More examples on running time

Ex) Find the running time without line-by-line analysis

1. \( x = 0 \)
2. \( \text{for } i = 1 \text{ to } n \) do
3. \( \text{for } j = 1 \text{ to } n \) do
4. \( x = x + (i - j) \)
5. return(x)

*Instead of doing line-by-line analysis, save time by focusing on statement #4!*

If \( n = 3 \), statement 4 runs \( \frac{9}{n^2} \) times

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} C = \sum_{i=1}^{n} C(n) = Cn^2
\]
Ex 2) Find running time
(1) \( X = 0 \)
(2) for \( i = 1 \) to \( n \) do
(3) \[ \text{for } j = 1 \text{ to } i \text{ do} \]
(4) \( X = X + (i - j) \); \( \text{Cost = } C \)
(5) return \( X \)

If \( n = 3 \), statement 4 runs \( 3 \) time

\[
\sum_{k=1}^{n} k = \frac{n(n+1)}{2}
\]

Running Time = \( C \left( \frac{n(n+1)}{2} \right) \approx \mathcal{O}(n^2) \)
Ex 3) Find running time

1. \( X = 0 \)
2. for \( i = 1 \) to \( n \) do
3. for \( j = i \) to \( n \) do
4. \( X = X + (i - j) \);
5. return \( X \);

Suppose \( n = 3 \) determine the times (4) executes

\[
\begin{align*}
\sum_{i=1}^{n} \sum_{j=i}^{n} (i-j) &= \sum_{i=1}^{n} \left( i(n) - \frac{n(n+1)}{2} - \frac{n(n-1)}{2} - \ldots - \frac{n(i+1)}{2} \right) \\
&= \frac{n(n+1)(n+2)}{6} - \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2} - \frac{n(n+1)}{2} \\
&= 0
\end{align*}
\]

Running Time is \( \left( \frac{n(n+1)}{2} \right)^2 \)
Upper/Lower Bounds for running times
- Some summations are difficult
- Bounds simplify

Requires - upper bound (UB) & lower bound (LB) must be the same function
only differing by a constant.

Constant for UB > constant for LB

Methods
Remove terms of expressions being subtracted, if helpful.
Substitute term (upper or lower bound of summation) into expression

LB: Split summations to reduce size.
Substitute term into expression
Example 3 (Again)

(1) \( X = 0 \)
(2) \textbf{for} \( i = 1 \) \textbf{to} \( n \) \textbf{do}
(3) \hspace{1em} \textbf{for} \( j = i \) \textbf{to} \( n \) \textbf{do}
(4) \hspace{3em} \( X = X + (i - j) \); \( \rightarrow C \)
(5) \textbf{return} \( X \);

\[
\sum_{i=1}^{n} \sum_{j=1}^{i} C = \sum_{i=1}^{n} C(n-1)
\]

\textbf{UB:} \( \sum_{i=1}^{n} C(n-1) \leq \sum_{i=1}^{n} Cn = Cn(n) = Cn^2 \)

\textbf{LB:} \( \sum_{i=1}^{n} C(n-1) \geq \sum_{i= \frac{n}{2}}^{n/2} C(n-1) \geq \sum_{i= \frac{n}{2}}^{n} C(n-n) \)

\( \geq \sum_{i=1}^{n/2} C(n-1) \geq \sum_{i=1}^{n/2} C(n-\frac{n}{2^i}) \)

\( = \sum_{i=1}^{n/2} C(n-\frac{n}{2^i}) \geq C(n-\frac{n}{2^i}) \)

\( = C(\frac{n}{2}) \cdot C(\frac{n}{2}) \cdot \frac{n}{2} \)

Split the summation (of the first sum) a good bound
Running Time: $c'n^2$

where $\frac{c}{4} \leq c' \leq c$
Homework: