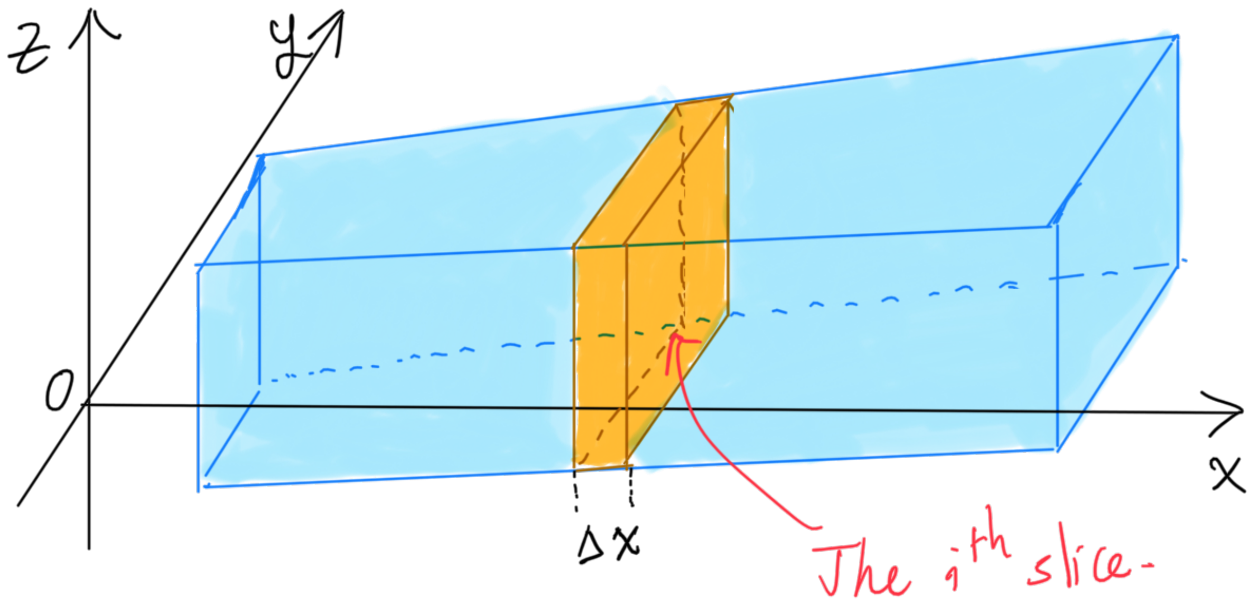
Volumes.

Volumes of solids with known cross sections.



$$V_i \cong A_i(x)\,\Delta x.$$

$$V = \sum_{i=1}^{n} V_i \cong \sum_{i=1}^{n} A_i(x)\,\Delta x.$$

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A_i(x)\,\Delta x = \int_a^b A(x)\,dx$$

The volume of the solid.

⊛ If the cross sections generated are perpendicular to the x-axis, then their areas will be functions of x, denoted by $A(x)$. The volume V of the solid on the interval $[a,b]$ is

$$\boxed{V = \int_a^b A(x)\,dx}.$$

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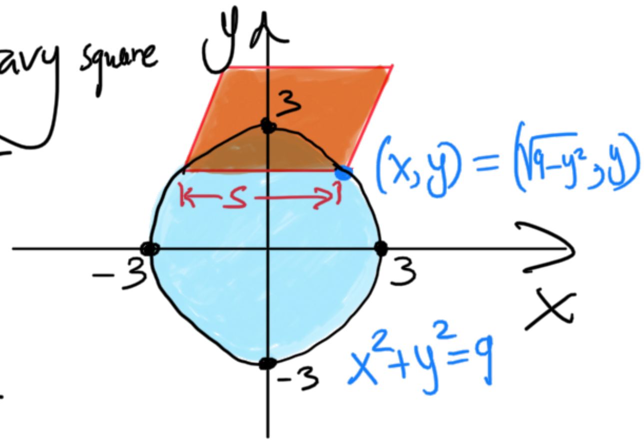
③ If the cross sections generated are perpendicular to the y-axis, then their areas will be functions of y, denoted by $A(y)$. The volume V of the solid on the interval $[c,d]$ is

$$\boxed{V = \int_c^d A(y)\,dy}.$$

Ex. Find the volume of the solid whose base is the region inside the circle $x^2+y^2=9$ if cross sections taken perpendicular to the y-axis are squares.

Soln. The area $A(y)$ of an arbitrary square cross section is $A(y)=s^2$, where $s = 2\sqrt{9-y^2}$, hence



$$A(y) = \left(2\sqrt{9-y^2}\right)^2 = 4(9-y^2).$$

The volume of the solid is

$$V = \int_{-3}^{3} 4(9-y^2)\,dy = 4\left[9y - \tfrac{1}{3}y^3\right]_{-3}^{3} = 4\left[18-(-18)\right] = 144.$$

Ex. Find the volume of the solid whose base is the region bounded by the lines $x+4y=4$, $x=0$ and $y=0$, if the cross sections taken perpendicular to the x-axis are semicircles.

