\[ \Theta(\text{Big Theta}) \]

\[ \Theta(g(n)) \text{ is the set of functions with the same order of growth as } g(n). \]

So \( g(n) \) is asymptotically tight bound for \( f(n) \).

\[ \Theta(g(n)) = \{ f(n) \text{ there exist positive constants } c_1, c_2 \text{ and } n \text{ such that for all } \]

\[ n \geq n_0, \quad 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \} \]
Ex. prove that \( n^2 + 5n + 7 = \Theta(n^2) \)

Pf. when \( n \geq 1 \), then

\[
1n^2 \leq n^2 + 5n + 7 \leq n^2 + 5n^2 + 7n^2 = 13n^2
\]

So, \( n^2 + 5n + 7 = \Theta(n^2) \) by def. of \( \Theta \) with \( n_0 = 1 \), \( c_1 = 1 \) \& \( c_2 = 13 \)

Ex. prove that \( n^5 + n^3 + 7n + 1 = \Theta(n^5) \)

Properties of the Notations

Thus \( f(n) = \Theta(g(n)) \) iff \( f(n) = O(g(n)) \) & \( f(n) = \Omega(g(n)) \)

Pf (Exc) Proof follows from the defns
Ex. Show that $\frac{1}{2}n^2 + 3n = \Theta(n^2)$ without using defn. of $\Theta$

**Prf.** If $n \geq 1$, $\frac{1}{2}n^2 + 3n \leq \frac{1}{2}n^2 + 3n = \frac{3n^2}{2}$

So, $\frac{1}{2}n^2 + 3n = \Theta(n^2)$ — (1)

When $n \geq 0$, $\frac{1}{2}n^2 \leq \frac{1}{2}n^2 + 3n$

So, $\frac{1}{2}n^2 + 3n = \Theta(n^2)$

\[ (1) + (2) + \text{TH} \quad \frac{1}{2}n^2 + 3n = \Theta(n^2) \]

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Relations b/w

O, $\Theta$

and $\Omega$

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**Thm (Transitivity)** If $f(n) = O(g(n))$ and $g(n) = \Theta(h(n))$, then $f(n) = \Theta(h(n))$

Same for Big $\Omega$ & $\Omega$

$f \sim g \sim h$
pf: Exc.

Ex. Since $4n^2 + 3n + 17 \overset{Exc.}{=} O(n^3)$

and $n^3 \overset{Def.}{=} O(n^4)$

we have $4n^2 + 3n + 17 \overset{th}{=} O(n^4)$

Thm (Scaling by a constant):

If $f(n) = O(g(n))$, then $Kf(n) = O(g(n))$

for any $K > 0$

pf. Assume that $f(n) = O(g(n))$

then by def. there exist positive constants $c$ and $n_0$ such that

$f(n) \leq cg(n)$ for all $n \geq n_0$

let $K > 0$, then $Kf(n) \leq Kcg(n) = \overset{C}{c'}$

for all $n \geq 0$

(Thus $Kf(n) = O(g(n))$)
FACT - let \( a, b \) and \( x \) be positive real numbers with \( a, b \neq 1 \) then \[ \log_a^x = \frac{\log_b^x}{\log_a^b} \]

Ex. Let \( a, b \neq 1 \) be positive numbers then

\[ \log_a n = O(\log_b n) \]

**Theorem (sums)** If \( f(n) = O(g(n)) \) and \( f_2(n) = O(g(n)) \)

then \( f_1(n) + f_2(n) = O(g_1(n) + g_2(n)) \)

\[ = O(\max \{g_1(n), g_2(n)\}) \]

Same for \( \Theta \) and \( \Omega \)

**Pf.** We prove that the result for \( O \) notation

\[ f_i(n) = O(g_i(n)) \text{ for } i=1,2 \]

**Def.** There exist positive constants \( c_i \) and \( n_i \) such that \( f_i(n) \leq c_i g_i(n) \), for all \( n \geq n_i \) (\( i=1,2 \))

Let \( n_0 = \max \{n_1, n_2\} \)

If \( n \geq n_0 \), we have

\[ f_1(n) + f_2(n) \leq c_1 g_1(n) + c_2 g_2(n) \]

\[ \leq \max \{c_1, c_2\} g_1(n) + \max \{c_1, c_2\} g_2(n) \]

\[ = \Theta(\max \{g_1(n), g_2(n)\}) \]
\[
\begin{align*}
&= \max \{c_1, c_2 \? (g_1(n) + g_2(n))
\leq \max \{c_1, c_2 \? (2 \? \max \{g_1(n), g_2(n)\})

(\text{Thus, } f_1 + f_2 \leq O(g_1 + g_2))

(\text{also } f_1 + f_2 \leq O(\max g_i))

(\text{Ex: previously we proved })
\quad n^2 + n = O(n^2) \text{ and } 3n^3 \ldots = O(n^3)

(\text{Then })
\quad (n^2 + n) + (3n^3 - 2n^2 + 12n - 15) = O(n^3)

(\text{and for } \Theta \text{ and } \Omega)
\quad (n^2 + n) (3n^3 - 2n^2 + 12n - 15) = O(n^5)
\end{align*}
\]
Thm (Symmetry) \( f(n) = O(g(n)) \) if and only if \( g(n) = O(f(n)) \).

Thm (Reflexive) \( f(n) = O(f(n)) \).

Homework: