LECTURE 22

$$c \ll \underline{\log n} \ll \log^2 n \ll \sqrt{n} \ll n \ll \underline{n \log n}$$
$$\ll n^{1.1} \ll n^2 \ll n^3 \ll n^4 \ll \underline{2^n \ll 3^n} \ll n! \ll \underline{n^n}$$

Ex. Show that $\log(n!) = O(n \log n)$

Pf. If $n \geq 1$, $n! = n(n-1)(n-2) \dots (3)(2)(1)$

$$\leq \underbrace{n, n, n \dots\dots n, n, n}_{n \text{ times}}$$

$$= n^n$$

Then $\log(n!) \leq \log(n^n) = n \log n$

$\therefore \log(n!) = O(n \log n)$

Recall:
1. $A \leq B \Rightarrow \log A < \log B$
2. $\log A^p = p \log A$
3. $B^{\log_B A} = A$
   In particular, $2^{\log A} = A$
4. $\log(A/B) = \log A - \log B$

Ex. Show that $(\sqrt{2})^{\log n} = O(\sqrt{n})$

Pf. $(\sqrt{n})^{\log n} = (2^{1/2})^{\log n} = 2^{\frac{1}{2}\log n} = \cancel{2}^{\cancel{\log} n^{1/2}}$

$= n^{1/2} = \sqrt{n}$

(Thus $\sqrt{2}^{\log n} = O(\sqrt{n})$

Same order

Ex. Prove that $n^8 + 7n^7 - 10n^5 - 2n^4 + 3n^2 - 17 = \Theta(n^8)$

Pf. To prove the result, we show that $f(n) = O(n^8)$ and $f(n) = \Omega(n^8)$

If $n \geq 1$, we have

① $n^8 + 7n^7 - 10n^5 - 2n^4 + 3n^2 - 17$

$\leq n^8 + 7n^7 + 3n^2 \leq n^8 + 7n^8 + 3n^8$

$= \underbrace{11}_{c} n^8$

$\therefore f(n) = O(n^8)$

② Prove that $f(n) = \Omega(n^8)$ by proving $cn^8 \leq f(n)$ for some $c > 0$ for all $n \geq n_0$

Since $f(n) = n^8 + 7n^7 - 10n^5 - 2n^4 + 3n^2 - 17$

Remove positive terms (keep n^8 for

powers)

$$\stackrel{n\geq 1}{\geq} n^8 - 10n^5 - 2n^4 - 17$$

$$\geq n^8 - 10n^7 - 2n^7 - 17n^7$$

$$= n^8 - 29n^7$$

If you take to the 8th, would be negative

Instead we work on $n^8 - 29n^7$ by finding c & n_0 so that

$$cn^8 \leq n^8 - 29n^7$$

Note that $cn^8 \leq n^8 - 29n^7 \iff 29n^7 \leq (1-c)n^8$

$$\iff 29 \leq (1-c)n$$

$$\iff c \leq 1 - \frac{29}{n}$$

If $n \geq 58$ (2×29), then $c = 1/2$ suffices

Thus, if $n \geq 58$, then $\frac{1}{2}n^8 \leq f(n)$

$$f(n) = \Omega(n^8)$$

c

$$cn^8 \leq n^8 - 29n^7$$

$$cn^8 - n^8 \leq -29n^7$$

$$n^8(c-1) \leq -29n^7$$

$$c - 1 \leq \frac{-29n^7}{n^8}$$

$$c \leq 1 - \frac{29}{n}$$

Proving that Big O Relationship does not hold

To prove $f(n)$ is not big-O of some other func. $g(n)$, we assume that witnesses n_0 and c exist & derive a contradiction

