Probability Concept.

Classical probability

If the sample space Ω of an experiment has a finite sample points which are equally likely, then we define the probability of any event A as

$$P(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } \Omega} = \frac{n(A)}{n(\Omega)}.$$

Ex. Tossing a coin twice, what is the probability that (1) both of them are H. ← Event A ~~(let)~~

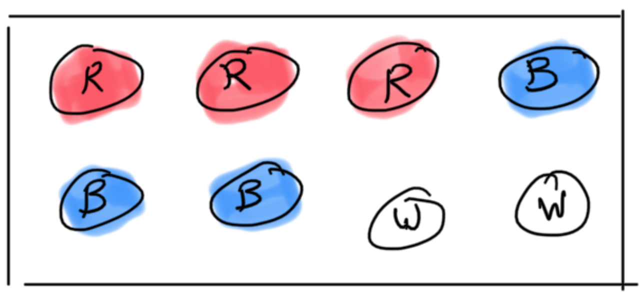
(2) first toss is H. ← Event B (let)

Soln. The possible outcomes is $\{HH, HT, TH, TT\}$.

(1) $A = \{HH\}$, so $P(A) = 1/4$.

(2) $B = \{HH, HT\}$, so $P(B) = 2/4 = 1/2$.

Ex. Selecting one ball randomly.



$$P(\{\text{the ball is blue}\}) = \frac{3}{8}.$$

$$P(\{\text{the ball is white}\}) = \frac{2}{8} = \frac{1}{4}.$$

$P(\{\text{the ball is red, white or blue}\}) = \frac{3+3+2}{8} = 1$

$P(\{\text{the ball is not red}\}) = \frac{3+2}{8} = \frac{5}{8}$.

Counting rules.

(1) Multiplication principle.

If a project consists of k stages, and the i^{th} stage can be performed in n_i way for each $i = 1, 2, ..., k$, then the whole project can be performed in $(n_1)(n_2)....(n_k)$ way

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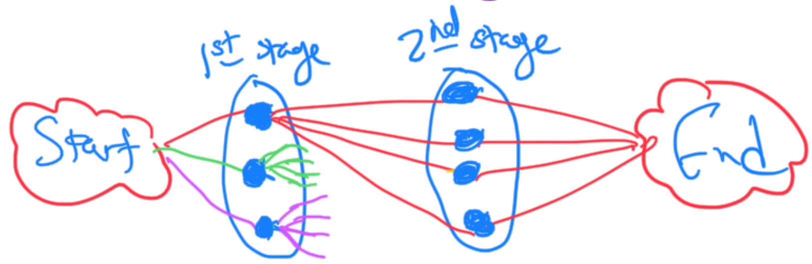
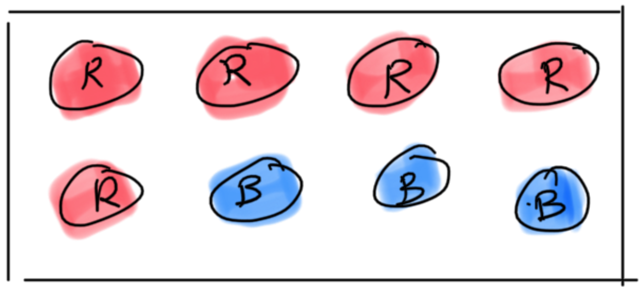
Ex. If we have 2 stages, and the first stage can be done in 3 ways, and the second stage can be done in 4 ways, then the project can be done in $3 \times 4 = 12$ ways.