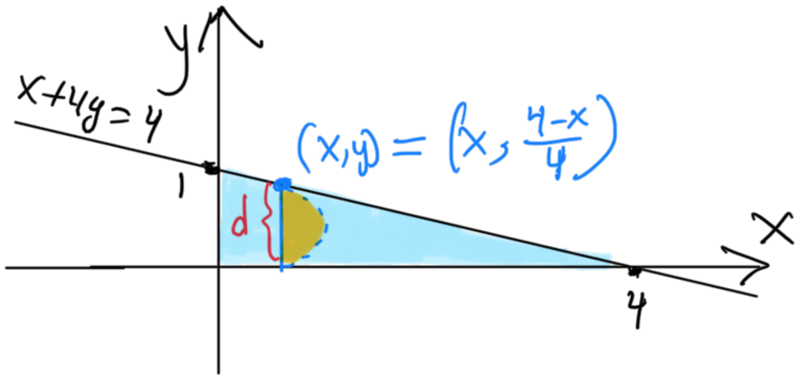
Soln. The area $A(x)$ of an arbitrary semicircle is

$$A(x) = \frac{1}{2}\pi\left(r(x)\right)^2$$

$$= \frac{1}{2}\pi\left(\frac{1}{2}d(x)\right)^2$$

$$= \frac{1}{2}\pi\left(\frac{1}{2}\left(\frac{4-x}{4}\right)\right)^2$$

$$= \frac{1}{128}\pi(4-x)^2.$$

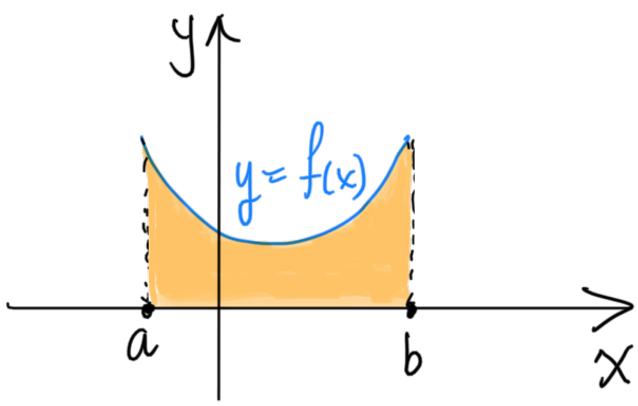


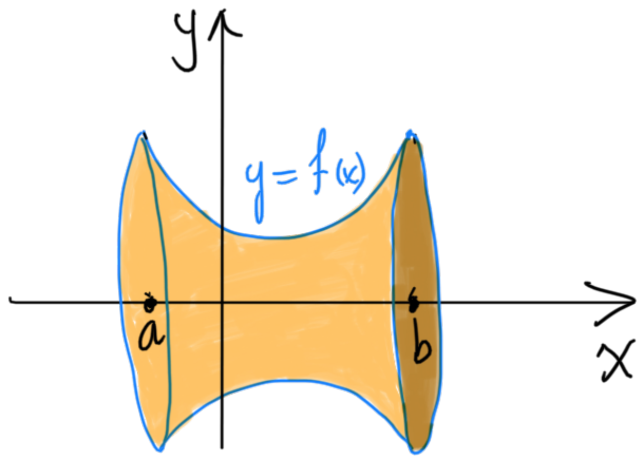
The volume of the solid is

$$V = \int_0^4 \frac{1}{128}\pi(4-x)^2\,dx = \frac{\pi}{128}\left.\frac{(4-x)^3}{-3}\right|_0^4 = \frac{\pi}{128}\left(\frac{64}{3}\right) = \frac{\pi}{6}.$$

Solids of revolution

Disk Method about the x-axis



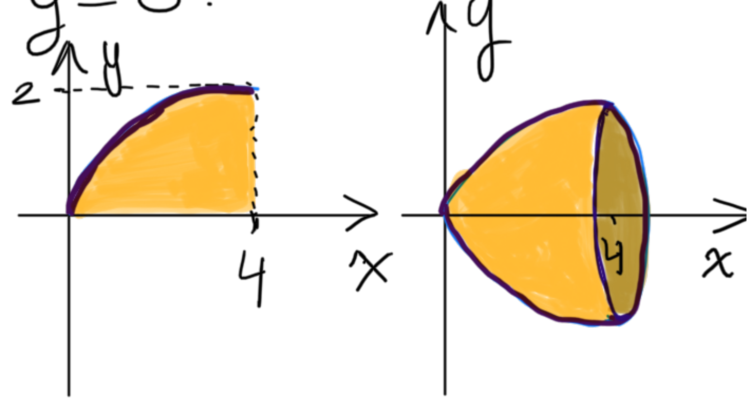


The volume of the solid generated by revolution about the x-axis (Disk Method) is given by

$$\boxed{V = \pi\int_a^b (f(x))^2\,dx}.$$

Ex. Sketch the region Ω bounded by the curves and find the volume of the solid generated by revolving this region about the x-axis.

(1) $y=\sqrt{x}$, $x=0$, $x=4$, $y=0$.



Soln. $V=\pi\int_0^4 (\sqrt{x})^2\,dx$

$= \pi\int_0^4 x\,dx$

$= \pi\left(\frac{x^2}{2}\right|_0^4$

$= 8\pi$.

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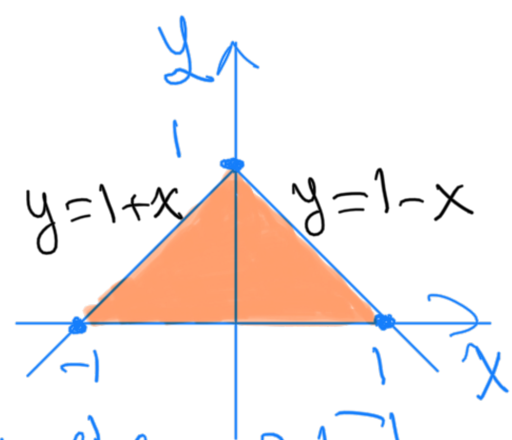
(2) $y=1-|x|$, $y=0$.



