Lecture 15

Induction

- Prove that the sum of the first n odd natural numbers is equal to n^2

BY INDUCTION:

$$P(n) = \text{"} 1+3+5+\ldots+(2n-1) = n^2 \text{"}$$
for all $n \in \mathbb{N}$

Base Case:

$P(1)$, substitute $n=1$

$$1 = 2(1)-1 = 1^2 \checkmark$$

Inductive Hypothesis: Assume that $P(k)$ holds. That is $1+3+5+\ldots+(2k+1) = k^2$ is true.

Inductive Step: Want: P(k+1) holds

That is: $\underbrace{1+3+5+\ldots+(2k-1)}_{k^2} + ((2(k+1)-1) \stackrel{?}{=} (k+1)^2$

$= k^2 + 2(k+1) - 1$

$= k^2 + 2k + 2 - 1$

$= k^2 + 2k + 1$

$= (k+1)^2$

$P(k+1)$ is true, thus $P(n)$ holds for all $n \in \mathbb{N}$

Ex) prove that for all $n \in \mathbb{N}$, we have $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Base Case: $n = 1$

$$1^2 \stackrel{?}{=} \frac{1(1+1)(2(1)+1)}{6}$$

$$1 = \frac{1(2)(3)}{6}$$

$1 = 1$ ✓

Recurrance - an equation or inequality that describes a function in terms of its values on smaller inputs.

Ex) ① $T(n) = T(n-1) + 1,\ T(0) = 1$

$T(0) = 1$

$T(1) = T(0) + 1 = 2$ ETC.

$T(2) = T(1) + 1 = 3$

② $T(n) = 3T(n-1) + 4,\ T(1) = 1$

③ $T(n) = T(n/2) + 1,\ T(1) = 0$

Ex) Let $T(n) = T(n-1) + 1$

$n \in \mathbb{N}$ and $T(0) = 1$.

Show that $T(n) = n+1$

By Induction on n

Base Case:

$T(0) = 1 = 0 + 1$

Inductive Hypothesis:

Assume that $T(k) = k+1,\ k \in \mathbb{N}$

Inductive Step:

WANT: $T(k+1) = (k+1)+1$

$$T(k+1) = T(k)+1 = (k+1)+1 = k+2$$

$T(k+1)$ is true

Summations

① Arithmetic Series

$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n+1}{2}$$

② $$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots n^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

③ $$\sum_{k=1}^{n} k^3 = 1^3+2^3+\ldots n^3$$

$$= \frac{n^2(n+1)^2}{4}$$

④ Geometric Series

$$\sum_{k=1}^{n} x^k = 1 + x + x^2 \ldots x^n = \frac{x^{n+1} - 1}{x - 1}$$

(5) Special case geometric series

IF $|x| < 1$

$$\sum_{k=0}^{n} x^k = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

(6) Harmonic Series

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} \ldots \frac{1}{n}$$

$$= \ln(n) + O(n)$$

(7) $$\sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2} \text{ for } |x| < 1$$

Homework:

https://u.osu.edu/alzalg.1/files/2019/09/hw6.pdf