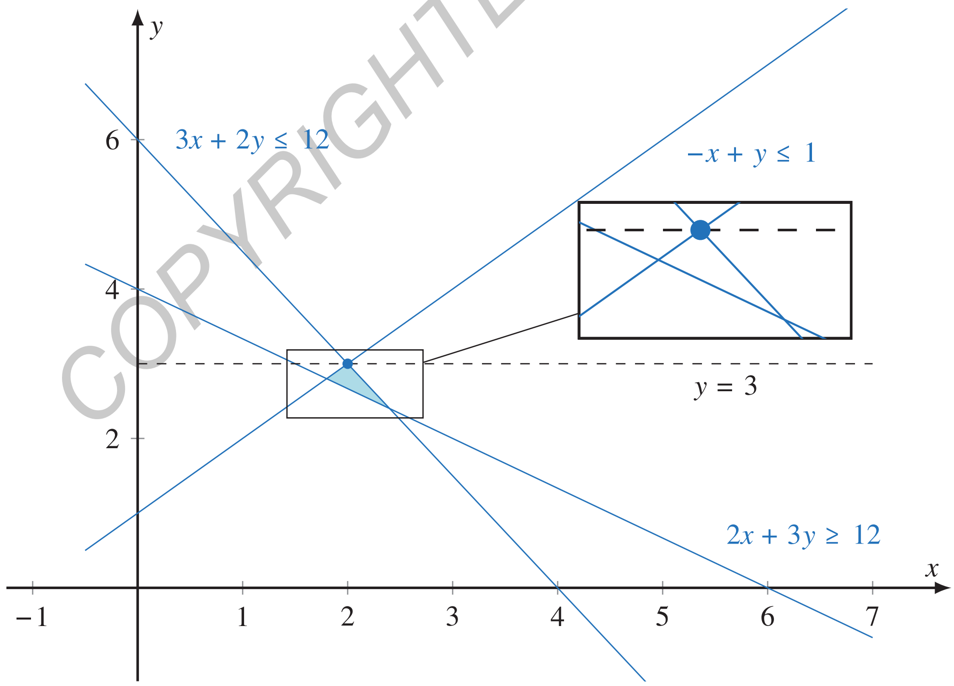


Figure 10.8 Graphical solution of the optimization problem in Example 10.14.

In this example, assume that x and y are restricted to be integer-valued. Sketch its feasible region and solve it graphically.

Solution

(a) The graphical representation of the LP is shown in Figure 10.8, with the feasible region shaded in cyan. We find that the optimal solution is 3 at $\boldsymbol{x} = (2, 3)^{\mathsf{T}}$.
(b) Introducing the condition $x, y \in \mathbb{Z}$ changes the feasible region, which is now indicated by the blue bullet shown in Figure 10.8. The optimal solution remains the same. ∎

10.3 Standard Form Linear Programs

Recall that the general form LP is:

$$\begin{aligned} \min\ & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s.t.}\ & \boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{x} \geq b_i,\ \ i = 1, 2, \dots, m_1, \\ & \boldsymbol{a}_j^{\mathsf{T}}\boldsymbol{x} \leq b_j,\ \ j = 1, 2, \dots, m_2, \\ & \boldsymbol{a}_k^{\mathsf{T}}\boldsymbol{x} = b_k,\ \ k = 1, 2, \dots, m_3, \\ & x_p \geq 0, \qquad p = 1, 2, \dots, m_4, \\ & x_q \leq 0, \qquad q = 1, 2, \dots, m_5. \end{aligned} \tag{10.2}$$

where $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$.

Recall also that there is no need to study maximization problems separately because maximizing $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ subject to some constraints is equivalent to minimizing $-\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ subject to the same constraints. In addition, because:

- $\boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{x} = b_i$ is equivalent to $\boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{x} \leq b_i$ and $\boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{x} \geq b_i$;
- $\boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{x} \leq b_i$ can be written as $(-\boldsymbol{a}_i)^{\mathsf{T}}\boldsymbol{x} \geq -b_i$;
- $x_i \geq 0$ and $x_i \leq 0$ are special cases of $\boldsymbol{u}^{\mathsf{T}}\boldsymbol{x} \geq 0$ and $(-\boldsymbol{u})^{\mathsf{T}}\boldsymbol{x} \geq 0$, respectively, where $\boldsymbol{u}$ is a unit vector in $\mathbb{R}^n$,

Problem (10.2) can be expressed exclusively in terms of inequality constraints of the form $\boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{x} \geq b_i$. As a result, Problem (10.2) can be formulated in vector form as

$$\begin{aligned} \min\ \ & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s.t.}\ \ & \boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{x} \geq b_i,\ \ i = 1, 2, \dots, m, \end{aligned} \tag{10.3}$$

or, more compactly, in matrix form as

$$\begin{aligned} \min\ & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s.t.}\ & A\boldsymbol{x} \geq \boldsymbol{b}, \end{aligned} \tag{10.4}$$

where $A \in \mathbb{R}^{m \times n}$ is the matrix whose rows are the row vectors $\boldsymbol{a}_1^{\mathsf{T}}, \boldsymbol{a}_2^{\mathsf{T}}, \dots, \boldsymbol{a}_m^{\mathsf{T}}$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_m)^{\mathsf{T}}$. We have the following example.

Example 10.15 The LP problem in Example 10.1 can be written as

$$
\begin{array}{lrcrcrcrcl}
\min & 2x_1 & - & x_2 & + & 4x_3 & & & & \\
\text{s.t.} & -x_1 & - & x_2 & & & - & x_4 & \geq & -2, \\
 & & & 3x_2 & - & x_3 & & & \geq & 5, \\
 & & & -3x_2 & + & x_3 & & & \geq & -5, \\
 & & & & & x_3 & + & x_4 & \geq & 3, \\
 & x_1 & & & & & & & \geq & 0, \\
 & & & & - & x_3 & & & \geq & 0.
\end{array}
$$

This can be also written in the matrix form (10.4) with

$$
\boldsymbol{c} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix}, \ A = \begin{bmatrix} -1 & -1 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \ \text{and} \ \boldsymbol{b} = \begin{bmatrix} -2 \\ 5 \\ -5 \\ 3 \\ 0 \\ 0 \end{bmatrix}.
$$

■

An LP problem of the form

$$
\begin{array}{ll}
\min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\
\text{s.t.} & A\,\mathbf{x} = \mathbf{b}, \\
 & \boldsymbol{x} \geq \mathbf{0}
\end{array}
\tag{10.5}
$$

is said to be the standard form LP problem.

We can convert an LP problem to the standard form by eliminating of free variables and eliminating of inequality constraints as detailed in the following workflow.

Workflow 10.3 We can convert an LP problem to the standard form by following three steps:

(i) Elimination of free variables: We replace each unrestricted variable x_i with $x_i^+ - x_i^-$, where $x_i^+, x_i^- \geq 0$.
(ii) Elimination of "≤" constraints: We replace $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ with $\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i$, where $s_i \geq 0$ is called a slack variable (see also Definition 10.11).
(iii) Elimination of "≥" constraints: We replace $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$ with $\sum_{j=1}^{n} a_{ij}x_j - e_i = b_i$, where $e_i \geq 0$ is called an excess variable (see also Definition 10.11).

Example 10.16 The LP problem:

$$
\begin{array}{lll}
\min & 3x_1 + 7x_2 & \\
\text{s.t.} & x_1 + x_2 & \geq 3, \\
 & 5x_1 + 3x_2 & = 19, \\
 & x_1 & \geq 0,
\end{array}
\quad
\begin{array}{l}
\text{is equivalent to} \\
\text{standard form} \\
\text{LP problem:} \\
(\text{letting } x_3 = s_3)
\end{array}
\quad
\begin{array}{lll}
\min & 3x_1 + 7x_2^+ - 7x_2^- & \\
\text{s.t.} & x_1 + x_2^+ - x_2^- - x_3 & = 3, \\
 & 5x_1 + 3x_2^+ - 3x_2^- & = 19, \\
 & x_1, x_2^+, x_2^-, x_3 & \geq 0.
\end{array}
$$

For instance, given the feasible solution $(x_1, x_2) = (2, 3)$ to the original problem, we obtain the feasible solution $(x_1, x_2^+, x_2^-, x_3) = (2, 3, 0, 2)$ to the standard form problem. In Exercise 10.12, we seek the point (x_1, x_2) for the original problem given the feasible solution $(x_1, x_2^+, x_2^-, x_3) = (4, 0, 1/3, 2/3)$ to the standard form problem. ■

10.4 Geometry of Linear Programming

The graphical method for linear optimization problems indicates that an optimal solution to an LP lies at a "corner" of a polyhedron. A vertex, an extreme point, and a basic feasible solution all describe corners of a polyhedron, with the first two being geometric definitions.

10.4.1 Extreme Points, Vertices, and Basic Feasible Solutions

In this part, we define a vertex, an extreme point, and a basic feasible solution of a given nonempty polyhedron.

Definition 10.2 Let P be a nonempty polyhedron. A vector $\boldsymbol{x} \in P$ is called a vertex of P if there is some $\boldsymbol{c}$ such that $\boldsymbol{c}^\top \boldsymbol{x} < \boldsymbol{c}^\top \boldsymbol{y}$ for all $\boldsymbol{y} \in P$ different from $\boldsymbol{x}$.

From Definition 10.2, we observe that $\boldsymbol{x}$ is a vertex of a polyhedron P if it is the optimal solution of some linear program with P as the feasible region. In Figure 10.9, we show two polyhedra. In each polyhedron P, the hyperplane $\{\boldsymbol{y} : \boldsymbol{c}^\top \boldsymbol{y} = \boldsymbol{c}^\top \boldsymbol{v}\}$ on the right-hand side touches P at a single point, $\boldsymbol{v}$, so the point $\boldsymbol{v}$ is a vertex. In contrast, the point $\boldsymbol{w}$ is not a vertex since there is no hyperplane intersecting solely at $\boldsymbol{w}$ within P.

Definition 10.3 Let P be a nonempty polyhedron. A vector $\boldsymbol{x} \in P$ is called an extreme point of P if there are no $\boldsymbol{y}, \boldsymbol{z} \in P$ and a scalar $\lambda \in (0, 1)$ such that $\boldsymbol{x} = \lambda \boldsymbol{y} + (1 - \lambda)\boldsymbol{z}$.

In Figure 10.10, we show three polyhedra. In each polyhedron, the vectors $\boldsymbol{v}_i$'s are extreme points, and the vector $\boldsymbol{w}$ is not an extreme point because $\boldsymbol{w}$ is a convex combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Definitions 10.2 and 10.3 are geometric, and hence intuitive. We need an equivalent definition that is algebraic, so that we can do computations. Before this, we need an intermediate concept for connecting geometry and algebra.

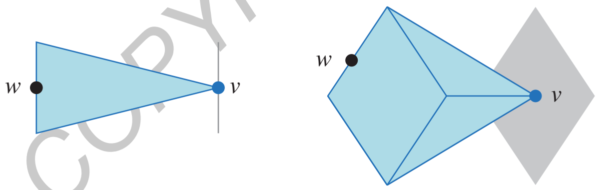


Figure 10.9 Vertices ($\boldsymbol{v}$'s) versus nonvertices ($\boldsymbol{w}$'s).

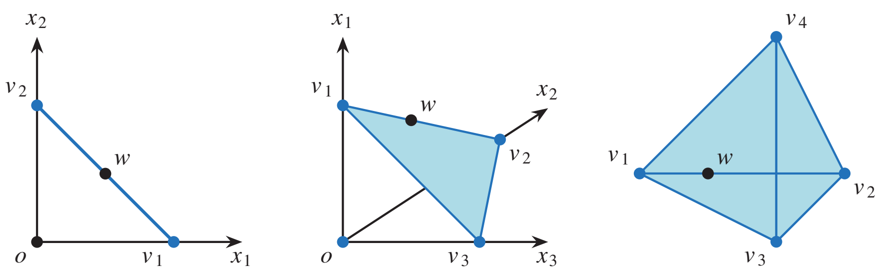


Figure 10.10 Extreme points ($\boldsymbol{v}_i$'s) versus nonextreme points ($\boldsymbol{w}$'s).

Definition 10.4 If a vertex $\boldsymbol{x}^\star$ satisfies an inequality $\boldsymbol{a}^\top \boldsymbol{x} \geq b$ (or $\boldsymbol{a}^\top \boldsymbol{x} \leq b$) as an equality, that is, $\boldsymbol{a}^\top \boldsymbol{x}^\star = b$, then we say that this inequality is active or binding at $\boldsymbol{x}^\star$.

If $P \subset \mathbb{R}^n$ is a polyhedron defined by linear equality and inequality constraints, then $\boldsymbol{x}^\star \in \mathbb{R}^n$ may or may not be feasible with respect the constraints. Now, if $\boldsymbol{x}^\star \in \mathbb{R}^n$ is feasible (i.e., $\boldsymbol{x}^\star \in P$; satisfying all the constraints), then from Definition 10.4 all the equality constraints are active at $\boldsymbol{x}^\star$.

We have the following example to more illustrate Definition 10.4.

Example 10.17 The polyhedron shown in the middle of Figure 10.10 is expressed as

$$P = \{(x_1, x_2, x_3)^\top : x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0\}. \tag{10.6}$$

There are three constraints that are binding at each of the points $\boldsymbol{v}_1, \boldsymbol{v}_2$, and $\boldsymbol{v}_3$. Namely the constraints $x_1 + x_2 + x_3 = 1, x_2 = 0$ and $x_3 = 0$ are active at $\boldsymbol{v}_1$, the constraints $x_1 + x_2 + x_3 = 1$, $x_1 = 0$ and $x_3 = 0$ are active at $\boldsymbol{v}_2$, and the constraints $x_1 + x_2 + x_3 = 1$, $x_1 = 0$ and $x_2 = 0$ are active at $\boldsymbol{v}_3$. Also, at the point $\boldsymbol{w}$, there are two constraints that are binding, which are $x_1 + x_2 + x_3 = 1$ and $x_3 = 0$. ■

If there are n constraints that are binding at a vector $\boldsymbol{x}^\star \in \mathbb{R}^n$, then $\boldsymbol{x}^\star$ satisfies a system of n linear equations in n unknowns. In view of Theorem 3.13, this system has a unique solution if and only if these n equations are linearly independent.

Now, we are ready to introduce the algebraic definition of a corner point.

Definition 10.5 Let P be a polyhedron defined by linear equality and inequality constraints, and $\boldsymbol{x}^\star \in \mathbb{R}^n$. We say that the vector

(a) $\boldsymbol{x}^\star$ is a basic solution if the following two statements hold:
   (i) All equality constraints are active.
   (ii) Out of the constraints that are active at $\boldsymbol{x}^\star$, there are n of them that are linearly independent.
(b) $\boldsymbol{x}^\star$ is a basic feasible solution if it is a basic solution, and satisfies all of the constraints (i.e., $\boldsymbol{x}^\star \in P$).

The following two examples illustrate Definition 10.5.

Example 10.18
In the polyhedron depicted in the middle of Figure 10.10, as represented in (10.6), we can identify the points $\boldsymbol{v}_i$'s as basic feasible solutions. However, point $\boldsymbol{o}$ fails to meet the equality constraint $x_1 + x_2 + x_3 = 1$, making it ineligible as a basic solution. On the other hand, point $\boldsymbol{w}$ is feasible but does not qualify as basic according to Definition 10.5. Nevertheless, if we replace the equality constraint $x_1 + x_2 + x_3 = 1$ with the inequality constraints $x_1 + x_2 + x_3 \leq 1$, then $\boldsymbol{o}$ transforms into a basic feasible solution, as shown in Figure 10.11.

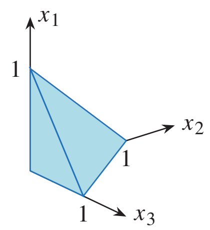


Figure 10.11 The polyhedron given in Example 10.18 with four corners.

■

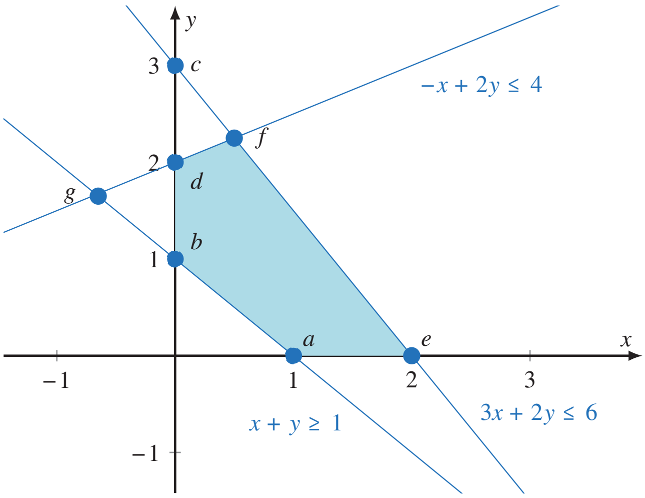


Figure 10.12 Basic solutions and basic feasible solutions.

Example 10.19 In Figure 10.12, the points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$, and $\boldsymbol{g}$ all represent basic solutions since they each have two linearly independent constraints that are active. Specifically, points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}$, and $\boldsymbol{f}$ are considered basic feasible solutions as they fulfill all imposed constraints. ■

We give, without proof, the following result in this context. For a proof, see, for example, Bertsimas and Tsitsiklis (1997).

Theorem 10.1 Let $\boldsymbol{x}^\star$ be a point in a nonempty polyhedron P. Then the following are equivalent:

(a) $\boldsymbol{x}^\star$ is a vertex.
(b) $\boldsymbol{x}^\star$ is an extreme point.
(c) $\boldsymbol{x}^\star$ is a basic feasible solution.

Definition 10.6 Two distinct basic solutions to a set of linear constraints in $\mathbb{R}^n$ are called adjacent if there are $n-1$ linearly independent constraints that are binding at both of them.

As an example, in Figure 10.12, the points $\boldsymbol{a}$ and $\boldsymbol{g}$ are adjacent to the point $\boldsymbol{b}$, and the points $\boldsymbol{d}$ and $\boldsymbol{e}$ are adjacent to $\boldsymbol{f}$.

In the subsequent development, we will see that we find an optimal corner point of an LP problem by moving from one basic feasible solution to an adjacent basic feasible solution that improves the objective function value, and so on, repeating this step until we cannot go to an adjacent basic feasible solution that improves the objective function value.

Let n and m be positive integers such that $m \leq n$. Let also $\boldsymbol{b} \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = m$ (i.e., A has a full-row rank). The set $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is called a polyhedron in standard form. Note that the number of equality constraints in P is m.

10.4.2 Finding Basic Feasible Solutions

The question that arises now is, how to find basic solutions of polyhedra in standard form? The system $A\boldsymbol{x} = \boldsymbol{b}$ gives m linearly independent constraints as $\text{rank}(A) = m$. Consequently, we need $n - m$ more binding constraints from $\boldsymbol{x} \geq \boldsymbol{0}$ (this is n nonnegativity constraints: $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$). Which $n - m$ (out of those n) constraints to select for our purpose? We cannot choose any $(n - m)$ x_i's. Theorem 10.2 helps in this task. Before this theorem, we give some definitions.

As a matter of notation, we use ";" for adjoining vectors and matrices in a column, and use "," or ":" for adjoining them in a row.

We write A as $A = [\boldsymbol{a}_1 : \boldsymbol{a}_2 : \cdots : \boldsymbol{a}_n]$ where $\boldsymbol{a}_j$ is the jth column of A. Since $\text{rank}(A) = m$, there exists an invertible matrix

$$A_B \triangleq [\boldsymbol{a}_{B_1} : \boldsymbol{a}_{B_2} : \cdots : \boldsymbol{a}_{B_m}] \in \mathbb{R}^{m \times m}. \tag{10.7}$$

Let $B \triangleq \{B_1, B_2, \dots, B_m\}$ and $N \triangleq \{1, 2, \dots, n\} - B$. We can permute the columns of A so that $A = [A_B : A_N]$. We can write the system $A\boldsymbol{x} = \boldsymbol{b}$ as $A_B\boldsymbol{x}_B + A_N\boldsymbol{x}_N = \boldsymbol{b}$ where $\boldsymbol{x} = (\boldsymbol{x}_B; \boldsymbol{x}_N)$ (equivalently, $\boldsymbol{x}^\mathsf{T} = (\boldsymbol{x}_B^\mathsf{T}, \boldsymbol{x}_N^\mathsf{T})$).

Definition 10.7 The $m \times m$ nonsingular matrix A_B is called a basis matrix. The vector $\boldsymbol{x}_B$ is called a basic solution (also called the vector of basic variables). The vector $\boldsymbol{x}_N$ is called a nonbasic solution (also called the vector of nonbasic variables).

We are now ready to state the following theorem which will be given without proof. For a proof, see, for example, Bertsimas and Tsitsiklis (1997).

Theorem 10.2 Let $\boldsymbol{b} \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ have linearly independent rows. Consider the constraints $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. A vector $\boldsymbol{x} \in \mathbb{R}^n$ is a basic solution if and only if we have

(a) The columns of A_B are linearly independent.
(b) $\boldsymbol{x}_N = \boldsymbol{0}$.

Since A_B is nonsingular, we can solve the system of m linear equations $A\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{x}_B$. The solution is given by $\boldsymbol{x}_N = \boldsymbol{0}$ and $\boldsymbol{x}_B = A_B^{-1}\boldsymbol{b}$. The three-step procedure in the following workflow, followed by an example, will teach us how to construct such basic solutions.

Workflow 10.4 We construct all basic solutions to a standard form polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ by following three steps:

(i) Choose m linearly independent columns $\boldsymbol{a}_{B(1)}, \boldsymbol{a}_{B(2)}, \dots, \boldsymbol{a}_{B(m)}$.
(ii) Set $\boldsymbol{x}_N = \boldsymbol{0}$.
(iii) Calculate $\boldsymbol{x}_B = A_B^{-1}\boldsymbol{b}$. If $\boldsymbol{x}_B \geq \boldsymbol{0}$, then the vector $\boldsymbol{x} = (\boldsymbol{x}_B; \boldsymbol{x}_N)$ is a basic feasible solution. Otherwise, $\boldsymbol{x} = (\boldsymbol{x}_B; \boldsymbol{x}_N)$ is a basic solution.

It is clear that the maximum number of basic feasible solutions is $\binom{n}{m}$. Note that, generally, not all of $\binom{n}{m}$ choices of m columns may produce a basis (i.e., a nonsingular matrix A_B).

Hence, the number of basic solutions may be smaller than $\binom{n}{m}$. Note also that not all of these $\binom{n}{m}$ bases may lead to basic feasible solutions.

In the following example, which is due to Krishnamoorthy (2023a), we have that $n = 5$ and $m = 3$, and that each of $\binom{5}{3} = 10$ choices produces a basic solution.

Example 10.20 Consider the linear system

$$\begin{aligned} x_1 + x_2 &\geq 2, \\ 3x_1 + x_2 &\geq 4, \\ 3x_1 + 2x_2 &\leq 10, \\ x_1, \quad x_2 &\geq 0. \end{aligned} \tag{10.8}$$

The resulting polyhedron is shown in Figure 10.13. In the standard form, we have

$$\begin{aligned} x_1 + x_2 - x_3 &= 2, \\ 3x_1 + x_2 - x_4 &= 4, \\ 3x_1 + 2x_2 + x_5 &= 10, \\ x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5 &\geq 0. \end{aligned}$$

Consequently, the following arrays draw the resulting polyhedron.

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 3 & 1 & 0 & -1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}.$$

The columns of A are

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \ \boldsymbol{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \ \boldsymbol{a}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{a}_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \boldsymbol{a}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

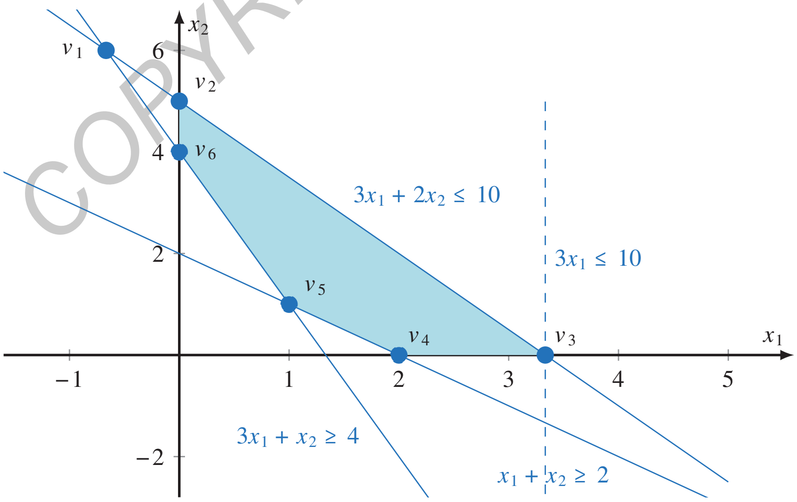


Figure 10.13 The polyhedron in Example 10.20.

Choosing $B = \{1, 2, 3\}$ (hence $N = \{4, 5\}$) gives

$$A_B = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}, \text{ and } \det(B) = -3 \neq 0 \text{ (hence } B \text{ is invertible).}$$

Let $\boldsymbol{x}_N = (x_4; x_5) = (0; 0)$. Finding A_B^{-1} and calculating $\boldsymbol{x}_B = A_B^{-1}\boldsymbol{b}$, we get

$$\boldsymbol{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 6 \\ 10/3 \end{bmatrix}.$$

This point is a basic solution, but it is not a basic feasible solution because not all entries are nonnegative. This point corresponds to the vertex $\boldsymbol{v}_1 = (-2/3; 6)$.

Choosing $B = \{2, 3, 4\}$ (hence $N = \{1, 5\}$) gives

$$A_B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 0 & 0 \end{bmatrix}, \text{ and } \det(B) = 2 \neq 0 \text{ (hence } B \text{ is invertible).}$$

Let $\boldsymbol{x}_N = (x_1; x_5) = (0; 0)$. Finding A_B^{-1} and calculating $\boldsymbol{x}_B = A_B^{-1}\boldsymbol{b}$, we get

$$\boldsymbol{x}_B = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}.$$

Thus $\boldsymbol{x} = (0; 5; 3; 1; 0)$. This point is a basic feasible solution because all the entries are nonnegative. This point corresponds to the vertex $\boldsymbol{v}_2 = (0; 5)$. Table 10.2 summarizes the correspondences between the basic feasible solutions in the standard form polyhedron and the vertices visualized in Figure 10.13. ■

We want to emphasize that we can identify the basic feasible solution within the standard form polyhedron for each corner point through a straightforward examination. In simpler terms, there is no need to systematically go through all possible combinations, such as the $\binom{n}{m}$ choices for bases. Take, for instance, Example 10.20, where at vertex $\boldsymbol{v}_5$, the constraints $x_1 + x_2 \geq 2$ and $3x_1 + x_2 \geq 4$ are active, while $3x_1 + 2x_2 \leq 10$ is not. Consequently,

Table 10.2 Correspondences between the basic feasible solutions in the standard form polyhedron and the vertices visualized in Figure 10.13

Vertex	B	$\det(A_B)$	The variable $\boldsymbol{x}$	Basic feasible solution?
$\boldsymbol{v}_1$	$\{1, 2, 3\}$	-3	$(-2/3; 6; 10/3; 0; 0)$	×
$\boldsymbol{v}_2$	$\{2, 3, 4\}$	2	$(0; 5; 3; 1; 0)$	✓
$\boldsymbol{v}_3$	$\{1, 3, 4\}$	3	$(10/3; 0; 4/3; 6; 0)$	✓
$\boldsymbol{v}_4$	$\{1, 4, 5\}$	-1	$(2; 0; 0; 2; 4)$	✓
$\boldsymbol{v}_5$	$\{1, 2, 5\}$	-2	$(1; 1; 0; 0; 5)$	✓
$\boldsymbol{v}_6$	$\{2, 3, 5\}$	1	$(0; 4; 2; 0; 2)$	✓

in the corresponding basic feasible solutions within the standard form polyhedron, we have $x_3 = x_4 = 0$ and $x_5 > 0$. Additionally, both x_1 and x_2 are strictly positive. Therefore, $\boldsymbol{x}_B = (x_1; x_2; x_5)$ forms the corresponding basis.

Likewise, consider vertex $\boldsymbol{v}_2$ where the constraints $x_1 + x_2 \geq 2$ and $3x_1 + x_2 \geq 4$ are not binding, but the constraints $3x_1 + 2x_2 \leq 10$ and $x_1 \geq 0$ are active. Consequently, in the corresponding basic feasible solutions within the standard form polyhedron, we find that x_3 and x_4 are both greater than 0, while x_5 equals 0. Additionally, it is worth noting that x_5 is strictly positive. Therefore, $\boldsymbol{x}_B = (x_2; x_3; x_4)$ forms the corresponding basis.

10.4.2.1 Degeneracy

At a basic solution, we must have n linearly independent active constraints. However, since no more than n constraints can be linearly independent in an nth-dimensional space, it is possible for more than n active constraints to exist at a basic solution. In such cases, this basic solution is referred to as degenerate.

Definition 10.8 A basic solution $\boldsymbol{x} \in \mathbb{R}^n$ is called degenerate if more than n of the constraints are active at $\boldsymbol{x}$. In a nonempty polyhedron in standard form with $A \in \mathbb{R}^{m \times n}$, $\boldsymbol{x}$ is a degenerate basic solution if more than $n - m$ of the components of $\boldsymbol{x}$ are zero.

Example 10.21 (Example 10.20 revisited)
Adding the constraint $x_1 \leq 10/3$ to System (10.8) results in three active constraints at $\boldsymbol{v}_3$ (see Figure 10.13). Therefore, $\boldsymbol{v}_3$ qualifies as a degenerate basic feasible solution. In the standard form, this constraint becomes $x_1 + x_6 = 10/3$, where x_6 represents the slack variable for the constraint $x_1 \leq 10/3$. In this case, with $n = 6$ and $m = 4$, and with $x_2 = x_5 = x_6 = 0$, we have more than $n - m = 2$ components of $\boldsymbol{x}$ equal to zero. ■

Degeneracy might not pose significant issues in small-scale problems, but it can introduce inefficiencies when dealing with large LP instances. In typical algorithms, the goal is to transition from one basic feasible solution to another nearby solution that either improves the objective function or, at the very least, maintains the current value. However, in the presence of degeneracy, the algorithm may cycle through several degenerate basic feasible solutions before finally reaching a vertex that genuinely enhances the objective function value. The extent of degeneracy largely hinges on how we represent the polyhedron. For instance, in Example 10.21, we could eliminate the constraint $x_1 \leq 10/3$ without altering the polyhedron, thereby resolving the degeneracy issue at $\boldsymbol{v}_3$. Additionally, if permissible, we could circumvent degeneracy by making minor adjustments to certain constraints, such as replacing $x_1 \leq 10/3$ with $x_1 \leq 10/3 + 0.001$. However, the feasibility of such modifications heavily relies on the specific problem application.

10.4.3 Pointedness

A polyhedron is pointed if it contains no lines (a line is a straight one-dimensional figure formed when two points are connected with minimum distance between them, and both the ends extended to infinity). Figure 10.14 shows two polyhedra, one of them (namely P_1)

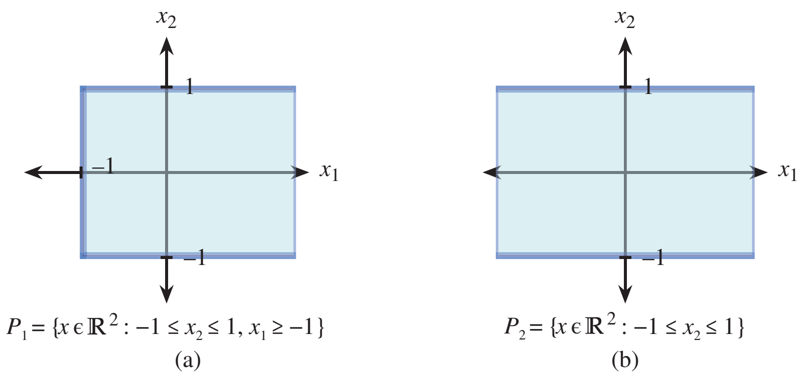


Figure 10.14 Pointed polyhedron (a) versus nonpointed polyhedron (b).

is pointed but the other (namely P_2) is nonpointed. Note that every nonempty polyhedron subset of a pointed polyhedron is pointed.

A good question to ask: Is every nonempty polyhedron pointed? We give the following theorem without proof. For a proof, see, for example, Bertsimas and Tsitsiklis (1997).

Theorem 10.3 Assume that the polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_i^\top \boldsymbol{x} \ge b_i, i = 1, \dots, m\}$ is nonempty. Then the following are equivalent:

(a) The polyhedron P is pointed.
(b) The polyhedron P has at least one extreme point.
(c) There exist n vectors out of the family $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$, which are linearly independent.

Note that, from Theorem 10.3, a bounded polyhedron is pointed. Similarly, the nonnegative orthant cone $\mathbb{R}^n_+ \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \ge \boldsymbol{0}\}$ is pointed. Since any standard form polyhedron is a subset of the nonnegative orthant cone, it is pointed too. The following two corollaries are now immediate.

Corollary 10.1 Every nonempty bounded polyhedron has at least one basic feasible solution.

Corollary 10.2 Every nonempty polyhedron in standard form has at least one basic feasible solution.

Note also that, from Theorem 10.3, every nonempty polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \ge \boldsymbol{b}\}$, with $A \in \mathbb{R}^{m \times n}$ and $m < n$, cannot have any basic feasible solution.

10.4.4 Optimality

In the above part, we have established the conditions for the existence of extreme points. In this part, we will see that if a nonempty polyhedron P has no corner points, then the

LP problem of minimizing a linear objective function over P cannot have a unique optimal solution. The following theorem presents the contrapositive of this statement.

Theorem 10.4 Consider the LP problem over a polyhedron P. If P has at least one extreme point and there exists an optimal solution, then there exists an extreme point of P which is optimal.

Proof: Let $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \geq \boldsymbol{b}\}$, and v be the optimal value of the cost $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ which we have assumed to be attained. Then $P_{\text{opt}} \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{c}^\mathsf{T}\boldsymbol{x} = v\}$ contains all optimal solutions in P. By assumption, P_{opt} is a nonempty polyhedron. From Theorem 10.3, P is pointed. Since $P_{\text{opt}} \subset P$, P_{opt} is pointed too. Using Theorem 10.3 again, P_{opt} has an extreme point, say $\boldsymbol{x}^\star$. Since $\boldsymbol{x}^\star \in P_{\text{opt}}$, we have $\boldsymbol{c}^\mathsf{T}\boldsymbol{x}^\star = v$, that is, $\boldsymbol{x}^\star$ is optimal. To complete the proof, it remains to show that $\boldsymbol{x}^\star$ is an extreme point of P.

Suppose, in the contrary, that $\boldsymbol{x}^\star$ is not an extreme point of P. Then, there exist $\boldsymbol{y}, \boldsymbol{z} \in P$ and a scalar $\lambda \in (0, 1)$ such that $\boldsymbol{x}^\star = \lambda\boldsymbol{y} + (1 - \lambda)\boldsymbol{z}$. Consequently, $v = \boldsymbol{c}^\mathsf{T}\boldsymbol{x}^\star = \lambda\boldsymbol{c}^\mathsf{T}\boldsymbol{y} + (1 - \lambda)\boldsymbol{c}^\mathsf{T}\boldsymbol{z}$. Furthermore, since v is the optimal cost, $\boldsymbol{c}^\mathsf{T}\boldsymbol{y} \geq v$ and $\boldsymbol{c}^\mathsf{T}\boldsymbol{z} \geq v$. It follows that $\boldsymbol{c}^\mathsf{T}\boldsymbol{y} = \boldsymbol{c}^\mathsf{T}\boldsymbol{z} = v$, and therefore $\boldsymbol{y}, \boldsymbol{z} \in P_{\text{opt}}$. But this contradicts the fact that $\boldsymbol{x}^\star$ is an extreme point of P_{opt}. Thus, $\boldsymbol{x}^\star$ is an extreme point of P. The proof is complete. ■

A more general result than that in Theorem 10.4 is stated in the following theorem, which will be given without proof. For a proof, see, for example, Bertsimas and Tsitsiklis (1997).

Theorem 10.5 Consider the LP problem over a polyhedron P. If P has at least one extreme point, then either the optimal cost is equal to $-\infty$, or there exists an extreme point of P which is optimal.

Theorems 10.4 and 10.5 specifically address polyhedra under the condition that they possess at least one extreme point. But what about polyhedra that don not satisfy this condition? Interestingly, any LP problem, whether dealing with a polyhedron with or without extreme points, can be converted into an equivalent problem in standard form. This transformation enables us to apply Theorem 10.5, as highlighted in Corollary 10.2. This insight leads to the following corollary.

Corollary 10.3 Consider the linear minimization problem over a nonempty polyhedron P. Then either the optimal cost is equal to $-\infty$, or there exists an optimal solution.

Generally, Corollary 10.3 does not hold for nonlinear programming problems. For example, the nonlinear optimization problem

$$\begin{aligned} &\min \;\; 1/x \\ &\text{s.t.} \quad x \geq 1, \end{aligned}$$

has no optimal solution, but the optimal cost is not $-\infty$.

10.5 The Simplex Method

We have introduced linear optimization problems and studied its geometry. Now, we are ready to introduce the simplex method. The word "simplex" is a general term of LP feasible region. Simplex method is used to solve LPs with any number of variables and constraints. The idea behind this method is to move from one basic feasible solution to an adjacent basic feasible solution so that the objective function value improves.

10.5.1 Simplex Method for Maximization

We begin by outlining the simplex method for solving LP problems with a focus on maximization.

The six-step procedure in Workflow 10.5, followed by Example 10.22, will teach us how to apply the simplex method for solving the maximization problem. First, we need the following definition.

Definition 10.9 Let A be an $m \times n$ matrix. Consider the standard form LP:

$$\begin{array}{ll} \max & z = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s.t.} & A\mathbf{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \mathbf{0}. \end{array} \tag{10.9}$$

The corresponding canonical form is linear system of $(m+1)$ equations:

$$\begin{aligned} z - \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} &= 0, \\ A\boldsymbol{x} &= \boldsymbol{b}. \end{aligned}$$

For example, the canonical form corresponding to the standard form LP:

$$\begin{array}{ll} \max & z = 2x_1 + 3x_2 \\ \text{s.t.} & x_1 + 2x_2 = 4, \\ & 2x_1 + x_2 + x_3 = 8, \\ & x_1, x_2, x_3 \geq 0, \end{array}$$

is the system

$$\begin{aligned} z - 2x_1 - 3x_2 &= 0, \\ x_1 + 2x_2 &= 4, \\ 2x_1 + x_2 + x_3 &= 8. \end{aligned}$$

Workflow 10.5 (The simplex method) We solve a maximization LP problem by following five steps:

(i) Write the given LP in the standard form.
(ii) Convert the standard form to a canonical form.
(iii) Find a basic feasible solution for the canonical form.

(iv) If the current basic feasible solution is optimal, stop. If not, find which basic variable must become nonbasic and which nonbasic variable must become basic, and apply elementary row operations in order to move to an adjacent basic feasible solution with a higher value for the objective function.
(v) Go to Step (*iv*).

For guidance on determining which variables should change from basic to nonbasic (or vice versa) and on assessing the optimality, refer to Remarks 10.1–10.3 below. The examples in this section, except the last two, are due to Krishnamoorthy (2023b).

Example 10.22 Use the simplex method to solve the following maximization LP.

$$\begin{array}{lll} \max & z = 2x_1 + 3x_2 & \\ \text{s.t.} & x_1 + 2x_2 & \leq 6, \\ & 2x_1 + x_2 & \leq 8, \\ & x_1, x_2 & \geq 0. \end{array} \tag{10.10}$$

Solution
We apply the steps in Workflow 10.5. Problem (10.10) in standard form is written as

$$\begin{array}{lll} \max & z = 2x_1 + 3x_2 & \\ \text{s.t.} & x_1 + 2x_2 + s_1 & = 6, \\ & 2x_1 + x_2 + s_2 & = 8, \\ & x_1, x_2, s_1, s_2 & \geq 0. \end{array} \tag{10.11}$$

Problem (10.11) in the canonical form is written as

$$\begin{array}{rl} z - 2x_1 - 3x_2 & = 0, \\ x_1 + 2x_2 + s_1 & = 6, \\ 2x_1 + x_2 \quad + s_2 & = 8. \end{array} \tag{10.12}$$

The canonical variables, which correspond to the unit columns, are the variables z, s_1 and s_2. Let BV denote the set of the basic variables. We select BV $= \{z, s_1, s_2\}$. Generally, we have $|\text{BV}| = m + 1$ ($m = 2$ in this example), and we choose $z \in$ BV always. Therefore, the BV contains the variable z plus m canonical variables. Let NBV denote the set of the nonbasic variables. Then NBV $= \{x_1, x_2\}$.

Fix $x_1 = x_2 = 0$. System (10.12) now reads $z = 0$, $s_1 = 6$, and $s_2 = 8$, which are a basic feasible solution.

Now, let us determine if the current basic feasible solution is optimal. An optimal solution is reached when we cannot further improve the value of z by increasing the value of any nonbasic variable (starting from zero). Currently, $z = 2x_1 + 3x_2 = 0$ as $x_1 = x_2 = 0$ (NBV $= \{x_1, x_2\}$). Increasing x_1 from 0 to 1 increases z from 0 to 2, while increasing x_2 from 0 to 1 increases z from 0 to 3 (we are increasing one variable at a time, while keeping the other nonbasic variable fixed at zero). It is more beneficial to increase x_2 here than x_1. Generally, we select the nonbasic variable with the largest positive coefficient in the z expression to enter the basis. In the canonical form, we choose the nonbasic variable with the most negative coefficient in row-0 to enter the basis. The following remark summarizes this discussion. We will move forward in solving Example 10.22 after this remark. ∎

Remark 10.1 (Criterion for choosing the entering variable in maximization) The entering variable in a maximization LP problem is the nonbasic variable having the most negative coefficient in the z-row.

Considering Remark 10.1, we designate x_2 as the entering variable in this step.

With the entering variable identified, our next task is to find a new neighboring basic feasible solution by also determining a leaving variable. Note that we cannot increase the entering variable, x_2, without bounds. As x_2 increases, s_1 or s_2 may decrease, and we must ensure that they remain nonnegative to maintain feasibility.

From row 1 and row 2, with $x_1 = 0$, we get

$$\text{Row 1: } 2x_2 + s_1 = 6, \quad \text{which implies} \quad s_1 = 6 - 2x_2 \geq 0,$$

$$\text{Row 2: } x_2 + s_2 = 8, \quad \text{which implies} \quad s_2 = 8 - x_2 \geq 0.$$

Note that s_1 and s_2 need to be nonnegative for feasibility. To keep $s_1 \geq 0$, we cannot increase x_2 beyond 6/2=3. To keep $s_2 \geq 0$, we cannot increase x_2 beyond 8/1=8.

Thus, we let $x_2 = 3$ which makes $s_1 = 0$. In this step, x_2 is called the entering variable, and s_1 is called the leaving variable.

The test in the following remark summarizes the above discussion. Applying this test guarantees that the basic solution remains feasible. We will move forward in solving Example 10.22 after this remark.

Remark 10.2 (Minimum ratio test for choosing the leaving variable) For each constraint row that has a positive coefficient[1] for the entering variable, we compute the ratio:

$$\text{Ratio} = \frac{\text{Right-hand side of the row}}{\text{Coefficient of entering variable in the row}}.$$

Among all these ratios, the nonbasic variable with smallest nonnegative ratio is the leaving variable.

Note that the smallest among all the ratios computed in Remark 10.2 is the largest value that the entering variable can take. Going back to Example 10.22, the ratios are:

$$\text{Row 1: } \tfrac{6}{2} = 3; \leftarrow \text{ The winner!}$$

$$\text{Row 2: } \tfrac{8}{1} = 8.$$

Therefore, s_1 leaves the basis, that is, it becomes nonbasic, and the entering variable x_2 takes its place.

1 We do not consider the row(s) with negative coefficients.

We use elementary row operations in order to make the entering variable basic in the row that the minimum ratio test meets the requirement outlined in Remark 10.2.

$$
\begin{array}{rl}
R_0 : & z - 2x_1 - 3x_2 = 0, \\
R_1 : & x_1 + 2x_2 + s_1 = 6, \\
R_2 : & 2x_1 + x_2 + s_2 = 8; \\
\hline
\frac{3}{2}R_1 + R_0 \to R_0 : & z - \frac{1}{2}x_1 + \frac{3}{2}s_1 = 9, \\
\frac{1}{2}R_1 \to R_1 : & \frac{1}{2}x_1 + x_2 + \frac{1}{2}s_1 = 3, \\
\frac{-1}{2}R_1 + R_2 \to R_2 : & \frac{3}{2}x_1 - \frac{1}{2}s_1 + s_2 = 5; \\
\hline
R_0 + \frac{1}{3}R_2 \to R_0 : & z + \frac{4}{3}s_1 + \frac{1}{3}s_2 = \frac{32}{3}, \\
R_1 - \frac{1}{3}R_2 \to R_1 : & x_2 + \frac{2}{3}s_1 - \frac{1}{3}s_2 = \frac{4}{3}, \\
\frac{2}{3}R_2 \to R_2 : & x_1 - \frac{1}{3}s_1 + \frac{2}{3}s_2 = \frac{10}{3}.
\end{array}
$$

We will complete the resolution of Example 10.22 once we address the following remark.

Remark 10.3 (Criterion for optimality in maximization) In a maximization LP problem, the optimum is reached at the iteration where all the z-row coefficient of the non-basic variables are nonnegative.

Note that in our example we cannot improve the value of z anymore (by making s_1 or s_2 basic). Hence, we have an optimal solution. The optimal solution is $(x_1; x_2) = (10/3; 4/3)$ with the optimal value $z = 32/3$. ■

10.5.2 The Full Tableau Method

The full tableau method provides a more convenient approach for conducting the necessary calculations required by the simplex method.

10.5.2.1 Simplex Tableau for Maximization

If we have a maximization problem, the structure of the simplex tableau is as follows:

z	$\boldsymbol{x}$	rhs	
1	$\boldsymbol{c}^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}} A_B^{-1} A$	$-\boldsymbol{c}_B^{\mathrm{T}} A_B^{-1} \boldsymbol{b}$	
$\boldsymbol{0}$	$A_B^{-1} A$	$A_B^{-1} \boldsymbol{b}$	$= \boldsymbol{x}_B$

Here A_B is defined in (10.7) and $\boldsymbol{c}_B$ is the cost vector corresponding to the basic variables. We keep maintaining and updating the above table till we reach the optimality. Example 10.23 resolves Example 10.22 using the simplex tableau method.

Example 10.23 (Example 10.22 revisited)
Use the simplex tableau method to solve the following maximization problem.

$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 6, \\ & 2x_1 + x_2 \leq 8, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solution
After introducing slack variables, we obtain the standard form problem given in (10.11). Note that $\boldsymbol{x} = (0, 0, 6, 8)$ is a basic feasible solution. Hence, we have the following initial tableau:

z	x_1	x_2	s_1	s_2	rhs	
1	−2	−3	0	0	0	
0	1	(2)	1	0	6	$= s_1$
0	2	1	0	1	8	$= s_2$

Since we are maximizing the objective function, we select a nonbasic variable with the greatest positive reduced cost to be the one that enters the basis. Indicating the pivot element with a circled number, we obtain the following tableau:

z	x_1	x_2	s_1	s_2	rhs	
1	$-1/2$	0	$3/2$	0	9	
0	$1/2$	1	$1/2$	0	3	$= x_2$
0	(3/2)	0	$-1/2$	1	5	$= s_2$

Note that we brought x_2 into the basis and s_1 exited. We then bring x_1 into the basis; s_2 exits and we obtain the following tableau:

z	x_1	x_2	s_1	s_2	rhs	
1	0	0	$4/3$	$1/3$	$32/3$	$= z$
0	0	1	$2/3$	$-1/3$	$4/3$	$= x_2$
0	1	0	$-1/3$	$2/3$	$10/3$	$= x_1$

The reduced costs in the zeroth row of the tableau are all nonnegative, so the current basic feasible solution is optimal. In terms of the original variables x_1 and x_2, this solution is $\boldsymbol{x} = (10/3; 4/3)$. The optimal value is $z = 32/3$.

The above series of tableaux can be combined in one single table as follows:

EROs	z	x_1	x_2	s_1	s_2	rhs	MR
R_0 :	1	−2	−3	0	0	0	
R_1 :	0	1	(2)	1	0	6	$6/2$
R_2 :	0	2	1	0	1	8	$8/1$
$R_0 + \frac{3}{2}R_1 \to R_0$:	1	$-1/2$	0	$3/2$	0	9	
$\frac{1}{2}R_1 \to R_1$:	0	$1/2$	1	$1/2$	0	3	$3/0.5$
$\frac{-1}{2}R_1 + R_2 \to R_2$:	0	(3/2)	0	$-1/2$	1	5	$5/1.5$
$R_0 + \frac{1}{3}R_2 \to R_0$:	1	0	0	$4/3$	$1/3$	$32/3$	
$R_1 - \frac{1}{3}R_2 \to R_1$:	0	0	1	$2/3$	$-1/3$	$4/3$	Optimal
$\frac{2}{3}R_2 \to R_2$:	0	1	0	$-1/3$	$2/3$	$10/3$	tableau!

■

Example 10.24 Use the simplex method to solve the following LP.

$$\begin{array}{lll} \max & z = 2x_1 - x_2 + x_3 & \\ \text{s.t.} & 3x_1 + x_2 + x_3 \le 60, & \text{(adding } s_1) \\ & x_1 - x_2 + 2x_3 \le 10, & \text{(adding } s_2) \\ & x_1 + x_2 - x_3 \le 20, & \text{(adding } s_3) \\ & x_1, x_2, x_3 \ge 0. & \end{array}$$

Solution
We have the following tableaux:

EROs	z	x_1	x_2	x_3	s_1	s_2	s_3	rhs	MR
R_0 :	1	−2	1	−1	0	0	0	0	
R_1 :	0	3	1	1	1	0	0	60	60/3
R_2 :	0	①	−1	2	0	1	0	10	10/1
R_3 :	0	1	1	−1	0	0	1	20	20/1
$R_0 + 2R_2 \to R_0$:	1	0	−1	3	0	2	0	20	
$R_1 - 3R_2 \to R_1$:	0	0	4	−5	1	−3	0	30	30/4
$R_2 \to R_2$:	0	1	−1	2	0	1	0	10	
$-R_2 + R_3 \to R_3$:	0	0	②	−3	0	−1	1	10	10/2
$R_0 + \frac{1}{2}R_3 \to R_0$:	1	0	0	3/2	0	3/2	1/2	25	
$R_1 - 2R_3 \to R_1$:	0	0	0	1	1	−1	−2	10	Optimal
$R_2 + \frac{1}{2}R_3 \to R_2$:	0	1	0	1/2	0	1/2	1/2	15	tableau!
$\frac{1}{2}R_3 \to R_3$:	0	0	1	−3/2	0	−1/2	1/2	5	

The optimal solution is given by $(x_1; x_2; x_3) = (15; 5; 0)$, and the optimal value is $z = 25$. ■

10.5.2.2 Detecting the Existence of Alternative Optimal Solutions

The simplex method can tell if alternative optimal solutions (i.e., infinitely many solutions) exist. The following remark signifies a pivotal insight into LP: A condition that opens the door to alternative optimal solutions that yield the same optimal value for the LP problem.

Remark 10.4 If the coefficient of a nonbasic variable in the zeroth row of the tableau is zero, then the LP problem has alternative optimal solutions.

We have the following example.

Example 10.25 Use the simplex tableau method to solve the following maximization problem.

$$\begin{array}{lll} \max & z = 4x_1 + x_2 & \\ \text{s.t.} & 8x_1 + 2x_2 \le 16, & \text{(adding } s_1) \\ & 5x_1 + 2x_2 \le 12, & \text{(adding } s_2) \\ & x_1, x_2 \ge 0. & \end{array}$$

Solution
We have the following tableaux:

EROs	z	x_1	x_2	s_1	s_2	rhs	MR
R_0 :	1	−4	−1	0	0	0	
R_1 :	0	⑧	2	1	0	16	$16/8 = 2$
R_2 :	0	5	2	0	1	12	$12/5 = 2.4$
$R_0 + \frac{1}{2}R_1 \to R_0$:	1	0	0	1/2	0	8	
$\frac{1}{8}R_1 \to R_1$:	0	1	1/4	1/8	0	2	This tableau
$\frac{-5}{8}R_1 + R_2 \to R_2$:	0	0	(3/4) circled	−5/8	1	2	is optimal!
$R_0 \to R_0$:	1	0	0	1/2	0	8	
$R_1 - \frac{1}{3}R_2 \to R_1$:	0	1	0	1/3	−1/3	4/3	This tableau
$\frac{4}{3}R_2 \to R_2$:	0	0	1	−5/6	4/3	8/3	is also optimal!

In view of Remark 10.4, we have alternative optimal solutions. An optimal solution is given by $(x_1; x_2) = (2; 0)$. Another optimal solution is given by $(x_1; x_2) = (4/3; 8/3)$. The optimal value is $z = 8$. As an exercise for the reader, use the graphical method to reach the same conclusion. ■

10.5.2.3 Detecting Unboundedness

The simplex method can be used to detect the unboundedness. The following remark tells us when we have an unbounded problem.

Remark 10.5 If there is no candidate for the minimum ratio test, then the LP problem is unbounded.

Example 10.26 Use the simplex tableau method to solve the following maximization problem.

$$\begin{array}{lll} \max & z = 2x_2 & \\ \text{s.t.} & x_1 - x_2 \le 4, & \text{(adding } s_1\text{)} \\ & -x_1 + x_2 \le 1, & \text{(adding } s_2\text{)} \\ & x_1, x_2 \ge 0. & \end{array}$$

Solution
We have the following tableaux:

EROs	z	x_1	x_2	s_1	s_2	rhs	MR
R_0 :	1	0	−2	0	0	0	
R_1 :	0	1	−1	1	0	4	$16/8 = 2$
R_2 :	0	−1	①	0	1	1	$12/5 = 2.4$
$R_0 + 2R_2 \to R_0$:	1	−2	0	0	2	2	
$R_1 + R_2 \to R_1$:	0	0	0	1	1	5	The LP is
$R_2 \to R_2$:	0	−1	1	0	1	1	unbounded!

Note that there is no candidate for the minimum ratio test. In view of Remark 10.5, we have an unbounded LP problem. As an exercise for the reader, use the graphical method to reach the same conclusion. ■

10.5.2.4 Breaking Ties

The following remark tells us how to break ties for entering or leaving variables if any.

Remark 10.6 If there are ties for entering or leaving, we can break them arbitrarily.

Later in this section, we will delve into more determined strategies for resolving tie-breakers when it comes to the selection of nonbasic variables. More specifically, we will refer to other remarks (Remarks 10.10 and 10.11) that provide more guidance on breaking ties, not only for the selection of nonbasic variables but also for deciding which variables should enter or leave the set of basic variables. By exploring these tie-breaking strategies, we aim to enhance more clarity and effectiveness of the decision-making process in LP problem-solving.

Example 10.27 Use the simplex tableau method to solve the following maximization problem.

$$\begin{array}{lll} \max & z = x_1 + x_2 & \\ \text{s.t.} & x_1 + x_2 + x_3 \leq 1, & (\text{adding } s_1) \\ & x_1 + 2x_3 \quad\ \leq 1, & (\text{adding } s_2) \\ & x_1, x_2, x_3 \quad \geq 0. & \end{array}$$

Solution

We have the following tableaux:

EROs	z	x_1	x_2	x_3	s_1	s_2	rhs	MR
R_0 :	1	−1	−1	0	0	0	0	
R_1 :	0	①	1	1	1	0	1	1 (A candidate)
R_2 :	0	1	0	2	0	1	1	1 (Another candidate!)
$R_0 + R_1 \to R_0$:	1	0	0	1	1	0	1	Alternative
$R_1 \to R_1$:	0	1	1	1	1	0	1	optimal
$-R_1 + R_2 \to R_2$:	0	0	−1	1	−1	1	0	solutions!

We have alternative optimal solutions. An optimal solution is given by $(x_1; x_2; x_3) = (1; 0; 0)$. The optimal value is $z = 1$. ■

10.5.2.5 Simplex Tableau for Minimization

Up to this point, our exploration has been centered on the simplex method as a means of tackling linear maximization problems. However, it is important to note that this method is versatile enough to be employed for solving linear minimization problems as well. When it comes to linear minimization LP problems, the guidelines pertaining to entering variables and determining optimality are diametrically opposite to those governing maximization LP problems. Further elaboration on these nuances is provided in the subsequent remarks.

Remark 10.7 (Criterion for choosing the entering variable in minimization) The entering variable in a minimization LP problem is the nonbasic variable having the most positive coefficient in the z-row.

Remark 10.8 (Criterion for optimality in minimization) In a minimization LP problem, the optimum is reached at the iteration where all the z-row coefficient of the non-basic variables are nonpositive.

In essence, the above remarks underscore the fundamental differences in approach between solving maximization and minimization LP problems using the simplex method, particularly when it comes to determining which variables to add to or remove from the basis and when to declare a solution as optimal. We have the following examples.

Example 10.28 Use the simplex tableau method to solve the following minimization problem.

$$\begin{array}{lll} \min & z = -x_1 - x_2 & \\ \text{s.t.} & x_1 - x_2 \leq 1, & (\text{adding } s_1) \\ & x_1 + x_2 \leq 2, & (\text{adding } s_2) \\ & x_1, x_2 \geq 0. & \end{array}$$

Solution

We have the following tableaux:

EROs	z	x_1	x_2	s_1	s_2	rhs	MR
R_0 :	1	1	1	0	0	0	
R_1 :	0	①	−1	1	0	1	$1/1 = 1$
R_2 :	0	1	1	0	1	2	$2/1 = 2$
$R_0 - R_1 \to R_0$:	1	0	2	−1	0	−1	
$R_1 \to R_1$:	0	1	−1	1	0	1	
$-R_1 + R_2 \to R_2$:	0	0	②	−1	1	1	$1/2$
$R_0 - (1/2)R_2 \to R_0$:	1	0	0	0	−1	−2	
$R_1 + (1/2)R_2 \to R_1$:	0	1	0	1/2	1/2	3/2	Optimal
$(1/2)R_2 \to R_2$:	0	0	1	−1/2	1/2	1/2	tableau!

The reduced costs in the zeroth row of the tableau are all nonpositive, so the current basic feasible solution is optimal. In terms of the original variables x_1 and x_2, this solution is $\boldsymbol{x} = (3/2; 1/2)$. The optimal value is $z = -2$. In addition, since the coefficient of the non-basic variable s_1 in the zeroth row of the last tableau is zero, we have alternative optimal solutions. ■

Example 10.29 Use the simplex tableau method to solve the following minimization problem.

$$\begin{array}{lllll} \min & z = -x_1 - x_2 & & \\ \text{s.t.} & 2x_1 + x_2 & \leq 4, & \text{(adding } s_1) \\ & 3x_1 + 5x_2 & \leq 15, & \text{(adding } s_2) \\ & x_1, x_2 & \geq 0. & \end{array}$$

Solution

We have the following tableaux:

EROs	z	x_1	x_2	s_1	s_2	rhs	MR
R_0 :	1	1	1	0	0	0	
R_1 :	0	②	1	1	0	4	$4/2 = 2$
R_2 :	0	3	5	0	1	15	$15/3 = 5$
$R_0 - \frac{1}{2}R_1 \rightarrow R_0$:	1	0	1/2	−1/2	0	−2	
$\frac{1}{2}R_1 \rightarrow R_1$:	0	1	1/2	1/2	0	2	$2/0.5 = 4$
$-\frac{3}{2}R_1 + R_2 \rightarrow R_2$:	0	0	(7/2)	−3/2	1	9	$9/3.5 \approx 2.57$
$R_0 - \frac{1}{7}R_2 \rightarrow R_0$:	1	0	0	−2/7	−1/7	−23/7	
$R_1 - \frac{1}{7}R_2 \rightarrow R_1$:	0	1	0	5/7	−1/7	5/7	Optimal
$\frac{2}{7}R_2 \rightarrow R_2$:	0	0	1	−3/7	2/7	18/7	tableau!

The optimal solution is $\boldsymbol{x} = (5/7; 18/7)$. The optimal value is $z = -23/7$. ■

10.5.2.6 Problems with Nonpositive Variables and/or Free Variables

Up to this point, we have explored the simplex method as a means to address linear optimization problems featuring variables constrained to be nonnegative. Handling scenarios where variables are required to be nonpositive is relatively straightforward. The approach involves introducing a new nonnegative variable that represents the negation of the original variable. In essence, if we encounter a variable, let us call it x_j, such that $x_j \leq 0$, we substitute it with $-x_j'$ while including the constraint $x_j' \geq 0$.

Similarly, when dealing with problems that include unrestricted-in-sign variables (often referred to as free variables), the solution approach remains uncomplicated. Here, the strategy is to introduce two fresh nonnegative variables, x_j' and x_j'', with the constraint that their difference equals the original variable x_j. Consequently, if we encounter an unrestricted-in-sign variable, denoted as x_j, we replace it with $x_j' - x_j''$ and supplement the model with the constraints $x_j', x_j'' \geq 0$. Notably, only one of x_j' or x_j'' can be part of the basis in a given tableau, but not both. To further elucidate this, an illustrative example follows.

Example 10.30 Use the simplex tableau method to solve the maximization LP:

$$\begin{array}{lll} \max & z = 2x_1 + x_2 & \\ \text{s.t.} & 3x_1 + x_2 & \leq 6, \\ & x_1 + x_2 & \leq 4, \\ & x_1 & \geq 0. \end{array}$$

Solution

Note that the variable x_2 is unrestricted-in-sign. An equivalent problem is

$$\begin{array}{llll} \max & z = 2x_1 + x_2' - x_2'' & & \\ \text{s.t.} & 3x_1 + x_2' - x_2'' & \leq 6, & \text{(adding } s_1\text{)} \\ & x_1 + x_2' - x_2'' & \leq 4, & \text{(adding } s_2\text{)} \\ & x_1, x_2', x_2'' & \geq 0. & \end{array}$$

We then have the following tableaux:

EROs	z	x_1	x_2'	x_2''	s_1	s_2	rhs	MR
R_0 :	1	−2	−1	1	0	0	0	
R_1 :	0	③	1	−1	1	0	6	$6/3 = 2$
R_2 :	0	1	1	−1	0	1	4	$4/1 = 4$
$R_0 + \frac{3}{2}R_1 \to R_0$:	1	0	−1/3	1/3	2/3	0	4	
$\frac{1}{3}R_1 \to R_1$:	0	1	1/3	−1/3	1/3	0	2	$2/(1/3) = 6$
$-\frac{1}{3}R_1 + R_2 \to R_2$:	0	0	(2/3)	−2/3	−1/3	1	2	$2/(2/3) = 3$
$R_0 + \frac{1}{2}R_2 \to R_0$:	1	0	0	0	1/2	1/2	5	
$R_1 - \frac{1}{2}R_2 \to R_1$:	0	1	0	0	1/2	−1/2	1	Optimal
$\frac{3}{2}R_2 \to R_2$:	0	0	1	−1	−1/2	3/2	3	tableau!

An optimal solution is given by $x_1 = 1$ and $x_2 = x_2' - x_2'' = 3 - 0 = 3$. The optimal value is $z = 5$. We point out that the columns corresponding to x_2' and x_2'' are always identical but with opposite signs. ■

10.5.3 The Big-M Method

Until this point, our exploration of the simplex method has centered on resolving linear optimization problems with inequality constraints of the form "≤." An intriguing question that arises is how to adapt this method to address maximization and minimization problems featuring inequality constraints of "≥" or "=" nature. To tackle such problem types, a widely employed technique is known as the big-M method. Essentially, the big-M method extends the applicability of the simplex algorithm to encompass problems encompassing "greater-than" and/or "equal" constraints.

10.5.3.1 Problems with "Greater-than" and/or "Equal" Constraints

In cases where we solely encounter "$\leq$" constraints, it is relatively straightforward to identify an initial basic feasible solution, typically involving the slack variables. However, a critical question emerges: How can we establish an initial basic feasible solution when confronted with "$\geq$" and/or "$=$" constraints? The big-M method offers a solution to this conundrum by introducing artificial variables for each "$\geq$" and "$=$" constraints, following the steps in Workflow 10.6.

Workflow 10.6 (The big-M method) We solve LP problems with "greater-than" and/or "equal" constraints by following six steps:

(i) Modify constraints as needed so that all the right-hand side values are nonnegative.
(ii) Add an artificial variable, say a_i, for constraint i if it is a "$\geq$" or "$=$" constraint. Then add the nonnegativity constraint $a_i \geq 0$.
(iii) Add $\pm Ma_i$ to the objective function, where M is a big positive number, as follows:
- For a maximization LP problem, add $-Ma_i$.
- For a minimization LP problem, add $+Ma_i$.

(iv) Convert the resulting LP into the standard form by adding slack/excess variables.
(v) Convert the LP into the canonical form and make the coefficient of a_i in the zeroth row zero by using elementary row operations involving M.
(vi) Operate Steps (iii)–(vi) in Workflow 10.5.

As a direct application of Workflow 10.6, we have the following example.

Example 10.31 Use the simplex tableau method to solve the following minimization problem.

$$
\begin{array}{lll}
\min & z = 2x_1 + 3x_2 & \\
\text{s.t.} & 2x_1 + x_2 & \geq 4, \\
& x_1 - x_2 & \geq -1, \\
& x_1, x_2 & \geq 0.
\end{array}
$$

Solution

Our initial course of action entails the execution of the procedure outlined in Steps (i) through (v) as laid out in Workflow 10.6. In this sequence, our primary objective is to bring about a modification to the constraints of the problem. Our aim is to ensure that all the right-hand side values within these constraints are adjusted to be nonnegative. Consequently, we get

$$
\begin{array}{lll}
\min & z = 2x_1 + 3x_2 & \\
\text{s.t.} & 2x_1 + x_2 & \geq 4, \\
& -x_1 + x_2 & \leq 1, \\
& x_1, x_2 & \geq 0.
\end{array}
$$

Then, we add an artificial variable a_i for constraint i if it is a "$\geq$" or "$=$" constraint. Then add $a_i \geq 0$. We also add Ma_i to the objective function, where M is a big positive number.

This yields

$$\begin{array}{lll} \min & z = 2x_1 + 3x_2 + Ma_1 & \\ \text{s.t.} & 2x_1 + x_2 + a_1 & \geq 4, \\ & -x_1 + x_2 & \leq 1, \\ & x_1, x_2, a_1 & \geq 0. \end{array}$$

Next, we convert the resulting LP into the standard form to get

$$\begin{array}{lll} \min & z = 2x_1 + 3x_2 + Ma_1 & \\ \text{s.t.} & 2x_1 + x_2 + a_1 - e_1 & = 4, \\ & -x_1 + x_2 + s_1 & = 1, \\ & x_1, x_2, s_1, e_1, a_1 & \geq 0. \end{array}$$

Now, our next step involves the conversion of the LP problem into its canonical form. To achieve this, we employ elementary row operations to manipulate the coefficients of the variable a_i in the zeroth row, making sure that they become zero. Following this preliminary step, we proceed to execute the subsequent steps, specifically Steps (*iii*) through (*vi*), as delineated in Workflow 10.5.

This can be seen in the subsequent tableaux.

EROs	z	x_1	x_2	e_1	s_1	a_1	rhs	
R_0 :	1	−2	−3	0	0	$\textcircled{-M}$	0	Not in
R_1 :	0	2	1	−1	0	1	4	canonical
R_2 :	0	−1	1	0	1	0	1	form
$R_0 + MR_1 \to R_0$:	1	$2M - 2$	$M - 3$	$-M$	0	0	$4M$	In
$R_1 \to R_1$:	0	$\textcircled{2}$	1	−1	0	1	4	canonical
$R_2 \to R_2$:	0	−1	1	0	1	0	1	form
$R_0 + (1 - M)R_1 \to R_0$:	1	0	−2	−1	0	$-M + 1$	4	
$\frac{1}{2}R_1 \to R_1$:	0	1	1/2	−1/2	0	1/2	2	Optimal
$\frac{1}{2}R_1 + R_2 \to R_2$:	0	0	3/2	−1/2	1	1/2	3	tableau!

The optimal solution is given by $(x_1; x_2) = (2; 0)$. The optimal value is given by $z = 4$. ■

Example 10.32 Use the simplex tableau method to solve the following minimization problem.

$$\begin{array}{lll} \min & z = 2x_1 - 3x_2 & \\ \text{s.t.} & x_1 + 3x_2 & \leq 9, \\ & 2x_1 + 5x_2 & \geq -6, \\ & x_2 & \geq 1, \\ & x_2 & \geq 0. \end{array}$$

Solution

Note that the variable x_1 does not have any sign restrictions, making it what is known as unrestricted-in-sign. We have the following tableaux:

EROs	z	x_1'	x_1''	x_2	s_1	s_2	e_1	a_1	rhs	
R_0 :	1	−2	2	3	0	0	0	$\textcircled{-M}$	0	
R_1 :	0	1	−1	3	1	0	0	0	9	
R_2 :	0	−2	2	−5	0	1	0	0	6	
R_3 :	0	0	0	1	0	0	−1	1	1	
$R_0 + MR_3 \rightarrow R_0$:	1	−2	2	$M+3$	0	0	$-M$	0	M	
$R_1 \rightarrow R_1$:	0	1	−1	3	1	0	0	0	9	
$R_2 \rightarrow R_2$:	0	−2	2	−5	0	1	0	0	6	
$R_3 \rightarrow R_3$:	0	0	0	$\textcircled{1}$	0	0	−1	1	1	
$R_0 - (M+3)R_3 \rightarrow R_0$:	1	−2	2	0	0	0	3	$-M-3$	−3	
$R_1 - 3R_3 \rightarrow R_1$:	0	1	−1	0	1	0	$\textcircled{3}$	−3	6	
$R_2 + 5R_3 \rightarrow R_2$:	0	−2	2	0	0	1	−5	5	11	
$R_3 \rightarrow R_3$:	0	0	0	1	0	0	−1	1	1	
$R_0 - R_1 \rightarrow R_0$:	1	−3	3	0	−1	0	0	$-M$	−9	
$\frac{1}{3}R_1 \rightarrow R_1$:	0	1/3	−1/3	0	1/3	0	1	−1	2	
$\frac{5}{3}R_1 + R_2 \rightarrow R_2$:	0	−1/3	$\textcircled{1/3}$	0	5/3	1	0	0	21	
$\frac{1}{3}R_1 + R_3 \rightarrow R_3$:	0	1/3	−1/3	1	1/3	0	0	0	3	
$R_0 - 9R_2 \rightarrow R_0$:	1	0	0	0	−16	−9	0	$-M$	−198	O
$R_1 + R_2 \rightarrow R_1$:	0	0	0	0	2	3	1	−1	23	P
$3R_2 \rightarrow R_2$:	0	−1	1	0	5	3	0	0	63	T
$R_2 + R_3 \rightarrow R_3$:	0	0	0	1	2	1	0	0	24	M

We contemplated an equivalent problem to establish a version of the problem with equality constraints while ensuring that all variables involved are nonnegative. Our focus has shifted to the following form:

$$
\begin{array}{lll}
\min & z = 2(x_1' - x_1'') - 3x_2 + Ma_1 & \\
\text{s.t.} & (x_1' - x_1'') + 3x_2 + s_1 & = 9, \\
 & -2(x_1' - x_1'') - 5x_2 + s_2 & = 6, \\
 & x_2 - e_1 + a_1 & = 1, \\
 & x_1', x_1'', x_2, s_1, s_2, e_1, a_1 & \geq 0.
\end{array}
$$

So, the optimal solution is given by $x_1 = x_1' - x_1'' = 0 - 63 = -63$, $x_2 = 24$ and $e_1 = 23$, hence $(x_1; x_2) = (-63; 24)$. The optimal value is $z = -198$. ■

10.5.3.2 Detecting Infeasibility

The big-M method can be used to detect the infeasibility. The following remark tells us when we have an infeasible problem.

Remark 10.9 If any artificial variable is basic in the optimal tableau, that is, $a_i > 0$ for some i, then the LP problem is infeasible.

As a direct application, we have the following example.

Example 10.33 Use the simplex tableau method to solve the following minimization problem.

$$\begin{array}{ll} \min & z = 3x_1 \\ \text{s.t.} & 2x_1 + x_2 \geq 6, \\ & 3x_1 + 2x_2 = 4, \\ & x_1, x_2 \geq 0. \end{array}$$

Solution

Considering an equivalent problem with equality constraints and nonnegative variables only, we are interested in a problem of the form

$$\begin{array}{ll} \min & z = 3x_1 + Ma_1 + Ma_2 \\ \text{s.t.} & 2x_1 + x_2 + a_1 - e_1 = 6, \\ & 3x_1 + 2x_2 + a_2 = 4, \\ & x_1, x_2, a_1, a_2, e_1 \geq 0. \end{array}$$

We then have the following tableaux:

EROs	z	x_1	x_2	e_1	a_1	a_2	rhs	
R_0 :	1	-3	0	0	($-M$)	($-M$)	0	
R_1 :	0	2	1	-1	1	0	6	
R_2 :	0	3	2	0	0	1	4	
$R_0 + MR_1 + MR_2 \to R_0$:	1	$5M - 3$	$3M$	$-M$	0	0	$10M$	
$R_1 \to R_1$:	0	2	1	-1	1	0	6	
$R_2 \to R_2$:	0	(3)	2	0	0	1	4	
$R_0 + (1 - \frac{5}{3}M)R_2 \to R_0$:	1	0	$-M/3 + 2$	$-M$	0	$-5M/3 + 1$	$10M/3 + 4$	O
$R_1 - \frac{2}{3}R_2 \to R_1$:	0	0	$-1/3$	-1	1	$-2/3$	$10/3$	P
$\frac{1}{3}R_2 \to R_2$:	0	1	$2/3$	0	0	$1/3$	$4/3$	T

The last tableau is optimal. Since $a_1 = 10/3 > 0$, the problem is infeasible. As an exercise for the reader, use the graphical method to reach the same conclusion. ■

10.5.3.3 Summary of the Simplex Method Steps

Considering all scenarios, we now summarize the above description of the simplex method.

Workflow 10.7 (Overview of the simplex method) We solve a linear optimization problem by operating the following steps:

(i) Modify constraints as needed so that all the right-hand side values are nonnegative.
(ii) Add an artificial variable, say a_i, for constraint i if it is a "≥" or "=" constraint. Then add the nonnegativity constraint $a_i \geq 0$.
(iii) Add $\pm Ma_i$ to the objective function, where M is a big positive number.
(iv) Convert the resulting LP into the standard form by adding slack/excess variables.
(v) Convert the LP into the canonical form and make the coefficient of a_i in the zeroth row zero by using elementary row operations involving M.
(vi) Find a basic feasible solution for the canonical form.
(vii) If the current basic feasible solution is optimal, stop. If not, move to an adjacent basic feasible solution with a higher value for the objective function by applying elementary row operations and noting that:
   - If the coefficient of a nonbasic variable in the zeroth row of the tableau is zero, then the LP has alternative optimal solutions.
   - If there is no candidate for the minimum ratio test, then the LP is unbounded.
   - If any artificial variable is basic in the optimal tableau, that is, $a_i > 0$ for some i, then the LP is infeasible.
(viii) Go to Step (*vii*).

10.5.4 Anticycling

The simplex method may encounter a phenomenon known as cycling, where it struggles to make progress. To address this issue and ensure that the simplex method always terminates, two anticycling rules have been developed. These rules are the lexicographic rule and Bland's rule, named after Robert Bland, who discovered it in 1976. In this part, we will focus our discussion on Bland's rule. Part of Exercise 10.21 targets the lexicographic rule. However, there are a number of good references to learn this and other anticycling rules, see for example (Bertsimas and Tsitsiklis, 1997, Section 3.4).

Below we outline the pivoting rule for choosing the entering and leaving variables.

Remark 10.10 (Pivoting rule) In the ordinary pivoting rule, we choose the entering variable with the most negative c_j and choose the leaving variable according to the minimum ratio test. If there are ties, break them by picking the variable with the smallest index.

Considering the above rule, we may get back to the starting tableau after some iterations in some LP problems, as in the following example attributed to Bertsimas and Tsitsiklis (1997).

Example 10.34 Consider the following LP problem.

$$
\begin{aligned}
\max \quad & z = \tfrac{3}{4}x_1 - 20x_2 + \tfrac{1}{2}x_3 - 6x_4 \\
\text{s.t.} \quad & \tfrac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \leq 0, \quad \text{(adding } x_5\text{)} \\
& \tfrac{1}{2}x_1 - 12x_2 - \tfrac{1}{2}x_3 + 3x_4 \leq 0, \quad \text{(adding } x_6\text{)} \\
& x_3 + 6x_4 \leq 1, \quad \text{(adding } x_7\text{)} \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned} \tag{10.13}
$$

If we use the simplex method to solve Problem 10.13 with the ordinary pivoting rule, we obtain the simplex tableau in Table 10.3.

Note that the ending tableau is identical to the starting tableau. This means that the simplex method is cycling here! ■

Example 10.34 will be revisited in order to avoid cycling after discussing Bland's rule.

10.5.4.1 Bland's Rule

Bland's rule (or the minimum index rule) is one of the algorithmic refinements of the simplex method to avoid cycling.

Remark 10.11 (Bland's rule) Bland's rule under which the simplex method for linear optimization terminates is as follows:

- Choose the entering variable x_j such that j is the smallest index with $c_j < 0$.
- Choose the leaving variable according to the minimum ratio test, and in the case of ties, choose the one with the smallest index.

An illustrative example follows to elucidate Bland's rule.

Example 10.35 (Example 10.34 revisited)

If we use the simplex method to solve Problem 10.13 with Bland's rule, we obtain the simplex tableau in Table 10.4.

Since we have applied Bland's rule, the simplex method has terminated. The last tableau is optimal, the optimal solution is $\boldsymbol{x} = (1; 0; 1; 0)$, and the optimal value is $z = 5/4$. ■

The question that remains now in this context is: How to prevent cycling when we solve linear maximization problems? One answer stems from the following remark.

Remark 10.12 If you start with a minimization problem, say min $f(\boldsymbol{x})$ subject to $\boldsymbol{x} \in S$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a function and S is a set, then an equivalent maximization problem is max $-f(x)$ subject to $\boldsymbol{x} \in S$. Similarly, if you start with a maximization problem, say max $f(\boldsymbol{x})$ subject to $\boldsymbol{x} \in S$, then an equivalent minimization problem is min $-f(x)$ subject to $\boldsymbol{x} \in S$.

Table 10.3 The simplex tableau of Example 10.34

EROs	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
R_0 :	1	−3/4	20	−1/2	6	0	0	0	0
R_1 :	0	1/4	−8	−1	9	1	0	0	0
R_2 :	0	1/2	−12	−1/2	3	0	1	0	0
R_3 :	0	0	0	1	6	0	0	1	1
$R_0 + 3R_1 \to R_0$:	1	0	−4	−7/2	33	3	0	0	0
$4R_1 \to R_1$:	0	1	−32	−4	36	4	0	0	0
$-2R_1 + R_2 \to R_2$:	0	0	4	3/2	−15	−2	1	0	0
$R_3 \to R_3$:	0	0	0	1	0	0	0	1	1
$R_0 + R_2 \to R_0$:	1	0	0	−2	18	1	1	0	0
$R_1 + 8R_2 \to R_1$:	0	1	0	8	−84	−12	8	0	0
$\frac{1}{4}R_2 \to R_2$:	0	0	1	3/8	−15/4	−1/2	1/4	0	0
$R_3 \to R_3$:	0	0	0	1	0	0	0	1	1
$R_0 + \frac{1}{4}R_1 \to R_0$:	1	1/4	0	0	−3	−2	3	0	0
$\frac{1}{8}R_1 \to R_1$:	0	1/8	0	1	−21/2	−3/2	1	0	0
$-\frac{3}{64}R_1 + R_2 \to R_2$:	0	−3/64	1	0	3/16	1/16	−1/8	0	0
$-\frac{1}{8}R_1 + R_3 \to R_3$:	0	−1/8	0	0	21/2	3/2	−1	1	1
$R_0 + 16R_2 \to R_0$:	1	−1/2	16	0	0	−1	1	0	0
$R_1 + 56R_2 \to R_1$:	0	−5/2	56	1	0	2	−6	0	0
$\frac{16}{3}R_2 \to R_2$:	0	−1/4	16/3	0	1	1/3	−2/3	0	0
$-56R_2 + R_3 \to R_3$:	0	5/2	−56	0	0	−2	6	1	1
$R_0 + \frac{1}{2}R_1 \to R_0$:	1	−7/4	44	1/2	0	0	−2	0	0
$\frac{1}{2}R_1 \to R_1$:	0	−5/4	28	1/2	0	1	−3	0	0
$-\frac{1}{6}R_1 + R_2 \to R_2$:	0	1/6	−4	−1/6	1	0	1/3	0	0
$R_1 + R_3 \to R_3$:	0	0	0	1	0	0	0	1	1
$R_0 + 6R_2 \to R_0$:	1	−3/4	20	−1/2	6	0	0	0	0
$R_1 + 9R_2 \to R_1$:	0	1/4	−8	−1	9	1	0	0	0
$3R_2 \to R_2$:	0	1/2	−12	−1/2	3	0	1	0	0
$R_3 \to R_3$:	0	0	0	1	6	0	0	1	1

In essence, the conversion between minimization and maximization problems provides an essential equivalence within optimization. By simply negating the objective function and retaining the integrity of the constraints, whether starting from a minimization or maximization problem, an equivalent problem with the opposite optimization objective is established.

Table 10.4 The simplex tableau of Example 10.35

EROs	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
R_0 :	1	−3/4	20	−1/2	6	0	0	0	0
R_1 :	0	1/4	−8	−1	9	1	0	0	0
R_2 :	0	1/2	−12	−1/2	3	0	1	0	0
R_3 :	0	0	0	1	6	0	0	1	1
$R_0 + 3R_1 \to R_0$:	1	0	−4	−7/2	33	3	0	0	0
$4R_1 \to R_1$:	0	1	−32	−4	36	4	0	0	0
$-2R_1 + R_2 \to R_2$:	0	0	4	3/2	−15	−2	1	0	0
$R_3 \to R_3$:	0	0	0	1	0	0	0	1	1
$R_0 + R_2 \to R_0$:	1	0	0	−2	18	1	1	0	0
$R_1 + 8R_2 \to R_1$:	0	1	0	8	−84	−12	8	0	0
$\frac{1}{4}R_2 \to R_2$:	0	0	1	3/8	−15/4	−1/2	1/4	0	0
$R_3 \to R_3$:	0	0	0	1	0	0	0	1	1
$R_0 + \frac{1}{4}R_1 \to R_0$:	1	1/4	0	0	−3	−2	3	0	0
$\frac{1}{8}R_1 \to R_1$:	0	1/8	0	1	−21/2	−3/2	1	0	0
$-\frac{3}{64}R_1 + R_2 \to R_2$:	0	−3/64	1	0	3/16	1/16	−1/8	0	0
$-\frac{1}{8}R_1 + R_3 \to R_3$:	0	−1/8	0	0	21/2	3/2	−1	1	1
$R_0 + 16R_2 \to R_0$:	1	−1/2	16	0	0	−1	1	0	0
$R_1 + 56R_2 \to R_1$:	0	−5/2	56	1	0	2	−6	0	0
$\frac{16}{3}R_2 \to R_2$:	0	−1/4	16/3	0	1	1/3	−2/3	0	0
$-56R_2 + R_3 \to R_3$:	0	5/2	−56	0	0	−2	6	1	1
$R_0 + \frac{1}{5}R_3 \to R_0$:	1	0	24/5	0	0	−7/5	11/5	1/5	1/5
$R_1 + R_3 \to R_1$:	0	0	0	1	0	0	0	1	1
$R_2 + \frac{1}{10}R_3 \to R_2$:	0	0	−4/15	0	1	2/15	−11/15	1/10	1/10
$\frac{2}{5}R_3 \to R_3$:	0	1	−112/5	0	0	−4/5	12/5	2/5	2/5
$R_0 + \frac{21}{2}R_2 \to R_0$:	1	0	2	0	21/2	0	3/2	5/4	5/4
$R_1 \to R_1$:	0	0	0	1	0	0	0	1	1
$\frac{15}{2}R_2 \to R_2$:	0	0	−2	0	15/2	1	−1/2	3/4	3/4
$6R_2 + R_3 \to R_3$:	0	1	−24	0	6	0	2	1	1

In consideration of Remark 10.12, to mitigate cycling in the context of maximizing $\boldsymbol{c}^\top \boldsymbol{x}$ while adhering to certain constraints, we employ the simplex method and implement Bland's rule to minimize $-\boldsymbol{c}^\top \boldsymbol{x}$ under the same constraints. This approach yields an identical optimal solution, albeit with the optimal value of the maximization problem being equal to the result of the minimization problem, multiplied by −1.

10.5.5 Complexity

Similar to any algorithmic method, the computational complexity of the simplex method is determined by the following two factors: (a) The computational complexity of each iteration. (b) The total number of iterations.

The following theorem is known to hold (we refer to Section 3.3 in Bertsimas and Tsitsiklis (1997)). It indicates that the amount of computation in each iteration of the full tableau method is propositional to the size of the coefficient matrix.

Theorem 10.6 The number of arithmetic operations in each iteration of the simplex tableau algorithm solving Problem (10.9) is $O(mn)$.

Note that the estimate of the computational complexity in Theorem 10.6 refers to a single iteration. This complexity estimate is for both the worst-case time and the best-case time.

In practice, the simplex method's advantage lies in the observation that it typically converges in just $O(m)$ iterations to discover an optimal solution. However, from a theoretical perspective, the method has its drawback, as this observation does not hold true for every LP problem. In fact, there exists a class of problems for which an exponential number of iterations is needed (Bertsimas and Tsitsiklis, 1997, Section 3.7). This phenomenon arises because the count of extreme points within the feasible set can grow exponentially with an increase in the number of variables and constraints.

10.6 Duality in Linear Programming

LP duality studies the relationships between pairs of linear programs and their solutions.

The LP problem in the primal standard form is defined as

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array} \qquad \text{(P|LP)}$$

where $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ constitute given data, and $\boldsymbol{x} \in \mathbb{R}^n$ is called the primal decision variable.

The LP problem in the dual standard form is the dual of (P|LP), which is defined as

$$\begin{array}{ll} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} \leq \boldsymbol{c}, \end{array} \qquad \text{(D|LP)}$$

where $\boldsymbol{y} \in \mathbb{R}^m$ is called the dual decision variable.

10.6.1 Lagrangian Duality and LP Duality

Problem (D|LP) can be derived from (P|LP) through the usual Lagrangian approach. The optimization problems are classified into two classes: Constrained optimization problems and unconstrained optimization problems. This classification is based on whether or not we have constraints on the variables. The Lagrangian approach is a technique by which

Table 10.5 Correspondence rules between primal and dual linear programs

Primal	Minimum	Maximum	Dual
C	$\geq \boldsymbol{b}$	$\geq \mathbf{0}$	V
N	$\leq \boldsymbol{b}$	$\leq \mathbf{0}$	A
S	$= \boldsymbol{b}$	urs	R
V	$\geq \mathbf{0}$	$\leq \boldsymbol{c}$	C
A	$\leq \mathbf{0}$	$\geq \boldsymbol{c}$	N
R	urs	$= \boldsymbol{c}$	S

a constrained optimization problem becomes an unconstrained optimization problem by adding Lagrangian multipliers for the equality constraints. The Lagrangian function is a function that combines the objective function being optimized with functions penalizing constraint violations linearly.

The Lagrangian function for (P|LP) is defined as

$$\mathcal{L}(\boldsymbol{x}, \lambda, \nu) \triangleq \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - \lambda^{\mathsf{T}}(A\boldsymbol{x} - \boldsymbol{b}) - \nu^{\mathsf{T}}\boldsymbol{x}.$$

The vectors λ and ν are called Lagrangian multipliers. The dual of (P|LP) has the objective function

$$q(\lambda, \nu) \triangleq \inf_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \lambda, \nu) = \lambda^{\mathsf{T}}\boldsymbol{b} + \inf_{\boldsymbol{x}} (\boldsymbol{c} - A^{\mathsf{T}}\lambda - \nu)^{\mathsf{T}}\boldsymbol{x}.$$

The dual problem is obtained by maximizing $q(\lambda, \nu)$ subject to $\boldsymbol{\nu} \geq \mathbf{0}$.

If $\boldsymbol{c} - A^{\mathsf{T}}\lambda - \nu \neq \mathbf{0}$, the infimum is clearly $-\infty$. So we can exclude λ for which $\boldsymbol{c} - A^{\mathsf{T}}\lambda - \nu \neq \mathbf{0}$. When $\boldsymbol{c} - A^{\mathsf{T}}\lambda - \nu = \mathbf{0}$, the dual objective function is simply $\lambda^{\mathsf{T}}\boldsymbol{b}$. Hence, we can write the dual problem as follows:

$$\begin{array}{lrcl} \max & \boldsymbol{b}^{\mathsf{T}}\lambda & & \\ \text{s.t.} & A^{\mathsf{T}}\lambda + \boldsymbol{\nu} & = & \boldsymbol{c}, \\ & \boldsymbol{\nu} & \geq & \mathbf{0}. \end{array} \tag{10.14}$$

Replacing λ and ν in (10.14) by $\boldsymbol{x}$ and $\boldsymbol{z}$, respectively, we get (D|LP).

In the realm of linear optimization, it is essential to acknowledge that there exist diverse formulations beyond the standard forms (P|LP) and (D|LP). Linear optimization problems can take on various alternative forms to suit specific problem requirements and constraints. When dealing with LPs that adopt different formulations, Table 10.5 offers a valuable resource. This table provides a summary of the correspondence rules that establish the relationships between the primal and dual LPs. In other words, it outlines how the parameters and components of the primal and dual formulations are interrelated, facilitating the translation and understanding of LP problems in their various forms.

In light of Table 10.5, we have the following remark.

Remark 10.13 The following are three typical pairs of primal and dual LP problems:

$$
\text{(P|LP)} \quad \begin{array}{ll} \min & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}; \end{array} \qquad \begin{array}{ll} \max & \mathbf{b}^{\mathsf{T}}\mathbf{y} \\ \text{s.t.} & A^{\mathsf{T}}\mathbf{y} \leq \mathbf{c}, \end{array} \quad \text{(D|LP)}
$$

$$
\overline{\text{(P|LP)}} \quad \begin{array}{ll} \min & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq \mathbf{b}; \end{array} \qquad \begin{array}{ll} \max & \mathbf{b}^{\mathsf{T}}\mathbf{y} \\ \text{s.t.} & A^{\mathsf{T}}\mathbf{y} = \mathbf{c}, \\ & \mathbf{y} \geq \mathbf{0}, \end{array} \quad \overline{\text{(D|LP)}}
$$

$$
\widehat{\text{(P|LP)}} \quad \begin{array}{ll} \min & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}; \end{array} \qquad \begin{array}{ll} \max & \mathbf{b}^{\mathsf{T}}\mathbf{y} \\ \text{s.t.} & A^{\mathsf{T}}\mathbf{y} \leq \mathbf{c}, \\ & \mathbf{y} \geq \mathbf{0}. \end{array} \quad \widehat{\text{(D|LP)}}
$$

The dual of the dual is the primal (see Proposition 10.1), so it does not matter which problem is called the primal.

Example 10.36 The following is a pair of primal–dual linear programs.

$$
\begin{array}{ll} \max & 5x_1 + 4x_2 - 3x_3 \\ \text{s.t.} & x_1 - 5x_3 \geq 4, \\ & 3x_1 + x_2 + 2x_3 \leq 5, \\ & x_1 \geq 0, x_2 \text{ urs}, x_3 \geq 0; \end{array} \qquad \begin{array}{ll} \min & 4y_1 + 5y_2 \\ \text{s.t.} & y_1 + 3y_2 \geq 5, \\ & y_2 = 4, \\ & -5y_1 + 2y_2 \geq -3, \\ & y_1 \leq 0, y_2 \geq 0. \end{array}
$$

If we take the dual of the dual, we get

$$
\begin{array}{ll} \max & 5z_1 + 4z_2 - 3z_3 \\ \text{s.t.} & z_1 - 5z_3 \geq 4, \\ & 3z_1 + z_2 + 2z_3 \leq 5, \\ & z_1 \geq 0, z_2 \text{ urs}, z_3 \geq 0, \end{array}
$$

which is the primal problem. ∎

The proof of the following proposition is left as an exercise for the reader.

Proposition 10.1 The dual of the dual is the primal.

10.6.2 The Duality Theorem

The duality theorem is a very powerful theoretical tool that is very useful in applications because it leads to an interesting class of optimization algorithms. In this part, we state and prove the weak and strong duality theorems for the primal–dual pair (P|LP) and (D|LP). All the results in this part are stated for the pair (P|LP) and (D|LP), but we indicate that all these results are satisfied for any primal–dual pair, including the pair $\overline{\text{(P|LP)}}$ and $\overline{\text{(D|LP)}}$ as well as the pair $\widehat{\text{(P|LP)}}$ and $\widehat{\text{(D|LP)}}$ outlined in Remark 10.13.

Recall that an optimization problem is called feasible if it has at least one feasible point, and infeasible otherwise. Recall also that an optimization problem is called unbounded if it is feasible and has unbounded optimal value. More specifically, a minimization (maximization) problem is called unbounded if it is feasible and has the optimal cost $-\infty$ (optimal cost $+\infty$).

We state the weak duality property in Theorem 10.7.

Theorem 10.7 (Weak duality in LP) Consider the primal–dual pair (P|LP) and (D|LP). Let (P|LP) and (D|LP) be both feasible. If $\boldsymbol{x}$ is a feasible solution to (P|LP) and $\boldsymbol{y}$ is a feasible solution to (D|LP), then $\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} \le \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$.

Proof: Note that, in (D|LP), the constraint $A^{\mathsf{T}}\boldsymbol{y} \le \boldsymbol{c}$ can be written as $A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c}$ with $\boldsymbol{s} \ge \boldsymbol{0}$. It follows that $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} = (A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s})^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} = \boldsymbol{y}^{\mathsf{T}}A\boldsymbol{x} + \boldsymbol{s}^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{y}^{\mathsf{T}}\boldsymbol{b} = \boldsymbol{y}^{\mathsf{T}}(A\boldsymbol{x} - \boldsymbol{b}) + \boldsymbol{s}^{\mathsf{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathsf{T}}\boldsymbol{s} \ge \boldsymbol{0}$, where the last equality follows from the constraint $A\boldsymbol{x} = \boldsymbol{b}$ stated in (P|LP), and the inequality follows because $\boldsymbol{x} \ge \boldsymbol{0}$ and $\boldsymbol{s} \ge \boldsymbol{0}$. The proof is complete. ■

The following corollary is now easy to obtain.

Corollary 10.4 Consider the primal–dual pair (P|LP) and (D|LP).

(a) If (P|LP) is unbounded, then (D|LP) is infeasible.
(b) If (D|LP) is unbounded, then (P|LP) is infeasible.

Proof: If we prove item (*a*), item (*b*) immediately follows by a symmetrical argument. Suppose, on the contrary, that Problem (P|LP) is feasible, with the optimal cost $-\infty$, and that Problem (D|LP) is also feasible. Let w be the optimal cost in (D|LP). By weak duality, we have $w \le -\infty$. That is, $w \le r$ for all $r \in \mathbb{R}$, which is impossible. This means that (D|LP) cannot have a feasible solution. This proves item (*a*), and hence completes the proof. ■

In Figure 10.15, we show visually how the duality gap between the primal and dual LP problems turns to zero. That is, the difference $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}$ becomes zero when $\boldsymbol{x}$ is an optimal solution to (P|LP) and $\boldsymbol{y}$ is an optimal solution to (D|LP). This is the essence of the strong duality property, which is stated below in Theorem 10.8.

Theorem 10.8 (Strong duality in LP) Consider the primal–dual pair (P|LP) and (D|LP). Assume that (P|LP) and (D|LP) are both feasible. If one of (P|LP) or (D|LP) has a finite optimal solution, so does the other, and their optimal values are equal.

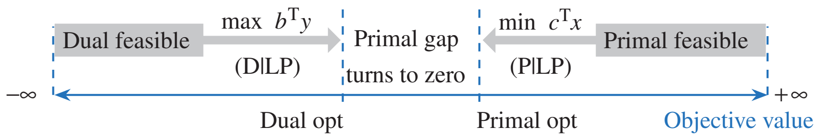


Figure 10.15 The duality gap between the primal and dual LP problems.

Proof: Let $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$ be feasible solutions to Problems (P|LP) and (D|LP), respectively. Starting from the weak duality (as presented in Theorem 10.7), we have the inequality $\boldsymbol{b}^{\mathsf{T}}\bar{\boldsymbol{y}} \leq \boldsymbol{c}^{\mathsf{T}}\bar{\boldsymbol{x}}$, which signifies that both Problems (P|LP) and (D|LP) are bounded. Let z and w represent the optimal values of (P|LP) and (D|LP), respectively. Using the weak duality once more, we have $w \leq z$. To establish that $w = z$, we can consider a contradiction. Suppose, to the contrary, that $w < z$. In that case, there exists no $\boldsymbol{y}$ that satisfies the inequalities $A^{\mathsf{T}}\boldsymbol{y} \leq \boldsymbol{c}$ and $\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} \geq z$, or equivalently

$$\begin{bmatrix} A^{\mathsf{T}} \\ -\boldsymbol{b}^{\mathsf{T}} \end{bmatrix} \boldsymbol{y} \leq \begin{bmatrix} \boldsymbol{c} \\ -z \end{bmatrix}. \tag{10.15}$$

Letting

$$\hat{A} \triangleq \left[\hat{A} \ \vdots -\boldsymbol{b}\right], \text{ and } \hat{\boldsymbol{c}} \triangleq \begin{bmatrix} \boldsymbol{c} \\ -z \end{bmatrix},$$

we can rewrite (10.15) as $\hat{A}^{\mathsf{T}}\boldsymbol{y} \leq \hat{\boldsymbol{c}}$. Using Farkas' lemma (Version II; see Theorem 3.16), there exists a vector $\hat{\boldsymbol{x}}$ satisfying

$$\hat{A}\hat{\boldsymbol{x}} = \boldsymbol{0}, \ \hat{\boldsymbol{c}}^{\mathsf{T}}\hat{\boldsymbol{x}} < 0, \ \text{and } \hat{\boldsymbol{x}} \geq \boldsymbol{0}. \tag{10.16}$$

Note that the vector $\hat{\boldsymbol{x}}$ can be written as $\hat{\boldsymbol{x}} \triangleq (\boldsymbol{x}; \alpha)$ with $\alpha \neq 0$. This rewrites (10.16) as

$$\left[A \ \vdots -\boldsymbol{b}\right] \begin{bmatrix} \boldsymbol{x} \\ \alpha \end{bmatrix} = \boldsymbol{0}, \ \begin{bmatrix} \boldsymbol{c} \\ -z \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \boldsymbol{x} \\ \alpha \end{bmatrix} < 0, \ \text{and } \begin{bmatrix} \boldsymbol{x} \\ \alpha \end{bmatrix} \geq \boldsymbol{0}. \tag{10.17}$$

To prove that $\alpha \neq 0$, suppose the contrary, that is, $\alpha = 0$. Then, from (10.17), we have $A\boldsymbol{x} = \boldsymbol{0}$, $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} < 0$, and $\boldsymbol{x} \geq \boldsymbol{0}$. Applying Farkas' Lemma (Version II) once again, we find that there is no vector $\boldsymbol{y}$ satisfying $A^{\mathsf{T}}\boldsymbol{y} \leq \boldsymbol{c}$. This implies that Problem (D|LP) is infeasible, which is in contradiction with our initial assumption.

It is now evident that $\alpha \neq 0$, and further analysis shows that $\frac{1}{\alpha}\boldsymbol{x} \geq \boldsymbol{0}$. Additionally, $A\boldsymbol{x} - \alpha\boldsymbol{c} = \boldsymbol{0}$, which can be written as $A(\frac{1}{\alpha}\boldsymbol{x}) = \boldsymbol{c}$. This implies that the vector $\frac{1}{\alpha}\boldsymbol{x}$ is feasible for (P|LP). However, from (10.17), we have $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - \alpha z < 0$, so $\boldsymbol{c}^{\mathsf{T}}(\frac{1}{\alpha}\boldsymbol{x}) < z$. This contradicts the fact that z is the optimal value of (D|LP). Hence, it is confirmed that $w = z$. The proof is complete. ■

The following example, which is due to Nemhauser and Wolsey (1988), is a direct application of Theorem 10.8.

Example 10.37 Consider the following primal–dual pair of problems.

$$\begin{array}{llcl} \min & 7x_1 + 2x_2 & & \\ \text{s.t.} & -x_1 + 2x_2 & \leq & 4, \\ & 5x_1 + x_2 & \leq & 20, \\ & -2x_1 - 2x_2 & \leq & -7, \\ & x_1, x_2 & \leq & 0; \end{array} \qquad \begin{array}{llcl} \max & 4y_1 + 20y_2 - 7y_3 & & \\ \text{s.t.} & -y_1 + 5y_2 - 2y_3 & \geq & 7, \\ & 2y_1 + y_2 - 2y_3 & \geq & 2, \\ & y_1, y_2, y_3 & \leq & 0. \end{array}$$

Let $\boldsymbol{x}^{\star} \triangleq (\frac{36}{11}; \frac{40}{11})$ and $\boldsymbol{y}^{\star} \triangleq (\frac{3}{11}; \frac{16}{11}; 0)$. One can easily see that $\boldsymbol{x}^{\star}$ and $\boldsymbol{y}^{\star}$ are feasible in the primal and dual problems, respectively. One can also easily see that $\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}^{\star} = 30\frac{2}{11}$ and $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^{\star} = 30\frac{2}{11}$. Based on the strong duality property (Theorem 10.8), since $\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}^{\star} = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^{\star}$, we conclude that $\boldsymbol{x}^\star$ and $\boldsymbol{y}^\star$ are optimal in the primal and dual problems, respectively, and their optimal value is $30\frac{2}{11}$. ■

It is a natural question to ask: Can Problems (P|LP) and (D|LP) be both infeasible? The following example answers this question positively.

Example 10.38 The following primal–dual pair of problems are both infeasible.

$$\begin{array}{llcl} \min & x_1 + 2x_2 & & \\ \text{s.t.} & x_1 + x_2 & = & 2, \\ & 3x_1 + 3x_2 & = & 4; \end{array} \qquad \begin{array}{llcl} \max & 2y_1 + 4y_2 & & \\ \text{s.t.} & y_1 + 3y_2 & = & 1, \\ & y_1 + 3y_2 & = & 2. \end{array}$$ ■

It is not hard now to establish the following corollary.

Corollary 10.5 Consider the primal–dual pair (P|LP) and (D|LP).

(a) If (P|LP) is infeasible, then (D|LP) is either infeasible or unbounded.
(b) If (D|LP) is infeasible, then (P|LP) is either infeasible or unbounded.

Proof: Note that the possibility that Problems (P|LP) and (D|LP) could be both infeasible has been grounded in Example 10.38. To establish item (*a*), it remains to demonstrate that if (P|LP) is infeasible and (D|LP) is feasible, then (D|LP) must be unbounded. Assume that (P|LP) is infeasible, and let $\bar{\boldsymbol{y}}$ be a feasible solution for (D|LP). Since (P|LP) is infeasible, there exists no $\boldsymbol{x}$ satisfying $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. Applying Farkas' Lemma (Version I, as presented in Theorem 3.15), we can conclude that there exists a vector $\hat{\boldsymbol{y}}$ that satisfies $A^\mathsf{T}\hat{\boldsymbol{y}} \geq \boldsymbol{0}$ and $\boldsymbol{b}^\mathsf{T}\hat{\boldsymbol{y}} < 0$. Now, due to the feasibility of $\bar{\boldsymbol{y}}$ in (D|LP), we know that $A^\mathsf{T}\bar{\boldsymbol{y}} \leq \boldsymbol{c}$. Let us define $\boldsymbol{y}_\alpha \triangleq \bar{\boldsymbol{y}} - \alpha\hat{\boldsymbol{y}}$ for $\alpha \geq 0$. We can then observe that

$$A^\mathsf{T}\boldsymbol{y}_\alpha = A^\mathsf{T}(\bar{\boldsymbol{y}} - \alpha\hat{\boldsymbol{y}}) = A^\mathsf{T}\bar{\boldsymbol{y}} - \alpha A^\mathsf{T}\hat{\boldsymbol{y}} \leq \boldsymbol{c} - \alpha A^\mathsf{T}\hat{\boldsymbol{y}} \leq \boldsymbol{c}.$$

This demonstrates that $\boldsymbol{y}_\alpha$ is feasible in (D|LP). Furthermore, because $\boldsymbol{b}^\mathsf{T}\hat{\boldsymbol{y}} < 0$, it is clear that $\boldsymbol{b}^\mathsf{T}\boldsymbol{y}_\alpha$ tends toward infinity as α approaches infinity:

$$\boldsymbol{b}^\mathsf{T}\boldsymbol{y}_\alpha = \boldsymbol{b}^\mathsf{T}(\bar{\boldsymbol{y}} - \alpha\hat{\boldsymbol{y}}) = \boldsymbol{b}^\mathsf{T}\bar{\boldsymbol{y}} - \alpha\boldsymbol{b}^\mathsf{T}\hat{\boldsymbol{y}} \longrightarrow \boldsymbol{b}^\mathsf{T}\bar{\boldsymbol{y}} + \infty = \infty.$$

This implies that Problem (D|LP) is unbounded, successfully proving item (*a*). To prove item (*b*), we can apply a symmetrical argument similar to that of item (*a*) and utilize Farkas' lemma (Version II). We leave the proof of this part as an exercise for the reader (see Exercise 10.11). With this, we conclude the proof. ■

Corollary 10.6 is now obvious. See also Table 10.6.

Corollary 10.6 There are only four possibilities for the primal–dual pair (P|LP) and (D|LP). Namely:

(a) Both (P|LP) and (D|LP) are feasible and their optimal values are finite and equal.
(b) (P|LP) is infeasible and (D|LP) is unbounded.
(c) (P|LP) is unbounded and (D|LP) is infeasible.
(d) Both (P|LP) and (D|LP) are infeasible.

Table 10.6 Possibilities for the primal and the dual linear programs

		(P|LP) Finite optimum	(P|LP) Unbounded	(P|LP) Infeasible
(D|LP)	**Finite optimum**	✓	×	×
(D|LP)	**Unbounded**	×	×	✓
(D|LP)	**Infeasible**	×	✓	✓

10.6.3 Complementary Slackness

Complementary slackness refers to the idea that for an optimal solution, the product of the decision variable in the primal problem and the corresponding slack variable in the dual problem is equal to zero. We have the following definition.

Definition 10.10 A slack variable is a variable that is added to an inequality constraint to transform it into an equality. Likewise, an excess (also called surplus or negative slack) variable is a variable that is subtracted to an inequality constraint to transform it into an equality.

Consider the pair $(\widehat{\text{P|LP}})$ and $(\widehat{\text{D|LP}})$ outlined in Remark 10.13.

$$(\widehat{\text{P|LP}}) \quad \begin{array}{rrcl} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\ \text{s.t.} & A\boldsymbol{x} & \geq & \boldsymbol{b}, \\ & \boldsymbol{x} & \geq & \boldsymbol{0}; \end{array} \qquad \begin{array}{rrcl} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} & \leq & \boldsymbol{c}, \\ & \boldsymbol{y} & \geq & \boldsymbol{0}. \end{array} \quad (\widehat{\text{D|LP}})$$

Let $\boldsymbol{s} \triangleq \boldsymbol{c} - A^{\mathsf{T}}\boldsymbol{y} \geq \boldsymbol{0}$ be the vector of slack variables of $(\widehat{\text{P|LP}})$, and $\boldsymbol{e} \triangleq A\boldsymbol{x} - \boldsymbol{b} \geq \boldsymbol{0}$ be the vector of excess variables of $(\widehat{\text{D|LP}})$. Then $(\widehat{\text{P|LP}})$ and $(\widehat{\text{D|LP}})$ are written as

$$\begin{array}{rrcl} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\ \text{s.t.} & A\boldsymbol{x} - \boldsymbol{e} & = & \boldsymbol{b}, \\ & \boldsymbol{x}, \boldsymbol{e} & \geq & \boldsymbol{0}; \end{array} \qquad \begin{array}{rrcl} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} & = & \boldsymbol{c}, \\ & \boldsymbol{y}, \boldsymbol{s} & \geq & \boldsymbol{0}. \end{array}$$

The complementary slackness conditions for LP are provided in the following theorem.

Theorem 10.9 **(Complementary slackness)** Consider the primal–dual pair $(\widehat{\text{P|LP}})$ and $(\widehat{\text{D|LP}})$. If $\boldsymbol{x}^\star$ is an optimal solution to $(\widehat{\text{P|LP}})$ and $\boldsymbol{y}^\star$ is an optimal solution to $(\widehat{\text{D|LP}})$, then $x_i^\star s_i^\star = 0$ for all i, and $y_j^\star e_j^\star = 0$ for all j, where $\boldsymbol{e}^\star \triangleq A\boldsymbol{x}^\star - \boldsymbol{b}$ and $\boldsymbol{s}^\star \triangleq \boldsymbol{c} - A^{\mathsf{T}}\boldsymbol{y}^\star$.

Proof: Note that

$$\begin{aligned} \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^\star &= (A^{\mathsf{T}}\boldsymbol{y}^\star + \boldsymbol{s}^\star)^{\mathsf{T}}\boldsymbol{x}^\star \\ &= \boldsymbol{y}^{\star\mathsf{T}}A\boldsymbol{x}^\star + \boldsymbol{s}^{\star\mathsf{T}}\boldsymbol{x}^\star \\ &= \boldsymbol{y}^{\star\mathsf{T}}(\boldsymbol{b} + \boldsymbol{e}^\star) + \boldsymbol{s}^{\star\mathsf{T}}\boldsymbol{x}^\star = \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}^\star + \boldsymbol{y}^{\star\mathsf{T}}\boldsymbol{e}^\star + \boldsymbol{s}^{\star\mathsf{T}}\boldsymbol{x}^\star. \end{aligned}$$

Note also that the strong duality property (Theorem 10.8) implies that $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^\star = \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}^\star$. It follows that $\boldsymbol{y}^{\star\mathsf{T}}\boldsymbol{e}^\star + \boldsymbol{s}^{\star\mathsf{T}}\boldsymbol{x}^\star = 0$. Because $\boldsymbol{x}^\star, \boldsymbol{e}^\star, \boldsymbol{y}^\star$ and $\boldsymbol{s}^\star$ are all nonnegative vectors, we have $x_i^\star s_i^\star = 0$ for all i, and $y_j^\star e_j^\star = 0$ for all j. The proof is complete. ■

Example 10.39 (Example 10.37 revisited)
To see how the complementary slackness conditions hold for the primal–dual pair in Example 10.37, note that the slack and excess variables are $\boldsymbol{s}^\star \triangleq (0; 0; 6\frac{9}{11})$ and $\boldsymbol{e}^\star \triangleq (0; 0)$, respectively. In this case, $x_i^\star e_i^\star = 0$ for $i = 1, 2$, and $y_j^\star s_j^\star = 0$ for $j = 1, 2, 3$. ■

Complementary slackness conditions are not only used to verify optimality but also serve as a foundation for duality theory, which is a powerful tool for solving and interpreting LP problems in various real-world applications.

10.6.4 The Dual Optimal Solution via the Primal Simplex Tableau

Recall that, from Theorem 10.8, if the primal and dual problems are both feasible and one of them has a finite optimal solution, so does the other, and their optimal values are equal. The question that emerges at this point is: How can one determine the optimal solution of a dual problem using the simplex tableau of the primal problem? We address this inquiry by presenting the following remark.

Remark 10.14 If we are given the simplex tableau of a primal maximization problem, then

$$\text{Optimal } y_i = \begin{cases} \text{Coefficient of } s_i \text{ in } R_0, & \text{if the } i\text{th constraint is “} \leq \text{”;} \\ -(\text{Coefficient of } e_i \text{ in } R_0), & \text{if the } i\text{th constraint is “} \geq \text{”;} \\ (\text{Coefficient of } a_i \text{ in } R_0) - M, & \text{if the } i\text{th constraint is “} = \text{.”}^{2} \end{cases}$$

If we are given the simplex tableau of a primal minimization problem, then

$$\text{Optimal } y_i = \begin{cases} \text{Coefficient of } s_i \text{ in } R_0, & \text{if the } i\text{th constraint is “} \leq \text{”;} \\ -(\text{Coefficient of } e_i \text{ in } R_0), & \text{if the } i\text{th constraint is “} \geq \text{”;} \\ (\text{Coefficient of } a_i \text{ in } R_0) + M, & \text{if the } i\text{th constraint is “} = \text{.”} \end{cases}$$

In the following example from Winston (1996), Remark 10.14 is applied.

Example 10.40 The dual problem of the maximization LP problem

$$\begin{array}{ll} \max & z = 30x_1 + 100x_2 \\ \text{s.t.} & x_1 + x_2 \leq 7, \\ & 4x_1 + 10x_2 \leq 40, \\ & 10x_1 \geq 30, \\ & x_1 \geq 0, x_2 \geq 0, \end{array} \tag{10.18}$$

is the minimization LP problem

$$\begin{array}{ll} \min & w = 7y_1 + 40y_2 + 30y_3 \\ \text{s.t.} & y_1 + 4y_2 + 10y_3 \geq 30, \\ & y_1 + 10y_2 \geq 100, \\ & y_1 \geq 0, y_2 \geq 0, y_3 \leq 0. \end{array} \tag{10.19}$$

2 This also holds when the ith constraint is “$\geq$.”

Solving Problem (10.18) by the simplex tableau method, we obtain

EROs	z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
R_0 :	1	-30	-100	0	0	0	M	0
R_1 :	0	1	1	1	0	0	0	7
R_2 :	0	4	10	0	1	0	0	40
R_3 :	0	10	0	0	0	-1	1	30
$R_0 - MR_3 \to R_0$:	1	$-30-10M$	-100	0	0	M	0	$-30M$
$R_1 \to R_1$:	0	1	1	1	0	0	0	7
$R_2 \to R_2$:	0	4	10	0	1	0	0	40
$R_3 \to R_3$:	0	10	0	0	0	-1	1	30
$R_0 + (3+M)R_3 \to R_0$:	1	0	-100	0	0	-3	$M+3$	90
$R_1 - \frac{1}{10}R_3 \to R_1$:	0	0	1	1	0	$1/10$	$-1/10$	4
$R_2 - \frac{2}{5}R_3 \to R_2$:	0	0	10	0	1	$3/5$	$-3/5$	28
$\frac{1}{10}R_3 \to R_3$:	0	1	0	0	0	$-1/10$	$1/10$	3
$R_0 + 10R_1 \to R_0$:	1	0	0	0	10	1	$M-1$	370
$R_1 \to R_1$:	0	0	0	1	$-1/10$	$3/50$	$-3/50$	1.2
$R_2 \to R_2$:	0	0	1	0	$-1/10$	$1/25$	$-1/25$	2.8
$R_3 \to R_3$:	0	1	0	0	0	$-1/10$	$1/10$	3

The last tableau is optimal. The optimal value is $z = 370$, the primal optimal solution is $(x_1; x_2) = (3; 2.8)$. According to Remark 10.14, the dual optimal solution is:

$$
\begin{aligned}
y_1 &= \text{Coefficient of } s_1 \text{ in } R_0 &&= 0;\\
y_2 &= \text{Coefficient of } s_2 \text{ in } R_0 &&= 10;\\
y_3 &= -(\text{Coefficient of } e_3 \text{ in } R_0) &&= -1,\\
\text{or } y_3 &= (\text{Coefficient of } a_3 \text{ in } R_0) - M = (M-1) - M = -1.
\end{aligned}
$$

To check this, note that the dual optimal value is:

$$w = 7y_1 + 40y_2 + 30y_3 = 370,$$

which is exactly the primal optimal value. ∎

10.7 A Homogeneous Interior-Point Method

Interior-point methods (Nesterov and Nemirovskii, 1994) represent a class of highly efficient techniques designed to solve both linear and nonlinear programming problems. In stark contrast to the simplex method, interior-point methods excel by moving through the interior of the feasible region to reach an optimal solution. Multiple interior-point algorithms have been devised for LP problems, offering versatile tools for optimization (Bertsimas and Tsitsiklis, 1997, Chapter 9).

Among the interior-point methods, homogeneous self-dual algorithms stand out as a valuable method for solving both linear and nonlinear programming. This section introduces a homogeneous interior-point algorithm tailored for solving (P|LP) and (D|LP) problems outlined in Section 10.6. The content presented here draws from previous work found in Tucker (1957) and Ye et al. (1994).

We define the following feasibility sets for the primal–dual pair (P|LP) and (D|LP).

$$
\begin{aligned}
\mathcal{F}_{\mathrm{P|LP}} &\triangleq \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\},\\
\mathcal{F}_{\mathrm{D|LP}} &\triangleq \left\{(\boldsymbol{y}, \boldsymbol{s}) \in \mathbb{R}^m \times \mathbb{R}^n : A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c},\ \boldsymbol{s} \geq \boldsymbol{0}\right\},\\
\mathcal{F}^{\circ}_{\mathrm{P|LP}} &\triangleq \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} > \boldsymbol{0}\},\\
\mathcal{F}^{\circ}_{\mathrm{D|LP}} &\triangleq \left\{(\boldsymbol{y}, \boldsymbol{s}) \in \mathbb{R}^m \times \mathbb{R}^n : A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c},\ \boldsymbol{s} > \boldsymbol{0}\right\},\\
\mathcal{F}^{\circ}_{\mathrm{LP}} &\triangleq \mathcal{F}^{\circ}_{\mathrm{P|LP}} \times \mathcal{F}^{\circ}_{\mathrm{D|LP}}.
\end{aligned}
$$

We also make the following assumptions about the primal–dual pair (P|LP) and (D|LP).

Assumption 10.1 The m rows of the matrix A are linearly independent.

Assumption 10.2 The set $\mathcal{F}^{\circ}_{\mathrm{LP}}$ is nonempty.

Assumption 10.1 is introduced for the sake of convenience. On the other hand, Assumption 10.2 imposes the requirement that both Problem (P|LP) and its dual counterpart (D|LP) must possess strictly feasible solutions. This condition serves as a guarantee, ensuring the existence of strong duality within the context of the LP problem.

The following primal–dual LP model provides sufficient conditions (but not always necessary) for an optimal solution of (P|LP) and (D|LP).

$$
\begin{aligned}
A\boldsymbol{x} &= \boldsymbol{b},\\
A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} &= \boldsymbol{c},\\
\boldsymbol{x}^{\mathsf{T}}\boldsymbol{s} &= 0,\\
\boldsymbol{x}, \boldsymbol{s} &\geq \boldsymbol{0}.
\end{aligned} \tag{10.20}
$$

The homogeneous LP model for the pair (P|LP) and (D|LP) is as follows:

$$
\begin{array}{rrrrrl}
A\boldsymbol{x} & & & -\boldsymbol{b}\tau & & = \boldsymbol{0},\\
 & -A^{\mathsf{T}}\boldsymbol{y} & -\boldsymbol{s} & +\boldsymbol{c}\tau & & = \boldsymbol{0},\\
-\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & +\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & & -\kappa & = 0,\\
\boldsymbol{x} & & & & & \geq \boldsymbol{0},\\
 & & \boldsymbol{s} & & & \geq \boldsymbol{0},\\
 & & & \tau & & \geq 0,\\
 & & & & \kappa & \geq 0.
\end{array} \tag{10.21}
$$

The first two equations in (10.21), with $\tau = 1$, represent primal and dual feasibility (with $\boldsymbol{x}, \boldsymbol{s} \geq \boldsymbol{0}$) and reversed weak duality. So they, together with the third equation after forcing $\kappa = 0$, define primal and dual optimal solutions. Note that homogenizing τ (i.e., making it a variable) adds the required variable dual to the third equation, introducing the artificial variable κ achieves feasibility, and adding the third equation in (10.21) achieves self-duality.

One can show that $\boldsymbol{x}^{\mathsf{T}}\boldsymbol{s} + \tau\kappa = 0$ (see Exercise 10.22). The next theorem relates (10.20) to (10.21), and it is easily proved. Here, as defined previously, $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \boldsymbol{0}\}$ is the nonnegative orthant cone.

Theorem 10.10 The primal–dual LP model (10.20) has a solution if and only if the homogeneous LP model (10.21) has a solution

$$
\left(\boldsymbol{x}^{\star}, \boldsymbol{y}^{\star}, \boldsymbol{s}^{\star}, \tau^{\star}, \kappa^{\star}\right) \in \mathbb{R}^n_+ \times \mathbb{R}^m \times \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+
$$

such that $\tau^{\star} > 0$ and $\kappa^{\star} = 0$.

The main step at each iteration of the homogeneous interior-point algorithm for solving (P|LP) and (D|LP) is the computation of the search direction $(\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s})$ from the Newton equations defined by the following system.

$$\begin{array}{rrrrr} A\,\Delta\boldsymbol{x} & & -\boldsymbol{b}\,\Delta\tau & & = \eta\boldsymbol{r}_p, \\ & -A^{\mathsf{T}}\,\Delta\boldsymbol{y} - \Delta\boldsymbol{s} & +\boldsymbol{c}\,\Delta\tau & & = \eta\boldsymbol{r}_d, \\ -\boldsymbol{c}^{\mathsf{T}}\,\Delta\boldsymbol{x} + & \boldsymbol{b}^{\mathsf{T}}\,\Delta\boldsymbol{y} & & -\quad\Delta\kappa & = \eta r_g, \\ & & \kappa\,\Delta\tau + & \tau\,\Delta\kappa & = \gamma\mu - \tau\kappa, \\ S\Delta\boldsymbol{x} & +X\Delta\boldsymbol{s} & & & = \gamma\mu\mathbf{1} - X\boldsymbol{s}, \end{array} \tag{10.22}$$

where $\mathbf{1}$ is a vector of ones with an appropriate dimension, η and γ are two parameters, $X \triangleq \mathrm{Diag}(\boldsymbol{x}) \in \mathbb{R}^{n\times n}$ is the diagonal matrix with the vector $\boldsymbol{x} \in \mathbb{R}^n$ on its diagonal, same for $S \triangleq \mathrm{Diag}(\boldsymbol{s}) \in \mathbb{R}^{n\times n}$, and

$$\begin{array}{llll} \boldsymbol{r}_p \triangleq \boldsymbol{b}\tau - A\boldsymbol{x}, & & \boldsymbol{r}_d \triangleq A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{s} - \tau\boldsymbol{c}, \\ r_g \triangleq \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} + \kappa, & & \mu \triangleq \frac{1}{n+1}(\boldsymbol{x}^{\mathsf{T}}\boldsymbol{s} + \tau\kappa). \end{array}$$

We state the homogeneous algorithm for (P|LP) and (D|LP) in Algorithm 10.1.

Algorithm 10.1: Generic homogeneous self-dual algorithm for LP

Input: Data in Problems (P|LP) and (D|LP), $(x, y, s, \tau, \kappa) \triangleq (\mathbf{1}, \mathbf{0}, \mathbf{1}, 1, 1)$

Output: An approximate optimal solution to Problem (P|LP)

while a stopping criterion is not satisfied **do**
- choose η, γ
- compute the solution $(\Delta x, \Delta y, \Delta s, \Delta\tau, \Delta\kappa)$ of the linear system (10.22)
- compute a step length θ so that
  $x + \theta\Delta x > \mathbf{0}$
  $s + \theta\Delta s > \mathbf{0}$
  $\tau + \theta\Delta\tau > 0$
  $\kappa + \theta\Delta\kappa > 0$
- set the new iterate according to
  $(x, y, s, \tau, \kappa) \triangleq (x, y, s, \tau, \kappa) + \theta(\Delta x, \Delta y, \Delta s, \Delta\tau, \Delta\kappa)$

end

The following theorem is known to hold (see Ye et al. (1994)). It gives the computational complexity (worst behavior) of Algorithm 10.1 in terms of the dimension of the decision variable (n).

Theorem 10.11 Let $\epsilon_0 > 0$ be the residual error at a starting point, and $\epsilon > 0$ be a given tolerance. Under Assumptions 10.2 and 10.1, if the pair (P|LP) and (D|LP) has a solution $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{s}^\star)$, then Algorithm 10.1 finds an ϵ-approximate solution (i.e., a solution with residual error less than or equal to ϵ) in at most

$$O\left(\sqrt{n}\ln\left(\mathbf{1}^{\mathsf{T}}\left(\boldsymbol{x}^\star + \boldsymbol{s}^\star\right)\left(\frac{\epsilon_0}{\epsilon}\right)\right)\right) \text{ iterations.}$$

Theoretically, the advantage of this interior-point method is maintaining the iteration complexity of $O(\sqrt{n}\ln(L))$, where L is the data length of the underlying LP problem. Practically, the disadvantage of this method is the doubled dimension of the system of equations, which must be solved at each iteration.

Exercises

10.1 Choose the correct answer for each of the following multiple-choice questions/items.

(a) All LP problems may be solved using the graphical method.

(i) True. (ii) False.

(b) Al-Akhawayn Inc. manufactures two varieties of paper towels, known as "regular" and "super-soaker." The marketing department has established a requirement that the total monthly production of regular paper towels should not exceed twice the monthly production of super-soaker paper towels. In this context, let us denote by x_1 the quantity of regular paper towels produced per month and by x_2 the quantity of super-soaker paper towels produced per month. The relevant constraint(s) can be expressed as:

(i) $2x_1 \leq x_2$. (iii) $x_1 \leq 0.5x_2$. (v) $x_1 - 0.5x_2 \geq 0$.
(ii) $x_1 \leq 2x_2$. (iv) $x_1 - x_2 \leq 0$.

(c) Problem A is a given formulation of an LP problem with an optimal solution. Problem B is a formulation obtained by multiplying the objective function of Problem A by a positive constant and leaving all other things unchanged. Problems A and B will have

(i) the same optimal solution and same objective function value.
(ii) the same optimal solution but different objective function values.
(iii) different optimal solutions but same objective function value.
(iv) different optimal solutions and different objective function values.

(d) Consider the following LP problem:

$$\begin{array}{ll} \max & 12x + 10y \\ \text{s.t.} & 4x + 3y \leq 480, \\ & 2x + 3y \leq 360, \\ & x, y \geq 0. \end{array}$$

Which of the following points (x, y) could be a feasible corner point?

(i) $(40, 48)$. (iii) $(180, 120)$. (v) None of the above.
(ii) $(120, 0)$. (iv) $(30, 36)$.

(e) Al-Akhawayn Inc. manufactures two categories of printers, which are labeled as "regular" and "high-speed." The regular printers utilize 2 units of recycled plastic per unit produced, while the high-speed printers consume 1 unit of recycled plastic per unit of production. The company has a monthly supply of 5000 units of recycled plastic. To produce these printers, a critical machine is essential, with each unit of regular printers requiring 5 units of machine time and each unit of high-speed printers necessitating 3 units of machine time. The total available machine time per month amounts to 10,000 units. In this context, let us denote the number of units of regular printers produced per month as x_1 and the number of units of high-speed printers produced per month as x_2. The relevant constraint(s) can be expressed as:

(i) $2x_1 + x_2 = 5000$.
(ii) $2x_1 + x_2 \leq 5000$.
(iii) $5x_1 + 3x_2 \leq 10,000$.
(iv) (i) and (iii).
(v) (ii) and (iii).

(f) Problem A is a given formulation of an LP with an optimal solution and its constraint 1 is $\leq$ type. Problem B is a formulation obtained from Problem A by replacing the $\leq$ constraint by an equality constraint and leaving all other things unchanged. Problems A and B will have

(i) the same optimal solution and same objective function value.
(ii) the same optimal solution but different objective function values.
(iii) different optimal solutions but same objective function value.
(iv) same or different solution profile depending on the role of the constraints in the solutions.

(g) Consider the following LP problem:

$$\begin{array}{lll} \max & 12x + 10y & \\ \text{s.t.} & 4x + 3y & \leq 480, \\ & 2x + 3y & \leq 360, \\ & x,\ y & \geq 0. \end{array}$$

Which of the following points (x, y) is not in the feasible region?

(i) $(30, 60)$.
(ii) $(105, 0)$.
(iii) $(0, 110)$.
(iv) $(100, 10)$.
(v) None of the above.

(h) In any graphically solvable LP problem, if two feasible points exist, then any nonnegative weighted average of these points (with weights summing up to 1) is also feasible.

(i) True.
(ii) False.

(i) In a two-variable graphical LP problem, if the coefficient of one of the variables in the objective function is changed (while the other remains fixed), then the slope of the objective function expression will change.

(i) True.
(ii) False.

(j) Al-Akhawayn Inc. engages in the production of two printer variants, denoted as "regular" and "high-speed." The regular printers consume 2 units of recycled plastic per unit of production, while the high-speed printers utilize 1 unit of recycled plastic per unit manufactured. As part of its commitment to sustainability, Al-Akhawayn ensures that a minimum of 5000 units of recycled plastic are used each month. The manufacturing process requires a crucial machine, with each unit of regular printers demanding 5 units of machine time and each unit of high-speed printers necessitating 3 units of machine time. The total available machine time per month is limited to 10,000 units. Given this context, we can represent the number of units of regular printers produced per month as x_1 and the number of units of high-speed printers produced per month as x_2.

By imposing these constraints, along with the nonnegativity constraints, we can identify one of the feasible corner points as (assuming the first number in the parenthesis is x_1 and the second number in the parenthesis is x_2):

(i) (0, 0). (ii) (2000, 0). (iii) None exists. (iv) (0, 5000). (v) (2500, 0).

(k) If a graphically solvable LP problem is unbounded, then it can always be converted to a regular bounded problem by removing a constraint.

(i) True. (ii) False.

(l) A point that satisfies all of a problem's constraints simultaneously is a(n):

(i) optimal solution.
(ii) corner point.
(iii) intersection of the profit line and a constraint.
(iv) intersection of two or more constraints.
(v) None of the above.

(m) In a two-variable graphical LP problem, if the RHS of one of the constraints is changed (keeping all other things fixed) then the plot of the corresponding constraint will move in parallel to its old plot.

(i) True. (ii) False.

(n) Two models of a product – Regular (x) and Deluxe (y) – are produced by a company. An LP model is used to determine the production schedule. The formulation is as follows:

$$\begin{array}{llrl} \max & 50x + 60y & & \text{(maximum profit)} \\ \text{s.t.} & 8x + 10y & \leq 800 & \text{(labor hours)}, \\ & x + y & \leq 120 & \text{(total units demanded)}, \\ & 4x + 5y & \leq 500 & \text{(raw materials)}, \\ & x, y & \geq 0 & \text{(nonnegativity)}. \end{array}$$

The optimal solution is $x = 100, y = 0$. How many units of the labor hours must be used to produce this number of units?