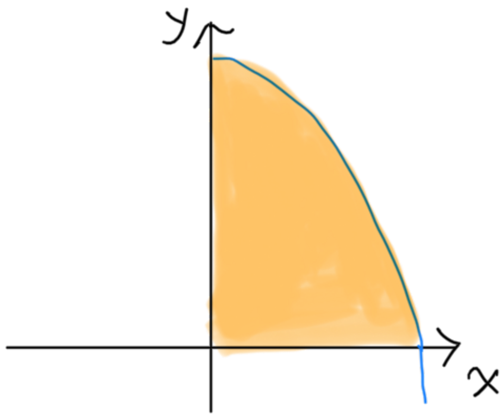
Soln. $1-|x| = \begin{cases} 1+x, & \text{for } x>0. \\ 1-x, & \text{for } x<0. \end{cases}$

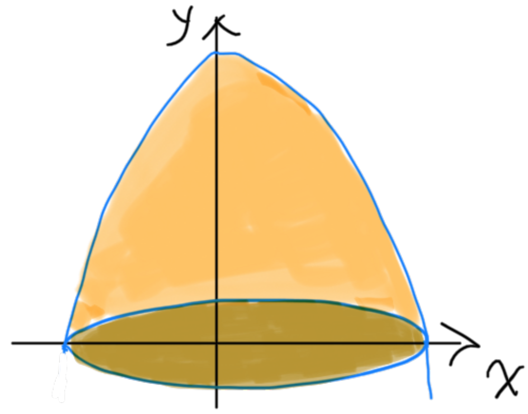
$$V=\pi\int_{-1}^{1}(f(x))^2\,dx = \pi\left[\int_{-1}^{0}(1+x)^2\,dx + \int_0^1 (1-x)^2\,dx\right]$$

$$= \pi\left[\frac{(1+x)^3}{3}\right]_{-1}^{0} + \left[\frac{(1-x)^3}{3}\right]_0^1 = \frac{2}{3}\pi.$$

The volume of the solid generated by revolution about the y-axis (Disk Method) is

$$\boxed{V = \pi \int_c^d \left(F(y)\right)^2 dy}.$$



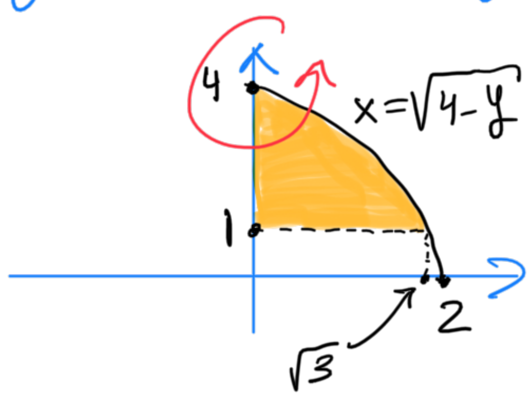


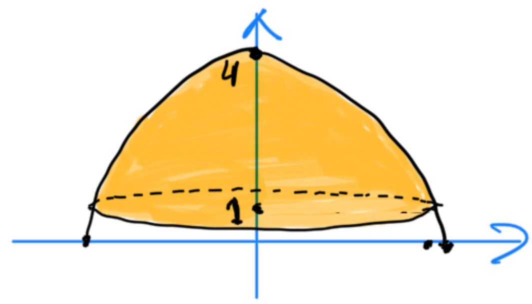
Ex. Sketch the region Ω bounded by the curves

$y = 4 - x^2$, $x = 0$, $x = \sqrt{3}$, and $y = 1$,

and find the volume of the solid generated by revolving this region about the y-axis.

Soln.



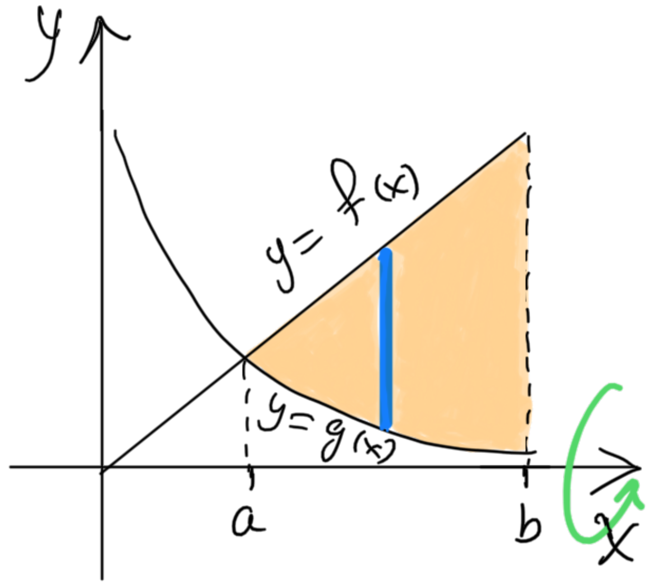


If $y = 4 - x^2$, then $x = \pm\sqrt{4-y}$.

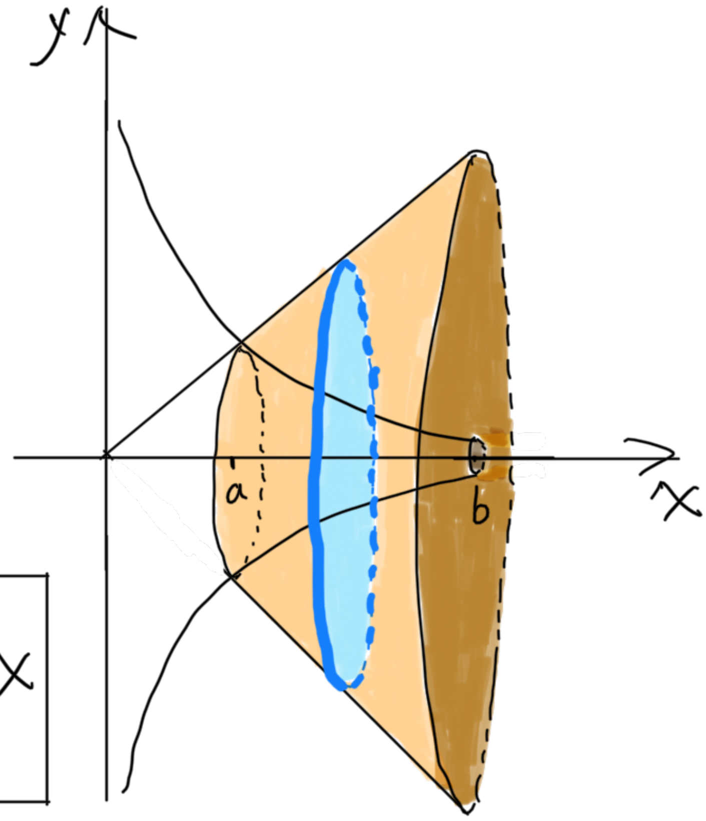
Now, when $\sqrt{4-y} = \sqrt{3}$ when $4 - y = 3$, that is $y = 1$.

$$V = \int_1^4 \pi \left(\sqrt{4-y}\right)^2 dy = \pi \int_1^4 (4-y)\, dy = \pi \left[4y - \frac{y^2}{2}\right]_1^4 = \frac{9\pi}{2}.$$

Washer method about the x-axis

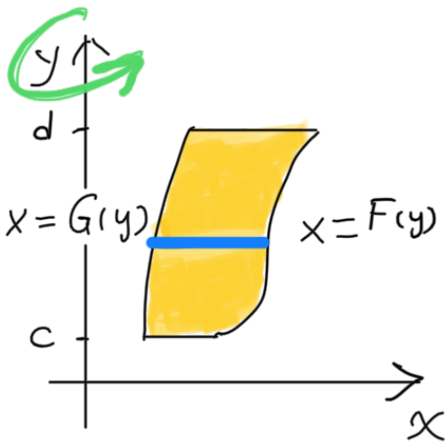


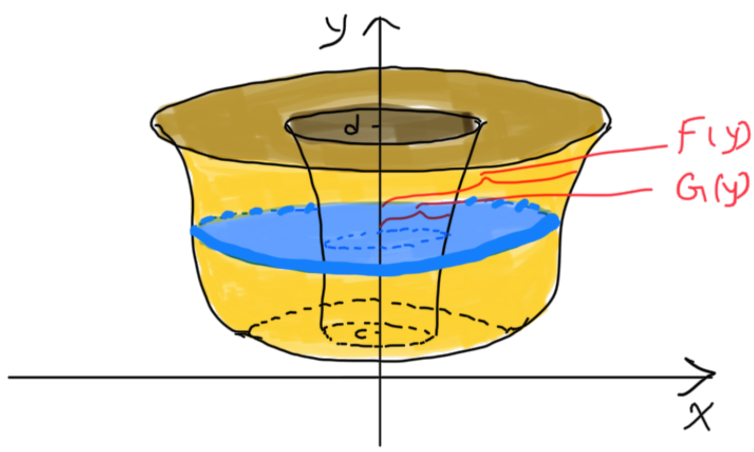
$$\boxed{V = \pi \int_a^b \left[(f(x))^2 - (g(x))^2 \right] dx}$$



