CSE 2321 WorkSheet 6

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Question 1.

[5+5 points] Complete the proof of the following well-known theorem.

Theorem: Let G be an undirected graph on n vertices and m edges. The following statements are equivalent.
(a) G is a tree.
(b) Any two vertices in G are connected by a unique simple path.
(c) G is connected but each edge is a bridge.
(d) G is connected and $m = n - 1$.
(e) G is acyclic and $m = n - 1$.
(f) G is acyclic but if any edge is added to G it creates a cycle.

Proof. We need to show that $(a) \to (b) \to (c) \to (d) \to (e) \to (f) \to (a)$.

$(a) \to (b)$: Since every tree is connected, there is at least one path between any two vertices in G. To show that there is a unique path between any two vertices in G, we use a contradiction. Suppose that there are at least two paths between some pair of vertices of G. The union of these two paths contains a cycle. So, G contains a cycle. This contradicts the fact that G is a tree. Thus, there is a unique path between any two vertices in G.

$(b) \to (c)$: The proof of this part is missing!

$(c) \to (d)$: By assumption, G is connected. So we need only to show that $m = n - 1$. We prove this by induction. It is clear that a connected graph with $n = 1$ or $n = 2$ vertices has $n - 1$ edges. Assume that every graph with fewer than n vertices satisfying (c) also satisfies (d). Let G be an n-vertex connected graph but $G - e$ is disconnected for every edge e of G. Let e' be any edge of G. Now, $G - e'$ is disconnected. Hence $G - e'$ has two connected components. Let G_1 and G_2 be the connected components of G. Let n_i and m_i be the number of vertices and edges in G_i, for $i = 1, 2$. Now, each component satisfies (c) because $n_i < n$ for $i = 1, 2$. By induction hypothesis, we have $m_i = n_i - 1$, for $i = 1, 2$. So, $m = m_1 + m_2 + 1 = (n_1 - 1) + (n2 - 1) + 1 = n - 1$. Thus, by induction principle, G has exactly $n - 1$ edges.

$(d) \to (e)$: We have to show that every connected graph G with n vertices and $n - 1$ edges is acyclic. We prove this by induction. For $n = 1, 2$, it is clear that all connected graphs with n vertices and $n - 1$ edges are acyclic. Assume that every connected graph with fewer than n vertices satisfying (d) is acyclic. Let G be an n-vertex connected graph with $n - 1$ edges. Because G is connected and has $n - 1$ edges, G has a vertex, say x, of degree 1. Let $G' = G - \{x\}$.Then, G' is connected and has $n - 1$ vertices and $n - 2$ edges. By induction hypothesis, G' is acyclic. Because x is a 1-degree vertex in G, x can not be in any cycle of G. Since $G' = G - \{x\}$ is acyclic, G must be acyclic. So, by induction, every n-vertex connected graph with $n - 1$ edges is acyclic.

$(e) \to (f)$: The proof of this part is missing!

$(f) \rightarrow (a)$: Assume that G is acyclic but adding (x, y) to G creates a cycle for every $x, y \in V$ with $(x, y) \notin E$. To prove that G is a tree we must show that G is connected. Let u and v be arbitrary vertices in G. If u and v are not already adjacent, adding the edge (u, v) creates a cycle in which all edges but (u, v) belong to G. Thus, there is a path from u to v. Since u and v were chosen arbitrarily, G is connected.

Question 2.

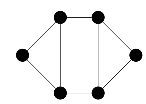
[5 points] The most important formula for studying planar graphs is Euler's formula.[1] The following is a well-known theorem by Euler.

Theorem [Euler's formula]: Let $G = (V, E)$ be a finite, connected, planar graph that is drawn in the plane without any edge intersections. Let also F be the set of faces (regions bounded by edges, including the outer, infinitely large region) of G, then

$$|V| - |E| + |F| = 2.$$

As an illustration, in the graph shown to the right, we have

$$|V| - |E| + |F| = 6 - 8 + 4 = 2.$$



Prove this theorem by induction on m where $m = |E|$.

[1] Euler's formula was first proved by Leonhard Euler (1707 - 1783), a Swiss mathematician who made important and influential discoveries in many branches of mathematics.