Ex. Selecting four balls randomly with replacement. Then



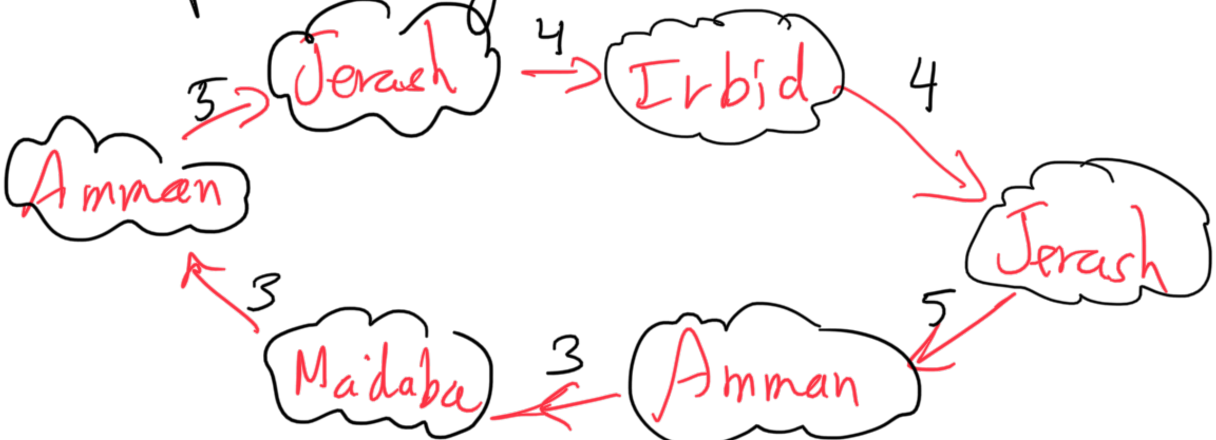
$$P(\{4 \text{ balls are red}\}) = \frac{\text{Number of ways of selecting 4 balls}}{\text{Number of ways of having 4 red balls}}$$

(4 stages)

$$= \frac{5 * 5 * 5 * 5}{8 * 8 * 8 * 8}$$

$$= \left(5/8\right)^4 .$$

Ex. Assume that there are 4 buses Irbid-Jerash, 5 buses Jerash-Amman, and 3 buses Amman-Madaba. We wish to take the following round-trip starting from Irbid.



Find the probability P of not using the same bus more once.

Soln. $$P = \frac{\cancel{4} * \cancel{5} * \cancel{3} * 2 * \cancel{4} * \cancel{3}}{\cancel{4} * \cancel{5} * \cancel{3} * \cancel{3} * 5 * \cancel{4}} = \frac{2}{5} .$$

(2) Permutations.

A permutation is an arrangement of outcomes in which the order does matter.

Keywords; arrangement, schedule, order, rank (1^{st}, 2^{nd}, 3^{rd} place).

Fact: The number of permutations of n distinct objects is $n! = n(n-1)(n-2) \cdots 3 * 2 * 1$.

$4! = 4*3*2*1$
$3! = 3*2*1$
$2! = 2*1$
$1! = 1$
$0! = 1$

Ex. In how many ways can we assign 4 problems to 4 students to solve so that each student has one problem.

Soln. $4! = 4*3*2*1 = 24$.

Fact: The number of permutations of n distinct objects taken r at a time is $P^n_r = n(n-1) * \cdots * (n-r+1) = \frac{n!}{(n-r)!}$.

For instance, $P^6_3 = \frac{6!}{(6-3)!} = \frac{6*5*4*\cancel{3!}}{\cancel{3!}} = 120$.

Ex. In how many ways 4 distinct letter passwords can we form from:
(1) the letters A, B, C, D.
(2) the letters A, B, C, D, E, F.

Soln. (1) $4! = 4*3*2*1 = 24$.
(2) $P^6_4 = 6*5*4*3 = 360$.

Ex. In how many ways 10 boys have 10 different birthdays. Ignore the existence of leap year, so a year has 365 days.

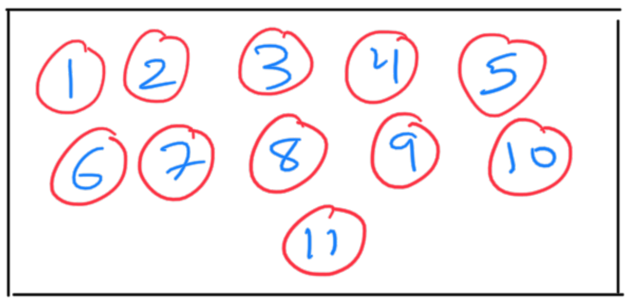
Soln. $P_{10}^{365} = (365)(364)(363)\cdots(357)(356)$.

Ex. $P(\{\text{10 boys have different birthdays}\}) = \dfrac{P_{10}^{365}}{(365)^{10}}$

No two boys have the same birthday

$$= \frac{\cancel{(365)}(364)\cdots(357)(356)}{\cancel{(365)}(365)\cdots(365)(365)}.$$

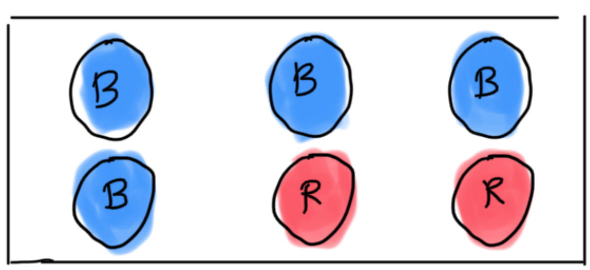
Ex. A box has 11 balls numbered $1, 2, \ldots, 11$.



An ordered sample of size 3 is taken with replacement.

$P(\{\text{balls have different numbers}\}) = \dfrac{P_3^{11}}{(11)^3} = \dfrac{\cancel{(11)}(10)(9)}{(11)^{\cancel{3}\,2}} = \dfrac{90}{121}$.

Ex. Selecting 2 balls without replacement from a box containing 6 balls (4 blue and 2 red).



(1) $P(\{\text{both balls will be blue}\}) = \dfrac{4 * 3}{5 * 6} = \dfrac{12}{30}$.

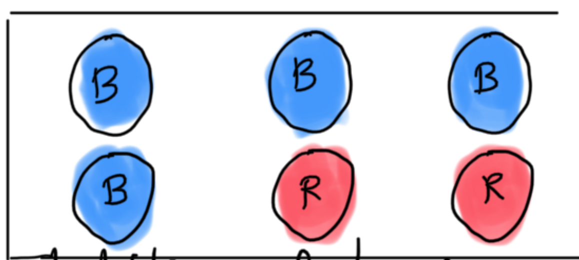
(2) $P(\{\text{both will be of the same color}\}) = \frac{4 * 3}{5 * 6} + \frac{2 * 1}{5 * 6} = \frac{14}{30}.$

(3) $P(\{\text{both will be of different color}\}) = \frac{4 * 2}{5 * 6} + \frac{2 * 4}{5 * 6} = \frac{16}{30}.$

(4) $P(\{\text{at least one will be white}\}) = 1 - P(\{\text{both are red}\})$

$= 1 - \frac{2 * 1}{30} = \frac{28}{30}.$

Exc. Selecting 4 balls without replacement from a box containing 6 balls (4 blue and 2 red). Find the probability of having:



(1) The 1st ball is red, 2nd is blue, 3rd is red, and 4th is blue. ← A

(2) The 1st is red and 2nd is blue ← B

(3) At least one of them is red. ← C

Soln. (1) $P(A) = \frac{2 * 4 * 1 * 3}{6 * 5 * 4 * 3}$

(2) $P(B) = \frac{2 * 4 * \cancel{4 * 3}}{6 * 5 * \cancel{4 * 3}}$

(3) $P(C) = 1 - P(\{\text{All of them are blue}\})$

$= 1 - \frac{\cancel{4} * \cancel{3} * 2 * 1}{6 * 5 * \cancel{4} * \cancel{3}}$

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(3) Combinations.

A combination is a grouping of outcomes in which the order does not matter.

Keywords: group, sample, selection, no rank (e.g., top 10 get gifts).

Fact: The number of combinations of n distinct objects taken r at a time is given by

$$C_r^n = \binom{n}{r} = \frac{P_r^n}{r!} = \frac{n!}{r!\,(n-r)!}.$$

Ex. $C_2^6 = \binom{6}{2} = \frac{6!}{2!\,(6-2)!} = \frac{6(5)\cancel{4!}}{2!\,\cancel{4!}} = 15$

Ex. How many pairs can be made from a group of 6 people?

Soln. $C_2^6 = \binom{6}{2} = 15$

Ex. How many groups can we form from 10 students if the group consists of 4 students?

Soln. $C_4^{10} = \binom{10}{4} = \frac{10!}{4!\,(10-4)!} = \frac{10!}{4!\,6!} = \frac{10\cdot \cancel{9}^{3}\cdot \cancel{8}\cdot 7\cdot \cancel{6!}}{\cancel{24}_{\cancel{3}}\cdot \cancel{6!}}$

$= 210.$

Ex. There are 6 boys and 4 girls in a class. A group of 5 students will be selected from the class. Then

(1) $P(\{\text{the group has 3 boys and 2 girls}\}) = \dfrac{\binom{6}{3}\binom{4}{2}}{\binom{10}{5}}$

$$= \frac{\frac{6!}{3!\,3!} \cdot \frac{4!}{2!\,2!}}{\frac{10!}{5!\,5!}} = \frac{120}{252} = \frac{10}{21}.$$

(2) $P(\{\text{the group has at least one girl}\})$

$= 1 - P(\{\text{the group has only boys}\})$

$$= 1 - \frac{\binom{6}{5}}{\binom{10}{5}} = 1 - \frac{6}{252} = \frac{41}{42}.$$

Or $= 1 - P(\{\text{not having any girl}\})$

$$= 1 - \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6} = 1 - \frac{1}{42} = \frac{41}{42}.$$

Ex. A lot consists of 1100 distinct items. There are 4 percent defective items in the lot. What is the probability that a random sample of size 50 contains 5 defective items?

Soln. In the lot

Number of defective items $= \frac{4}{100}(1100) = 44$.

Number of non-defective items $= 1100 - 44 = 1056$.

Then

$$P\left(\begin{Bmatrix}\text{Random sample of size 50 items} \\ \text{contains 5 defective items}\end{Bmatrix}\right) = \frac{\binom{44}{5}\binom{1056}{45}}{\binom{1100}{50}}.$$

Axiomatic probability

If the sample space is not finite, or the outcomes are not equally likely, we cannot use classical definition of probability. We consider axiomatic definition of probability.

Def. Let Ω be a sample space, and $\mathcal{C}$ be the collection of all subsets of Ω, then the probability function is the function

$$P : \underbrace{\mathcal{C}}_{\text{Domain}} \longrightarrow \underbrace{[0,1]}_{\text{Range}}$$

satisfying the following axioms:

(1) $P(A) \geq 0$, for any event A.

(2) $P(\Omega) = 1$.

(3) If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

We can prove the following facts:-

Facts: Let $A, B \subseteq \Omega$, we have

(1) $P(\phi) = 0$.

(2) $P(A) + P(\bar{A}) = 1$.

(3) $P(A - B) = P(A) - P(A \cap B)$.

(4) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(5) If $A \subseteq B$, then $P(A) \leq P(B)$.

Let us prove (2), for instance:

Note that $\Omega = A \cup \bar{A}$ and $A \cap \bar{A} = \phi$ mutually disjoint

Then $1 = P(\Omega) = P(A \cup \bar{A}) = P(A) + P(\bar{A})$

By Axiom (3) in the definition.

Ex- If A and B are two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$, find $P(\bar{A} \cap \bar{B})$.

<u>Soln.</u> $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - [\frac{1}{2} + \frac{1}{3} - \frac{1}{4}]$

$= \frac{12 - 6 - 4 + 3}{2}$

$= \frac{5}{12}.$

<u>Ex.</u> Let A and B be two events such that $P(\bar{A} \cap B) = 0.2$ and $P(\bar{A}) = 0.6$. Find $P(A \cup B)$.

<u>Soln.</u> $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(A - B)$

$= (1 - 0.6) + 0.2$

$= 0.6.$

<u>Exc.</u> Prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Searching keywords:

- Classical probability, find the probability of.
- Counting rules, find the number of.
- Multiplication principle, permutations, combinations.
- Axiomatic probability.
- The University of Jordan الجامعة الأردنية
- Principles of Statistics مبادئ الإحصاء
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/131.aspx

For any comments or concerns, please use my email to contact me.

د. بهاء محمود الزالق
The University of Jordan
Dr. Baha Alzalg
baha2math@gmail.com