Washer method about the y-axis

$$\boxed{V = \pi \int_c^d \left[(F(y))^2 - (G(y))^2 \right] dy}$$

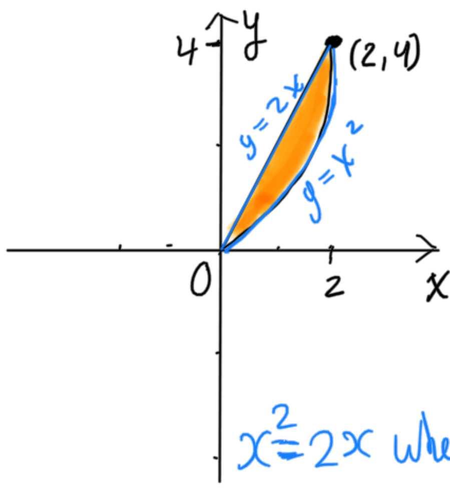


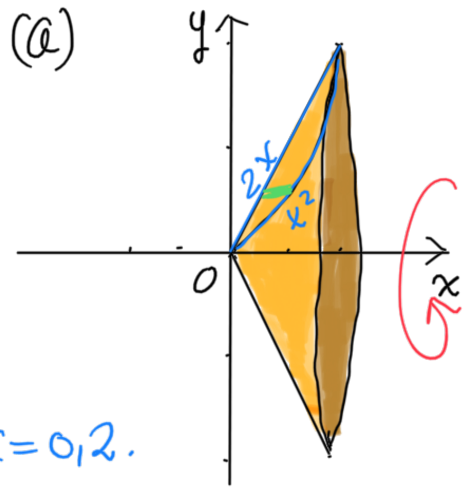


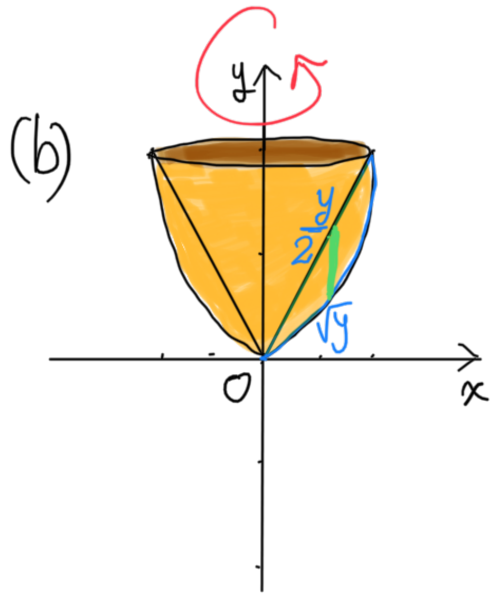
Ex. Sketch the region Ω bounded by the curves and find the volume of the solid generated by revolving this region about

(a) the x-axis. (b) the y-axis.

① $y = x^2$ and $y = 2x$.







Soln. (a) $V = \pi \int_0^2 \left[(2x)^2 - (x^2)^2 \right] dx$

$= \pi \int_0^2 (4x^2 - x^4)\, dx$

$= \pi \left[\frac{4}{3} x^3 - \frac{1}{5} x^5 \right]_0^2$

$= \frac{64}{15} \pi$.

(b) $V = \pi \int_0^4 \left[(\sqrt{y})^2 - (y/2)^2 \right] dy$

$= \pi \int_0^4 (y - y^2/4)\, dy$

$= \pi \left[\frac{1}{2} y^2 - \frac{1}{12} y^3 \right]_0^4$

$= \frac{8}{3} \pi$.

② $y = \frac{1}{4} x^2$, $x = 0$, and $y = 1$. Exc.

Ex- Let Ω be the region bounded by $y=4-x^2$ and $y=0$. Find the volume of the solid obtained by revolving Ω about each of the following:-

(a) The x-axis (the line $y=0$).

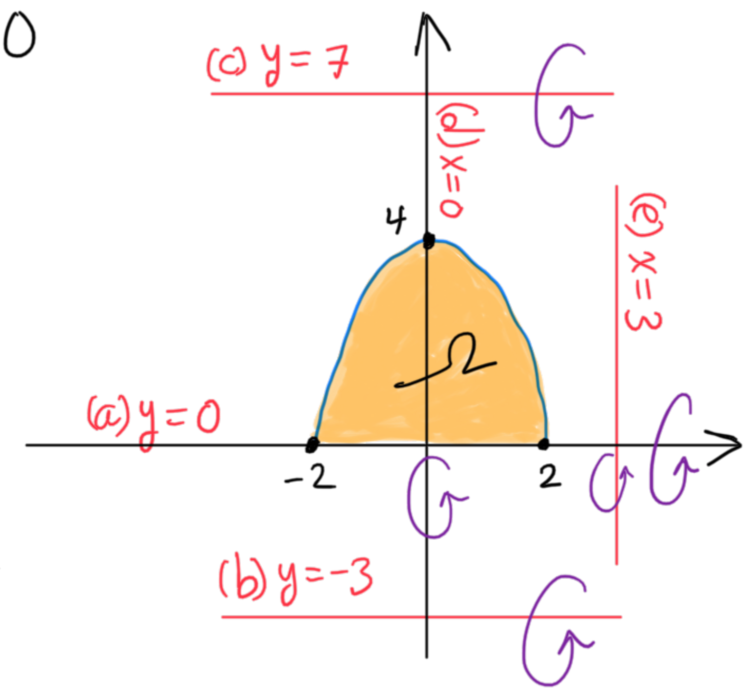
(b) The line $y=-3$.

(c) The line $y=7$.

(d) The line y-axis (the line $x=0$).

(e) The line $x=3$.

Soln. $\boxed{V = \pi \int_a^b \left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2 \right] d(x \text{ or } y)}$.



(a) Outer Radius (O.R.) $= (4-x^2) - 0$
$= 4-x^2$.

Inner Radius (I.R.) $= 0-0$
$= 0$.

$$V = \pi \int_{-2}^{2} \left[(4-x^2)^2 - (0)^2 \right] dx$$
$$= \pi \int_{-2}^{2} (4-x^2)^2 \, dx = \frac{512}{15}\pi.$$

(b) O.R. $= (4-x^2) - (-3) = 7-x^2$.
I.R. $= 0-(-3) = 3$.
$$V = \pi \int_{-2}^{2} \left[(7-x^2)^2 - (3)^2 \right] dx = \frac{1472}{15}\pi.$$

(c) O.R. $= 7-0=7$.

I.R. $= 7-(4-x^2) = 3+x^2$.

$$V = \pi \int_{-2}^{2} \left[(7)^2 - (3+x^2)^2\right] dx = \frac{576}{5}\pi .$$

(d) O.R. $= \sqrt{4-y} - 0 = \sqrt{4-y}$.

