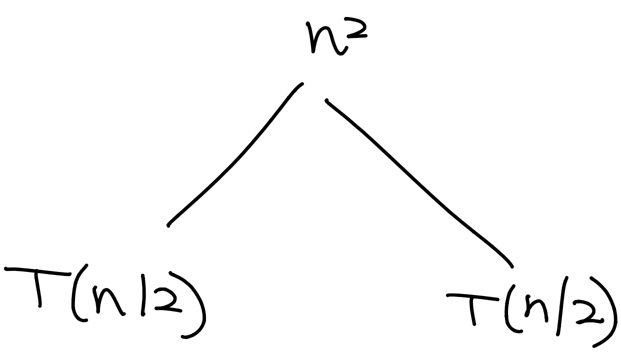
Lecture 27

The Recursion-Tree method for recurrances

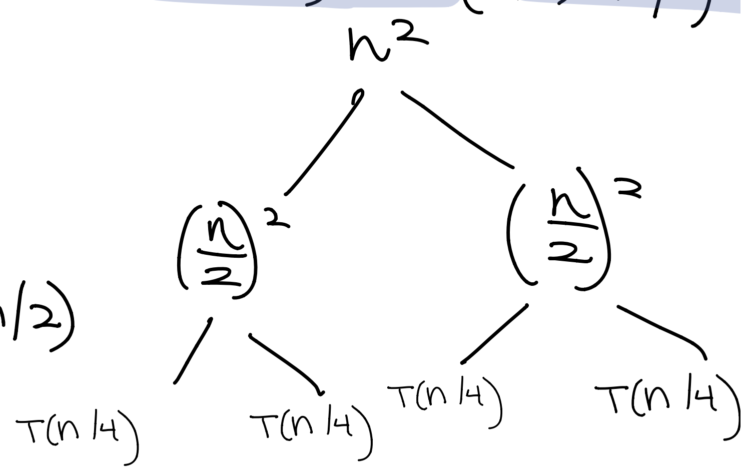
IDEA: Convert recurrance into a tree so each node represents the cost incurred at various levels of recursion. Then sum the costs of all levels.

Ex. Solve $T(n) = 2T(n/2) + n^2$ using a recursion-tree

$T(n) = 2T(n/2) + n^2$

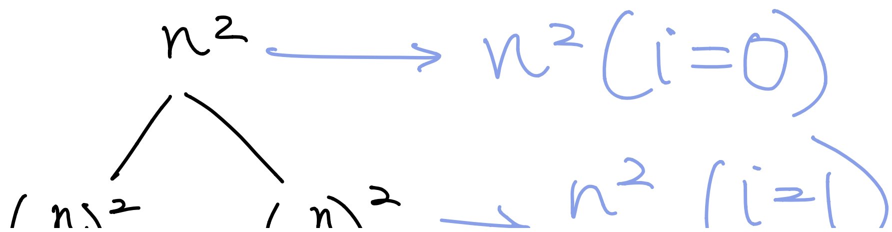


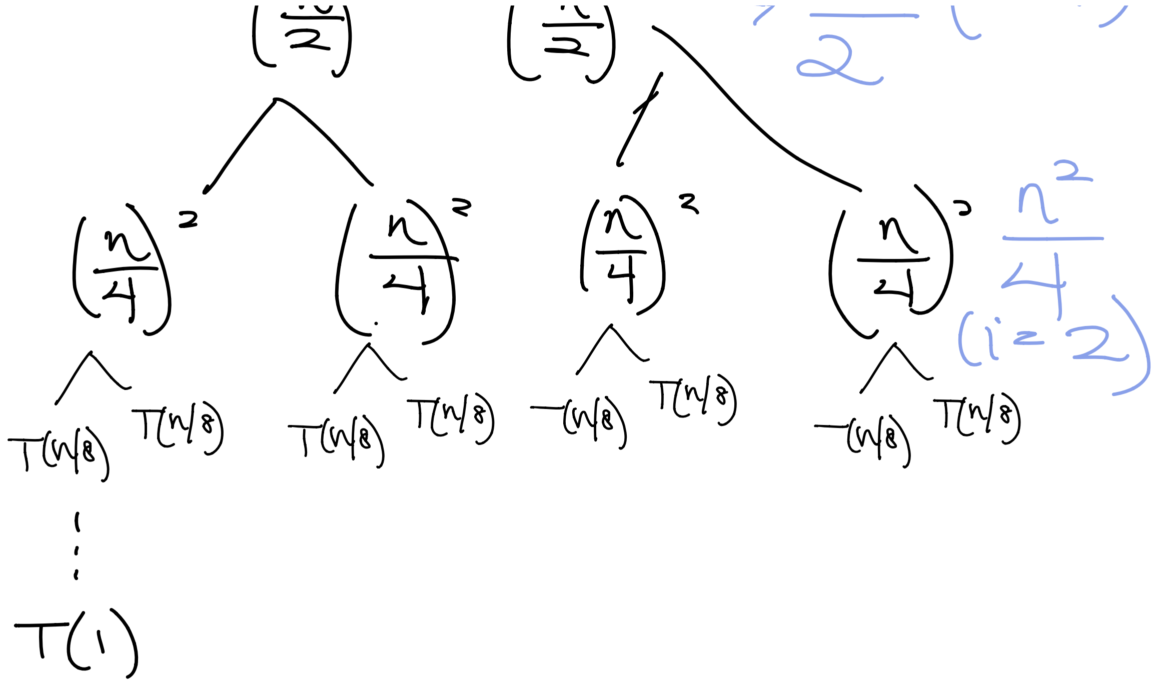
$T(n/2) = 2T(n/4) + (n/2)^2$



$T(n/4) = 2T(n/8) + (n/4)^2$

① write this tree





$$\#\text{ of nodes @ level } i = 2^i$$

$$\text{Cost of node} = \left(\frac{n}{2^i}\right)^2$$

2

$$\begin{array}{c}\text{Total cost}\\ \text{@ level } i\end{array} = \left(\begin{array}{c}\#\text{ nodes}\\ \text{at level } i\end{array}\right)\left(\begin{array}{c}\text{cost}\\ \text{of each}\\ \text{node @ level } i\end{array}\right) = \left(2^i\right)\left(\frac{n}{2^i}\right)^2$$

$$= \frac{n^2}{2^i}$$

3

Total Cost at all Levels =

$$T(n) = \sum_{i=0}^{\text{height of tree}} (\text{total cost at level } i)$$

How to find the height of the tree?

The subproblem size decreases by a factor of 2 everytime we go down a level, eventually we must reach a boundary condition

How far from the root do we reach 1?

Subproblem size for node @ level h

$= n/2^h$

Thus we reach one when

$\frac{n}{2^h} = 1$, hence $h = \log n$

Recall $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ when } |x| < 1$$

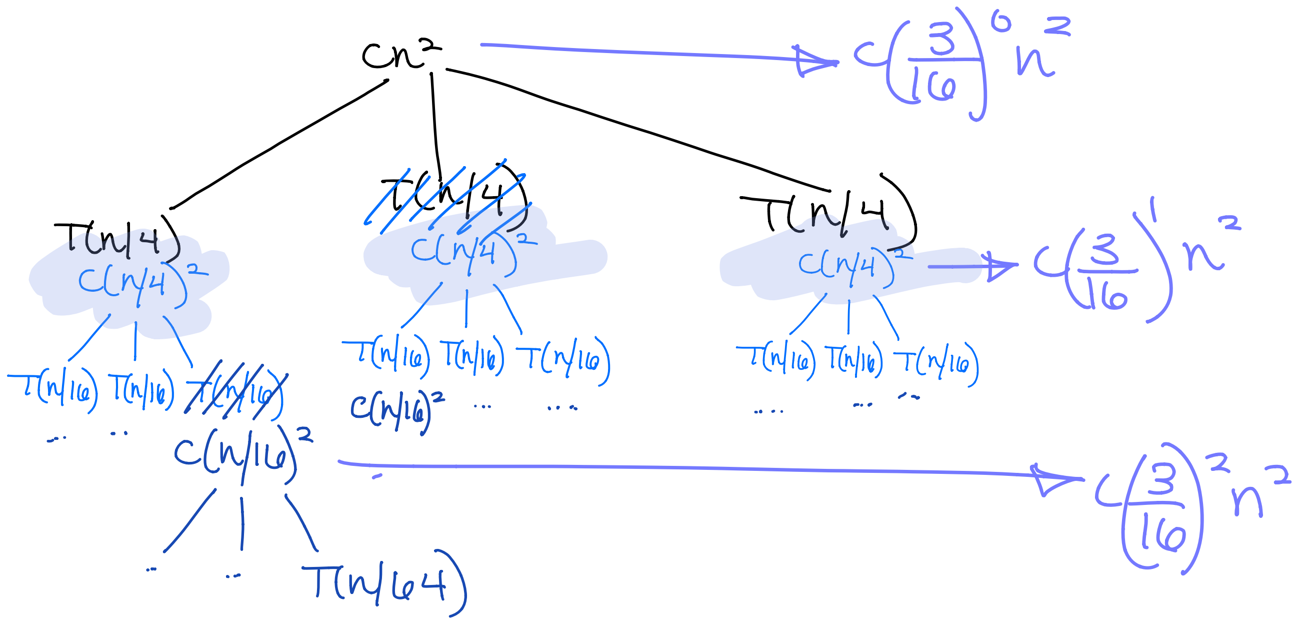
(Geometric Series)

Then $T(n) = \sum_{i=0}^{\log n} \frac{n^2}{2^i} \le n^2$

$$\sum_{i=0}^{\infty} \frac{1}{2} \overset{G.S}{=} n^2\left(\frac{1}{1-\frac{1}{2}}\right) = 2n^2$$

Thus, $T(n) = O(n^2)$

Ex. Solve $T(n) = 3T(n/4) + cn^2$



At level i, the total cost is

$$c\left(\frac{3}{16}\right)^i n^2$$

h: The height of the tree : $\frac{n}{4^h} = 1$

$$h = \log_4 n$$

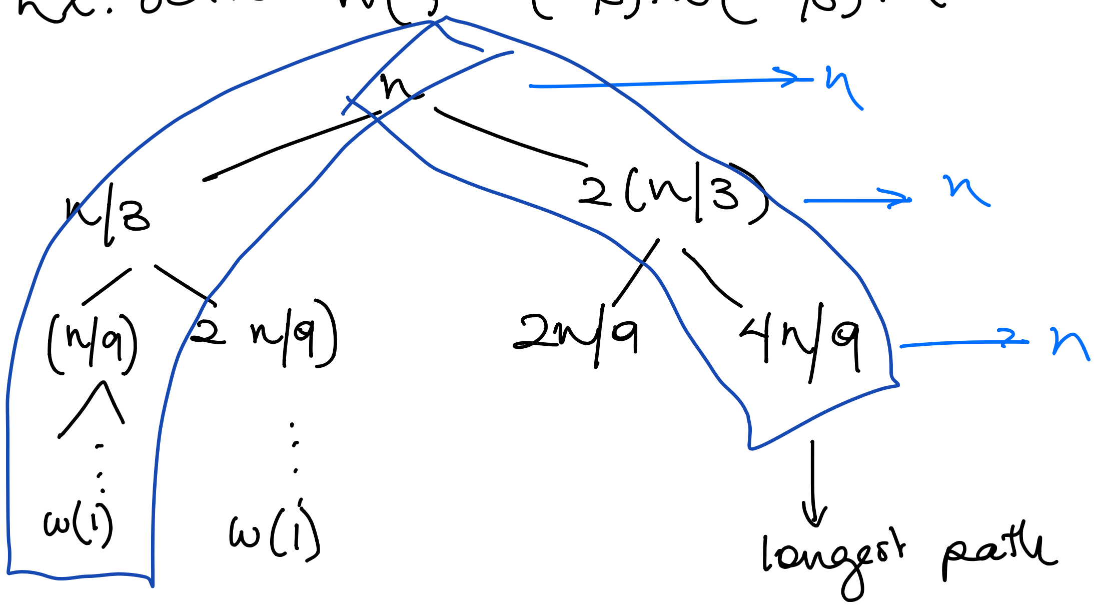
Total Cost at ALL levels

$$T(n) = \sum_{i=0}^{\log_4 n} \left(\frac{3}{16}\right)^i cn^2 \leq cn^2 \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i$$

$$= cn^2\left(\frac{1}{1-\frac{3}{16}}\right) = \frac{16}{13}cn^2$$

Thus $T(n) = O(n^2)$

Ex. Solve $W(n) = W(n/3) + W(2n/3) + n$



Cost @ each = n

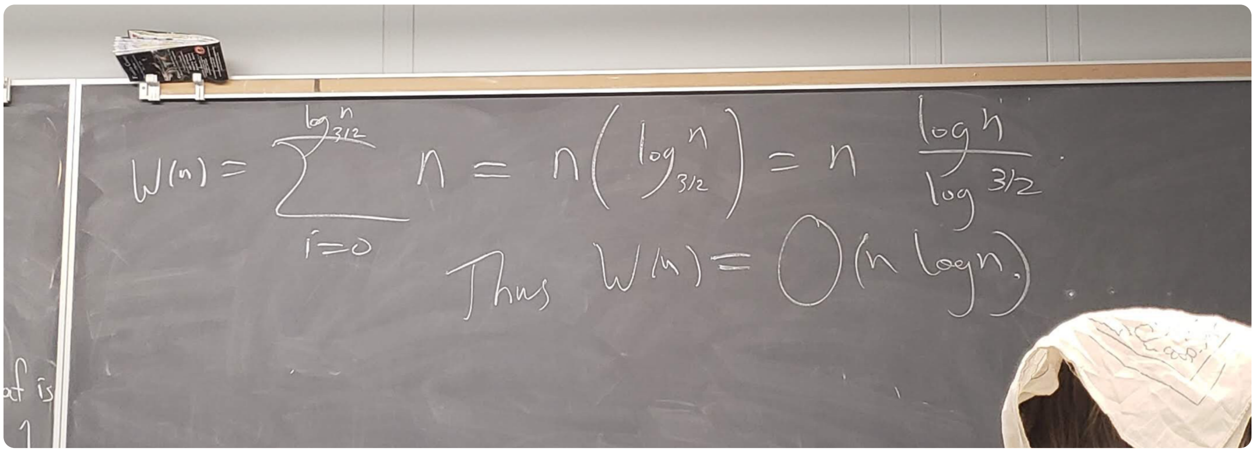
$$W(n) = \sum_{i=0}^{n} n$$

The longest path from root to a leaf is

$$n \to 2/3\,n \to (2/3)^2 n \to \dots \to 1$$

$$\left(\frac{2}{3}\right)^h n = 1$$

$$h = \log_{3/2} n$$



Homework:

https://u.osu.edu/alzalg.1/files/2019/10/updatehw11.pdf