Lecture 2₁

Θ (Big Theta)

$\Theta(g(n))$ is the set of functions w/ the same order of growth as $g(n)$.

So $g(n)$ is asymptotically tight bound for $f(n)$

$\Theta(g(n)) = \{ f(n)$ there exist positive constants c_1, c_2 and n such that for all $n \geq n_0$

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \}$$

$\rightarrow g(n)$

Ex. prove that $\underbrace{n^2+5n+7}_{f(n)} = \Theta(n^2)$

Pf. when $n \geq 1$, then

$$\underbrace{1n^2}_{g} \leq \underbrace{n^2+5n+7}_{f} \leq n^2+5n^2+7n^2 = \underbrace{13n^2}_{c_2\, g}$$

So, $n^2+5n+7 = \Theta(n^2)$ by def. of Θ with $n_0 = 1$, $c_1 = 1$ & $c_2 = 13$

Ex. prove that $n^5+n^3+7n+1 = \Theta(n^5)$

Properties of the Notations

<u>Thm</u> $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ & $f(n) = \Omega(g(n))$

<u>Pf</u> (Exc) Proof follows from the defns.

Ex. Show that $\frac{1}{2}n^2+3n=\Theta(n^2)$

w/out using defn. of Θ

Pf. If $n \geq 1$, $\frac{1}{2}n^2+3n \leq \frac{1}{2}n^2+3n^2=\frac{7}{2}n^2$

So, $\frac{1}{2}n^2+3n=O(n^2)$ — (1)

when $n \geq 0$, $\frac{1}{2}n^2 \leq \frac{1}{2}n^2+3n$

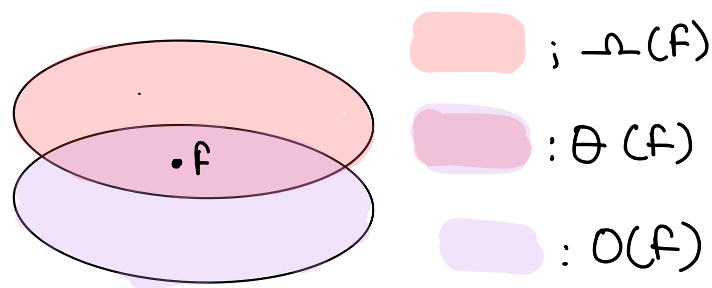
So, $\frac{1}{2}n^2+3n=\Omega(n^2)$

(1) + (2) + TH $\Rightarrow$ $\frac{1}{2}n^2+3n=\Theta(n^2)$

Relations b/w

O, Θ

and Ω



Thm (Transitivity) If $f(n)=O(g(n))$ and $g(n)=O(h(n))$, then $f(n)=O(h(n))$

Same for Big Θ & Ω

$$\begin{tikzcd} f \arrow[r, bend left] \arrow[rr, bend right] & g \arrow[r, bend left] & h \end{tikzcd}$$

Pf. Exc.

Ex. Since $4n^2 + 3n + 17 \overset{\text{Exc.}}{\underset{\text{Def.}}{=}} O(n^3)$

and $n^3 \overset{\text{Exc}}{\underset{\text{Def}}{=}} O(n^4)$

We have $4n^2 + 3n + 17 \overset{\text{TH}}{=} O(n^4)$

Thm (Scaling by a constant)
If $f(n) = O(g(n))$, then $K(f(n)) = O(g(n))$ for any $K > 0$

Pf. Assume that $f(n) = O(g(n))$ then by def. there exist positive constants c and n_0 such that

$$f(n) \le c\,g(n) \text{ for all } n \ge n_0$$

Let $K > 0$, then $Kf(n) \le \underbrace{Kc}_{c'} g(n)$
for all $n \ge 0$

Thus $Kf(n) = O(g(n))$

FACT - from Calculus let a, b and x be positive real #s with $a, b \neq 1$ Then $\log_a^x = \frac{\log_b^x}{\log_b^a}$

Ex Let $a, b \neq 1$ be positive #s then $\log_a^n = \Theta(\log_b^n)$

Thm (Sums) If $f(n) = O g(n)$ and $f_2(n) = O g(n)$ then $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$
$= O(\max\{g_1(n), g_2(n)\})$

Same for Θ & Ω

Pf. We prove that the result for O notation

$f_i(n) = O g_i(n)$ for $i = 1, 2$

Def - There exist + constants c_i and n_i such that $f_i(n) \leq c_i g_i(n)$, for all $n \geq n_i$ $(i = 1, 2)$

Let $n_0 = \max\{n_1, n_2\}$

If $n \geq n_0$, we have

$f_1(n) + f_2(n) \leq c_1 g_1(n) + c_2 g_n(n)$
$\leq \max\{c_1, c_2\} g_1(n)$
$+ \max\{c_1, c_2\} g_2(x)$

$$= \underbrace{\max\{c_1, c_2\}}_{\text{constant}} (g_1(n) + g_2(n))$$

$$\leq \underbrace{\max\{c_1, c_2\}}_{\text{constant}} (2 * \max\{g_1(n), g_2(n)\})$$

Thus, $f_1 + f_2 \in O(g_1 + g_2)$

also $f_1 + f_2 \in O(\max_i g_i)$

Ex. previously we proved
$n^2 + n = O(n^2)$ and $3n^3 \ldots = O(n^3)$

Then
$$(n^2+n) + (3n^3 - 2n^2 + 13n - 15) = \underbrace{O(n^3)}_{\max\{n^2, n^3\}}$$

Thm (Products)
If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$
then $f_1(n) f_2(n) = O(g_1(n) g_2(n))$
And for Θ and Ω

$$(n^2+n)(3n^3 - 2n^2 + 13n - 15) = O(n^5)$$

Thm (Symmetry) $f(n) = O(g(n))$
IFF $g(n) = \Omega(f(n))$

Thm (Reflexive) $f(n) = O(f(n))$

Homework:

https://u.osu.edu/alzalg.1/files/2019/10/HW9W.pdf