I.R. $= 0-0=0$.

$$V = \pi \int_0^4 \left[\left(\sqrt{4-y}\right)^2 - 0^2\right] dy = \pi \int_0^4 (4-y)\, dy = 8\pi .$$

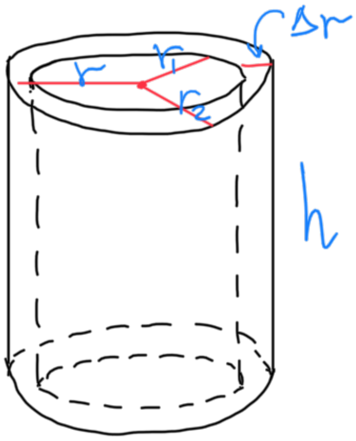
(e) O.R. $= 3-(-\sqrt{4-y}) = 3+\sqrt{4-y}$.

I.R. $= 3 - \sqrt{4-y}$.

$$V = \pi \int_0^4 \left[\left(3+\sqrt{4-y}\right)^2 - \left(3-\sqrt{4-y}\right)^2\right] dy = 64\pi .$$

Volume using cylindircal shells.

Volume of a cylindrical shell



The average radius $r = \frac{1}{2}(r_1+r_2)$.

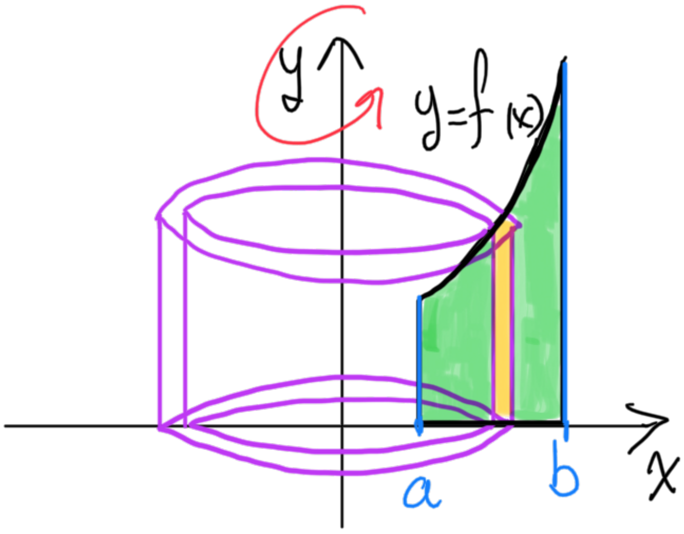
The thickness of the shell $\Delta r = r_2 - r_1$.

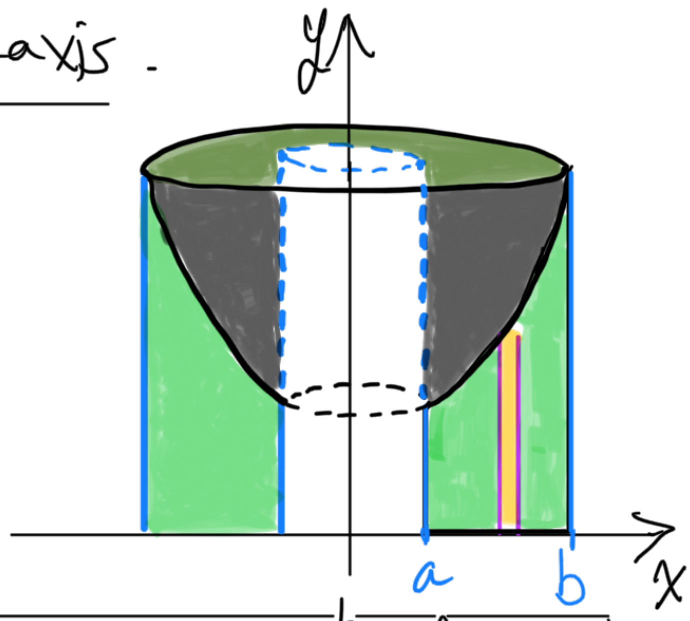
$$V = (\text{circumference})(\text{height})(\text{thickness}) .$$

$$= (2\pi r)(h)(\Delta r) .$$

Volume by the shell method.

Shell method about the y-axis.

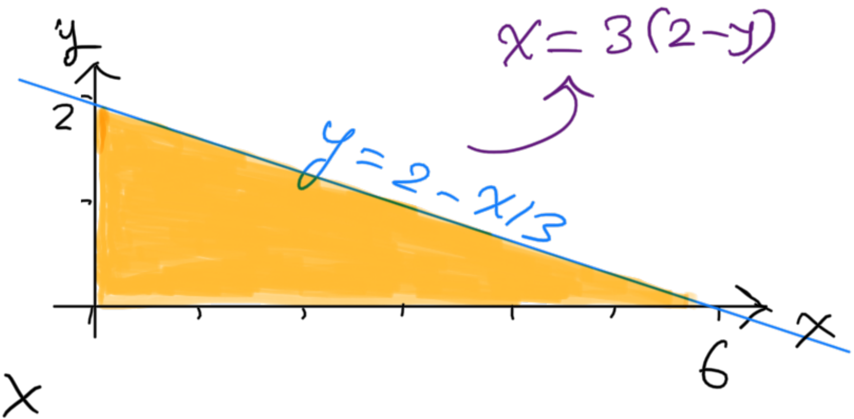




$$V_i \cong 2\pi \bar{x}_i f(\bar{x}_i)\,\Delta x.$$

$$\boxed{V = 2\pi \int_a^b x f(x)\,dx}.$$

Ex. The region bounded by $x+3y=6$, $y=0$, and $x=0$ is rotated about the y-axis. Find the volume of the solid that is generated.