(i) 400. (ii) 200. (iii) 500. (iv) 5000. (v) None of the above.

(o) LP theory states that the optimal solution to any problem will lie at:

(i) the origin.
(ii) a corner point of the feasible region.
(iii) the highest point of the feasible region.
(iv) the lowest point in the feasible region.
(v) none of the above.

(p) The dual of an LP problem with maximized objective function, all ≤ constraints and nonnegative variables, has minimized objective function, all ≥ constraints and nonnegative decision variables.

(i) True. (ii) False.

(q) The two objective functions (max $5x + 7y$, and min $-15x - 21y$) will produce the same solution to an LP problem.

(i) True. (ii) False.

(r) In order for an LP problem to have a unique solution, the solution must exist

(i) at the intersection of the nonnegativity constraints.
(ii) at the intersection of the objective function and a constraint.
(iii) at the intersection of two or more constraints.
(iv) none of the above.

(s) If a minimization problem has an objective function of $2x_1 + 5x_2$, which of the following corner points is the optimal solution?

(i) (0, 2). (iii) (3, 3). (v) (2, 0).
(ii) (0, 3). (iv) (1, 1).

(t) In an LP problem with a nonempty feasible region, when the objective function is parallel to one of the constraints, then

(i) the solution is not optimal.
(ii) multiple optimal solutions may exist.
(iii) a single corner point solution exists.
(iv) no feasible solution exists.
(v) none of the above.

(u) An LP problem cannot have

(i) no optimal solutions.
(ii) exactly two optimal solutions.
(iii) as many optimal solutions as there are decision variables.
(iv) an infinite number of optimal solutions.
(v) none of the above.

10.2 A homemaker intends to create a blend of two food types, denoted as F_1 and F_2, with the objective of ensuring that the vitamin composition of the mixture contains a minimum of 8 units of vitamin A and 11 units of vitamin B. The cost of Food F_1 is 60 per kg, while the cost of Food F_2 is 80 per kg. Food F_1 contains 3 units per kg of vitamin A and 5 units per kg of vitamin B, while Food F_2 contains 4 units per kg of vitamin A and 2 units per kg of vitamin B. Formulate this problem as an LP problem with the objective of minimizing the cost of the mixture.

10.3 A baker possesses 30 ounces of flour and 5 packages of yeast. For each loaf of bread, 5 ounces of flour and 1 package of yeast are required. The baker can sell each loaf of bread for 30 cents. Additionally, the baker has the option to purchase extra flour at a rate of 4 cents per ounce or sell any remaining flour at the same price. Formulate this an LP problem that can be used to assist the baker in maximizing profits, which are calculated as revenues minus costs.

10.4 A farmer owns a 126-acre farm and cultivates Radish, Onion, and Potato. When he sells his entire harvest in the market, he earns 5 per kg for Radish, 4 per kg for Onion, and 5 per kg for Potato. The average yield on his farm is 1500 kg of Radish per acre, 1800 kg of Onion per acre, and 1200 kg of Potato per acre. To grow 100 kg of Radish, 100 kg of Onion, and 80 kg of Potato, he needs to spend 12.5 on water. The labor requirement to cultivate each crop is 6 man-days per acre for Radish and Potato and 5 man-days per acre for Onion. He has a total of 500 man-days of labor available at a rate of 40 per man-day. Write an LP model that can help the farmer maximize his total profit.

10.5 Use the graphical method to solve the following LP problem.

$$
\begin{aligned}
\min \quad & z = 15x_1 + 10x_2 \\
\text{s.t.} \quad & 0.25x_1 + x_2 \leq 65, \\
& 1.25x_1 + 0.5x_2 \leq 90, \\
& x_1 + x_2 \leq 85, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

10.6 A company manufactures two products, denoted as X and Y, with a combined daily production capacity of 9 tons. Both products, X and Y, necessitate the same production capacity. The company has a standing contract to deliver a minimum of 2 tons of X and 3 tons of Y per day to another business. The production of one ton of X consumes 20 machine hours, while one ton of Y requires 50 machine hours. The maximum daily available machine hours amount to 360. The company can sell all its output, and it earns a profit of JD 80 per ton for X and JD 120 per ton for Y.

(a) Formulate this as an LP problem that can be used to maximize the total profit.
(b) Solve this optimization problem graphically.

10.7 A small paint company produces two paint types, labeled as P_1 and P_2, using two raw materials, denoted as M_1 and M_2. The table shown below contains the essential data for this scenario.

According to a market survey, the highest daily demand for product P_2 is limited to 2 tons. Additionally, the daily demand for product P_1 should not surpass that of P_2 by more than 1 ton.

Tons of raw material per ton of paints produced			
	P_1	P_2	Availability
M_1	6	4	24
M_2	1	2	6
Profit per ton (in $)	500	400	

(a) Write an LP formulation for the problem to maximize the total daily profit (in $).
(b) Solve the LP model obtained in item (*a*) using the graphical method.
(c) If the number of tons to be produced for P_2 is restricted to be integer-valued, the problem obtained in item (*a*) is called a mixed integer program. Sketch its feasible region and solve it graphically.
(d) If the number of tons to be produced for P_1 and P_2 are both restricted to be integer-valued, the problem obtained in item (*a*) becomes a pure integer program. Sketch its feasible region and solve it graphically.

10.8 Use the graphical method to solve the following optimization problems.

(a)
$$\begin{array}{lrl} \min & 5x + 7y & \\ \text{s.t.} & x + 3y & \geq 6, \\ & 5x + 2y & \geq 10, \\ & y & \leq 4, \\ & x, \quad y & \geq 0. \end{array}$$

(c)
$$\begin{array}{lrl} \max & x_2 & \\ \text{s.t.} & -x_1 + x_2 & \leq 1, \\ & 3x_1 + 2x_2 & \leq 12, \\ & 2x_1 + 3x_2 & \geq 12, \\ & x_1, \quad x_2 & \geq 0, \\ & x_1, \quad x_2 & \in \mathbb{Z}. \end{array}$$

(b)
$$\begin{array}{lrl} \max & 5x + 4y & \\ \text{s.t.} & 6x + 4y & \leq 24, \\ & x + 2y & \leq 6, \\ & -x + y & \leq 1, \\ & y & = 2, \\ & x, \quad y & \geq 0. \end{array}$$

10.9 Consider the following LP problem.

$$\begin{array}{lrl} \max & z = 5x_1 + 4x_2 & \\ \text{s.t.} & 3x_1 + 2x_2 & \leq 12, \\ & x_1 + 2x_2 & \leq 6, \\ & -x_1 + x_2 & \leq 1, \\ & x_2 & \leq 2, \\ & x_1, \quad x_2 & \geq 0. \end{array}$$

Sketch the feasible region and solve it graphically for each of the following cases:
(a) The variable x_2 is restricted to be integer-valued; in this case the problem becomes a mixed integer program.
(b) The variables x_1 and x_2 are both restricted to be integer-valued; in this case, the problem becomes a pure integer program.

10.10 Transform the following LP into the standard form.

$$\begin{array}{lrl} \min & z = 2x_1 - 4x_2 + 5x_3 - 30 & \\ \text{s.t.} & 3x_1 + 2x_2 - x_3 & \geq 10, \\ & -2x_1 + 4x_3 & \leq 35, \\ & 4x_1 - x_2 & \leq 20, \\ & x_1 \leq 6, \quad x_2 \leq 8, \quad x_3 & \leq 10. \end{array}$$

10.11 Prove item (b) in Corollary 10.5.

10.12 Find the feasible solution $(x_1; x_2)$ for the original LP problem in Example 10.16 given the feasible solution $(x_1; x_2^+; x_2^-; x_3) = (4; 0; 1/3; 2/3)$ to the same problem in the standard form.

10.13 Choose the correct answer for each of the following multiple-choice questions/items.

(a) A two-variable LP problem cannot be solved by the simplex method.
   (i) True. (ii) False.

(b) If, when we are using a simplex table to solve a maximization problem, we find that the ratios for determining the pivot row are all negative, then we know that the solution is:
   (i) unbounded.
   (ii) feasible.
   (iii) degenerate.
   (iv) optimal.
   (v) none of the above.

(c) In converting a greater-than-or-equal constraint for use in a simplex table, we must add:
   (i) an artificial variable.
   (ii) a slack variable.
   (iii) a slack and an artificial variable.
   (iv) an excess and an artificial variable.
   (v) a slack and an excess variable.

(d) For a minimization problem using a simplex table, we know we have reached the optimal solution when the row R_0:
   (i) has no numbers in it.
   (ii) has no positive numbers in it.
   (iii) has no negative numbers in it.
   (iv) has no nonzero numbers in it.
   (v) none of the above.

(e) A feasible solution requires that all artificial variables are:
   (i) greater than zero.
   (ii) less than zero.
   (iii) equal to zero.
   (iv) there are no special requirements on artificial variables; they may take on any value.
   (v) none of the above.

(f) If the right-hand side of a constraint is changed, the feasible region will not be affected and will remain the same.
   (i) True. (ii) False.

(g) With Bland's rule, the simplex algorithm solves feasible linear minimization problems without cycling when:
   (i) we choose the rightmost nonbasic column with a negative cost to select the entering variable.
   (ii) we choose the rightmost nonbasic column with a negative cost to select the leaving variable.
   (iii) we choose the leftmost nonbasic column with a negative cost to select the entering variable.

(iv) we choose the leftmost nonbasic column with a negative cost to select the leaving variable.

10.14 Use the simplex tableau method to solve the following maximization problems.

(a) max $z = x_1 + 1.5x_2$
s.t. $2x_1 + 4x_2 \leq 12$,
$3x_1 + 2x_2 \leq 10$,
$x_1, x_2 \geq 0$.

(b) max $z = 3x_1 + 5x_2 + 4x_3$
s.t. $2x_1 + 3x_2 \leq 8$,
$2x_2 + 5x_3 \leq 10$,
$3x_1 + 2x_2 + 4x_3 \leq 15$,
$x_1, x_2, x_3 \geq 0$.

(c) max $z = 2x_1 - x_2 + x_3$
s.t. $3x_1 + x_2 + x_3 \leq 6$,
$x_1 + x_2 + 2x_3 \leq 1$,
$x_1 + x_2 - x_3 \leq 2$,
$x_1, x_2, x_3 \geq 0$.

(d) max $z = 60x_1 + 30x_2 + 20x_3$
s.t. $8x_1 + 6x_2 + x_3 \leq 48$,
$4x_1 + 2x_2 + 1.5x_3 \leq 20$,
$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$,
$x_2 \leq 5$, $x_1, x_2, x_3 \geq 0$.

10.15 Consider the maximization problem presented by the following tableau. The parameters a and b are unknown.

x_1	x_2	s_1	s_2	s_3	rhs
0	0	17	$-3+2a$	0	10
1	0	3	-1	0	2
0	1	4	a	0	2
0	0	1	b	1	6

For each of the following cases, explicitly discuss how many optimal solutions, if any, there are to the LP problem. (If the LP is unbounded state that).

(a) $a = -2$ and $b = 0$. (b) $a = 2$ and $b = -1$. (c) $a = 3/2$ and $b = 1$.

10.16 Consider the following tableau of the simplex method for a maximization LP problem

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
1	0	0	0	c_1	c_2	c_3	$z^\star$
0	0	-2	a_3	a_5	-1	0	0
0	1	a_1	0	-3	0	a_7	2
0	0	a_2	a_4	-4	a_6	a_8	b

(a) There have to be three basic variables. Find them and give conditions on (all or some of) the unknowns $c_1, c_2, c_3, a_1, a_2, \dots, a_8$ that make these variables basic.
(b) Give a condition on b that makes the LP feasible and conditions on c_1, c_2, and c_3 that make the LP optimal.
(c) Do we have alternative optimal solutions? Justify your answer.

10.17 Consider the following optimization problem:

$$\begin{aligned} \max\ & z = 5x_1 - x_2 \\ \text{s.t.}\ & x_1 - 3x_2 \leq 1, \\ & x_1 - 4x_2 \leq 3, \\ & x_1,\quad x_2 \geq 0. \end{aligned}$$

Use the simplex algorithm to show that this LP is an unbounded LP problem.

10.18 Consider the following primal–dual pair of problems.

$$\begin{aligned} \min\ & 13x_1 + 10x_2 + 6x_3 \\ \text{s.t.}\ & 5x_1 + x_2 + 3x_3 = 8, \\ & 3x_1 + x_2 = 3, \\ & x_1,\quad x_2,\quad x_3 \geq 0; \end{aligned} \qquad \begin{aligned} \max\ & 8y_1 + 3y_2 \\ \text{s.t.}\ & 5y_1 + 3y_2 \leq 13, \\ & y_1 + y_2 \leq 10, \\ & 3y_1 \leq 6. \end{aligned}$$

Show that $\boldsymbol{x}^\star \triangleq (1;0;1)$ and $\boldsymbol{y}^\star \triangleq (2;1)$ are optimal in the primal and dual problems, respectively, and find the corresponding optimal values.

10.19 In Example 10.38, we gave a pair of problems with the property that the primal and dual problems are both infeasible. Give an example of another pair with this property.

10.20 Consider the following LP problem.

$$\begin{aligned} \min\ & z = 5x_1 + 3x_2 - 2x_3 \\ \text{s.t.}\ & x_1 + x_2 + x_3 \geq 4, \\ & 2x_1 + 3x_2 - x_3 \geq 9, \\ & x_2 + 3x_3 \leq 5, \\ & x_1,\quad x_2,\quad x_3 \geq 0. \end{aligned}$$

(a) Write down the corresponding dual LP problem.
(b) Suppose that the simplex method has been applied directly to the primal problem, and the resulting optimal tableau is:

z	x_1	x_2	x_3	e_1	a_1	e_2	a_2	s_3	rhs
1	−2.5	0	0	0	$-M$	−1.25	$1.25 - M$	−0.75	7.5
0	−0.5	0	1	0	0	0.25	−0.25	0.75	1.5
0	0.5	1	0	0	0	−0.25	0.25	0.25	3.5
0	−1	0	0	1	−1	0	0	1	1

(i) Deduce the optimal solution to the primal problem and the optimal value.
(ii) Deduce the optimal solution to the corresponding dual problem.

10.21 In this exercise, you are required to implement both the revised simplex method and the tableau simplex method using Octave/Matlab or another programming tool of your preference. Subsequently, you should conduct performance comparisons between your implemented algorithms and established standard optimization software. To evaluate your programs, you will apply them to a selection of LP problems for which you must generate random data. Finally, you are expected to present a well-organized and structured solution for this assignment.

(a) Write an Octave function capable of solving an LP in standard form using the revised simplex method. This function should accept the constraint matrix A, the right-hand side vector $\boldsymbol{b}$, and the cost vector $\boldsymbol{c}$ as input and provide as output an optimal solution vector $\boldsymbol{x}$ along with the optimal cost. In cases where the LP is unbounded or infeasible, the function should appropriately indicate this. Additionally, the number of simplex pivots or iterations employed should be part of the function's output. The function should offer flexibility in selecting both entering and leaving variables, with the following options available:
  - For choosing the entering variable, the function should provide the choice to implement the following options.
    - Smallest value rule: After calculating all reduced costs, choose the variable with the smallest value (i.e., the most negative reduced cost) to enter the basis. This should be the default option.
    - Smallest index rule/Bland's rule: Calculate the reduced costs one at a time and choose the variable that first gives a negative reduced cost to enter. In this option, you must not calculate all reduced costs.
  - For choosing the leaving variable, the function should implement the following rule:
    - Smallest index rule: From among the candidates, the variable x_j with the smallest index j leaves. This should be the default option.

(b) Write an Octave function capable of solving an LP in standard form using the tableau simplex method. This function should accept the constraint matrix A, the right-hand side vector $\boldsymbol{b}$, and the cost vector $\boldsymbol{c}$ as input and provide as output an optimal solution vector $\boldsymbol{x}$ along with the optimal cost. In cases where the LP is unbounded or infeasible, the function should appropriately indicate this. Additionally, the number of simplex pivots or iterations employed should be part of the function's output. The function should offer flexibility in selecting both entering and leaving variables, with the following options available:
  - For choosing the entering variable, the function should provide the choice to implement the following options:
    - Smallest value rule: After calculating all reduced costs, choose the variable with the smallest value (i.e., the most negative reduced cost) to enter the basis. This should be the default option.
    - Smallest index rule/Bland's rule: After calculating all reduced costs, choose the variable with the smallest index with a negative reduced cost to enter the basis.
  - For choosing the leaving variable, the function should provide the following options.

- Smallest index rule: From among the candidates, the variable x_j with the smallest index j leaves. This should be the default option.
- Lexicographic rule: The leaving variable corresponds to the lexicographically[3] smallest row, after scaling (see (Bertsimas and Tsitsiklis, 1997, Section 3.4)).

10.22 Use (10.21) to show that $\boldsymbol{x}^\top \boldsymbol{s} + \tau\kappa = 0$.

Notes and Sources

The history of optimization and LP is a blend of ancient and modern influences. The origins of "optimization" can be traced back to ancient civilizations, where early mathematicians formulated and solved various optimization problems. Early references to optimization can be found in the works of ancient mathematicians like Euclid and Archimedes, who sought to maximize or minimize certain geometric quantities. The term "calculus of variations" was introduced in the 18th century, with pioneers like Leonhard Euler making substantial contributions to the field. However, the formalization of "linear programming," a specific branch of optimization, emerged in the mid-20th century. George Dantzig is often credited with pioneering LP during World War II, when he developed the simplex method for solving LP problems (refer to Dantzig (2016)). His work, along with the contributions of John von Neumann and Leonid Kantorovich, marked a significant turning point in the history of optimization and LP.

In linear optimization problems, we optimize a linear function subject to linear equality and inequality constraints. In this chapter, we began our study of linear optimization with the graphical method. We delved into the intricacies of the geometry of LP. Subsequently, our focus shifted to the study of the simplex method, which is the most prevalent algorithm for solving linear optimization problems. After that, we delved into an exploration of the duality in LP. As we neared the conclusion of this chapter, we addressed the LP problems that extended beyond the scope of the simplex method by investigating an interior-point method.

As we conclude this chapter, it is worth noting that the cited references and others, such as Boyd et al. (2004); Chong and Zak (2013); Nocedal and Wright (2006); Panik (1996); Roos et al. (1998); Schrijver (1999); Mitchell et al. (2006); Ferris et al. (2007); Griva et al. (2008); Aggarwal (2020); Vavasis (1999); Luenberger (1973); Bazaraa et al. (2005); Solow (2008); Chandru and Rao (1999); Golub and Bartels (2007); Pan (2023); Taha (1971); Hillier and Lieberman (2001); Peressini et al. (2012), and Ackoff and Sasieni (1968), also serve as valuable sources of information pertaining to the subject matter covered in this chapter. Exercise 10.21 is due to Krishnamoorthy (2023a).

3 The lexicographic order, also referred to as lexical order or dictionary order, extends the concept of alphabetical ordering found in dictionaries to sequences of arranged symbols or, in a broader context, to elements within a totally ordered set.

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