Ex. Prove that n^2 is not $O(n)$

Pf. Suppose that $n^2 = O(n)$. Then there exist n_0 & c such that

$$n^2 \leq cn \text{ for all } n \geq n_0.$$

- Pick n_1 equal to the larger of $2c$ and n_0

Then $n_1^2 \leq cn_1 \xrightarrow{\text{divide by } n_1} n_1 \leq c$

$\rightarrow\leftarrow$

Contradiction tells us that n^2 is not $O(n)$

Proof using limits

Brief Review of Limits

Thm. Let a and c be real #s then

1. $\lim_{n\to\infty} = a$
2. If $a > 0$, $\lim_{n\to\infty} n^a = \infty$
3. If $a < 0$, $\lim_{n\to\infty} n^a = 0$
4. If $a > 1$, $\lim_{n\to\infty} a^n = \infty$
5. If $0 < a < 1$, $\lim_{n\to\infty} a^n = 0$
6. If $c > 0$, $\lim_{n\to\infty} \log_c^n = \infty$

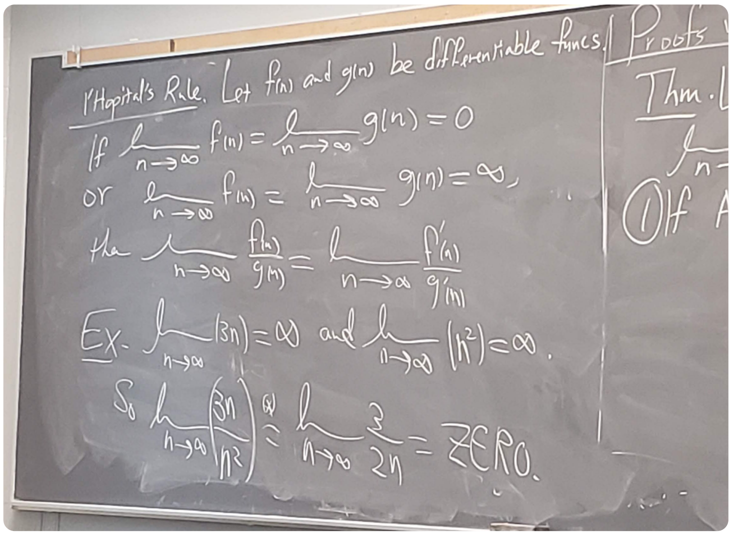
Thm: If $\lim_{n\to\infty} f(n) = \infty$, then $\lim_{n\to\infty} \frac{1}{f(n)} = 0$

Thm: Let a be a finite real #
and let $\lim_{n\to\infty} f(n) = A$ and $\lim_{n\to\infty} g(n) = B$
(A & B are finite real #s) Then

1. $\lim_{n\to\infty} a\, f(x) = a \lim_{n\to\infty} f(n) = aA$
2. $\lim_{n\to\infty} (f(n) + g(n)) = A \pm B$
3. $\lim_{n\to\infty} f(n)\, g(n) = A \cdot B$

④ If $B \neq 0$, $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \frac{A}{B}$

L'Hôpital's Rule



Proofs Using Limits

<u>Thm</u> let $f(n)$ & $g(n)$ be funcs such that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = A \text{ then}$$

① If $A = 0$, then $f(n) = O(g(n))$

② If $A = \infty$, then $f(n) =$

③ If $0 < A < \infty$, then $f(n) = \Theta(g(n))$

Use limits to prove that

① $5n^8 = \Theta(n^8)$ make $\frac{f(n)}{g(n)}$

pf. $\lim_{n\to\infty} \frac{5\cancel{n^8}}{\cancel{n^8}} = 5 = A$

So $5n^8 = \Theta(n^8)$

② $n^2 = O(n^4)$

pf. $\lim_{n\to\infty} \frac{n^2}{n^4} = \lim_{n\to\infty} \frac{1}{n^2} = \text{ZERO}$

③ $n^4 - 23n^3 + 12^2 + 15n - 21 = \Theta(n^4)$

pf. $\lim_{n\to\infty} \frac{n^4 - 23n^3 + 12n^2 + 15n - 21}{n^4}$

$= \lim_{n\to\infty} \left(\cancel{\frac{n^4}{n^4}}^{\,1} - \cancel{\frac{23n^3}{n^4}}^{\,0} + \cancel{\frac{12n^2}{n^4}}^{\,0} + \cancel{\frac{15n}{n^4}}^{\,0} - \cancel{\frac{21}{n^4}}^{\,0} \right)$

$= 1$ Case 3