Soln. By the shell method

$$V = 2\pi \int_0^6 x f(x)\,dx$$

$$= 2\pi \int_0^6 x\,(2 - x/3)\,dx$$

$$= 2\pi \int_0^6 \left(2x - x^2/3\right) dx$$

$$= 2\pi \left[x^2 - x^3/9\right]_0^6$$

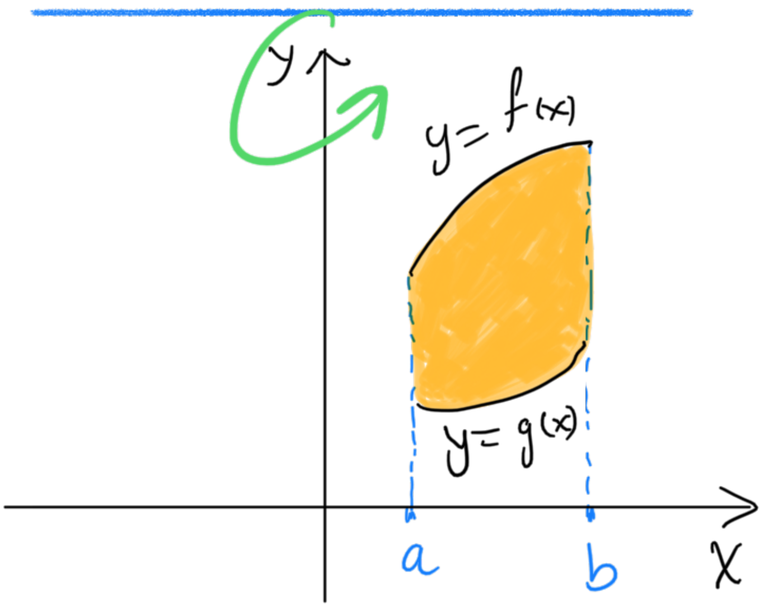
$$= 24\pi.$$

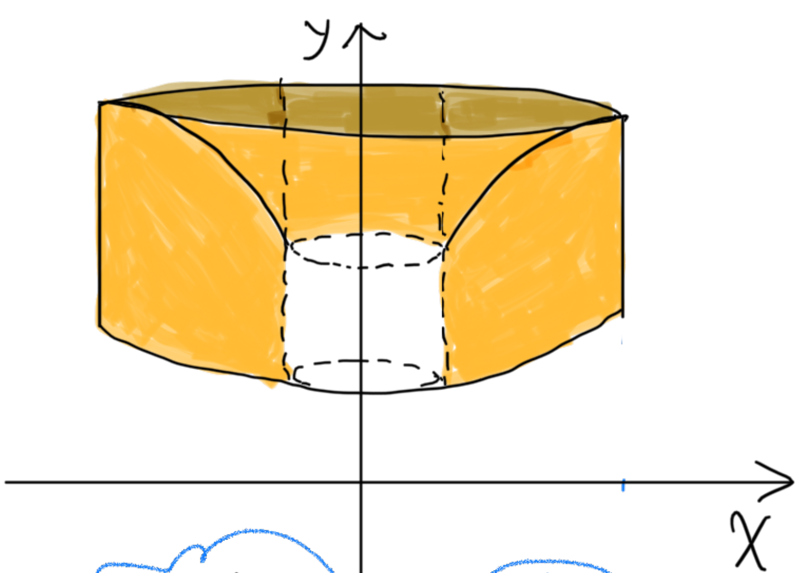
By the washer method

$$V = \pi \int_0^2 [3(2-y)]^2 \, dy = 9\pi \left[\frac{(2-y)^3}{-3} \right]_0^2 = 24\pi .$$

So $$V = \underbrace{2\pi \int_c^d (F(y))^2 \, dy}_{\text{W.M.}} = \underbrace{2\pi \int_a^b x f(x) \, dx}_{\text{SH.M.}}$$

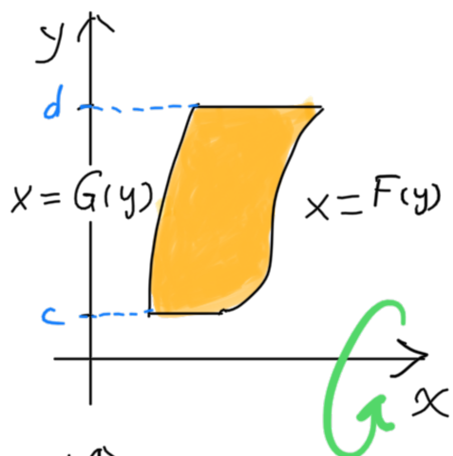
Shell method about y-axis

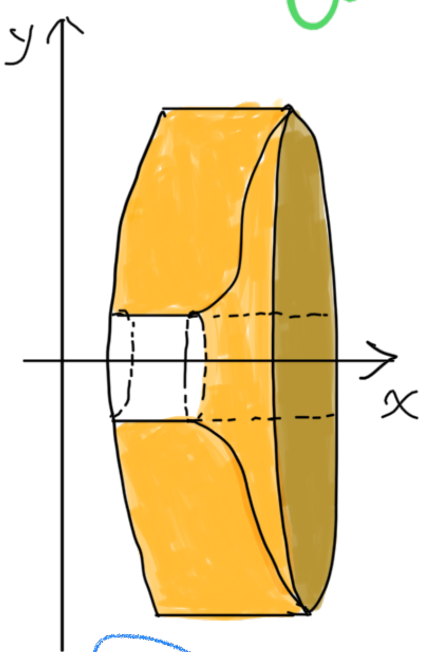




$$V = 2\pi \int_a^b x [f(x) - g(x)] \, dx$$

Shell method about x-axis

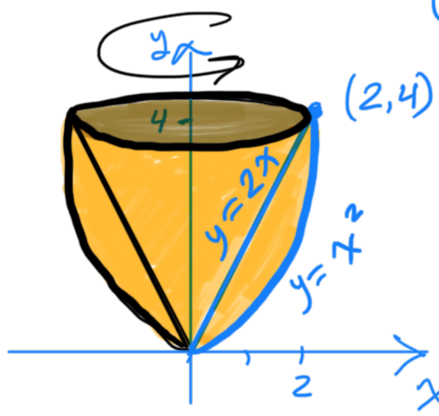




$$V = 2\pi \int_c^d y [F(y) - G(y)] \, dy$$

Ex: Find the volume of the solid generated by revolving the region between $y=x^2$ and $y=2x$ about:

(a) The y-axis (b) The x-axis (c) The line $y=-2$.

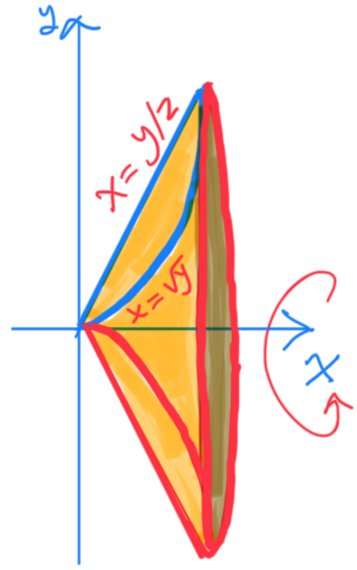


Soln.

(a) $V = 2\pi \int_0^2 x(2x - x^2)\,dx$

$= 2\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2$

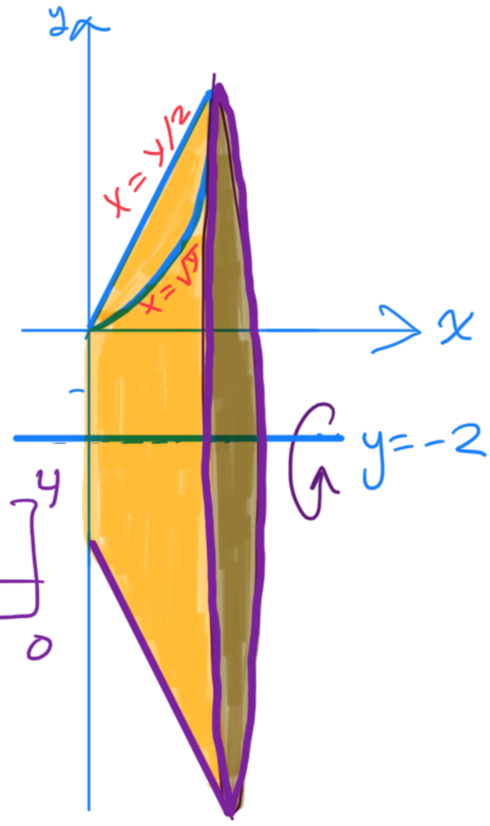
$= \frac{8}{3}\pi.$



(b) $V = 2\pi \int_0^4 \underline{y\left(\sqrt{y} - y/2\right) dy}$

$= 2\pi \left[\frac{2}{5}y^{5/2} - \frac{1}{6}y^3\right]_0^4$

$= \frac{64}{15}\pi.$



(c) $V = 2\pi \int_0^4 \underline{[y-(-2)]\left(\sqrt{y} - \frac{y}{2}\right) dy}$

$= 2\pi \int_0^4 \underline{(y+2)\left(y^{1/2} - \frac{1}{2}y\right) dy}$

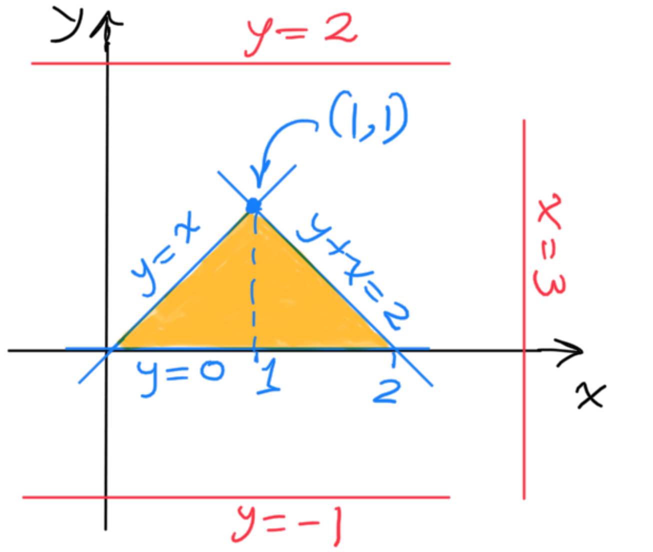
$= 2\pi \int_0^4 \left[y^{3/2} - \frac{1}{2}y^2 + 2y^{1/2} - y\right] dy$

$= 2\pi \left[\frac{2}{5}y^{5/2} - \frac{1}{6}y^3 + \frac{4}{3}y^{3/2} - \frac{1}{2}y^2\right]_0^4$

$= \cdots.$

Ex. Let Ω be the region bounded by the graphs of $y=x$, $y+x=2$, and $y=0$. Compute the volume of the solid formed by revolving Ω about the lines (a) $y=2$. (b) $y=-1$ and (c) $x=3$.

Soln. $x = 2-x$ when $x=1$.



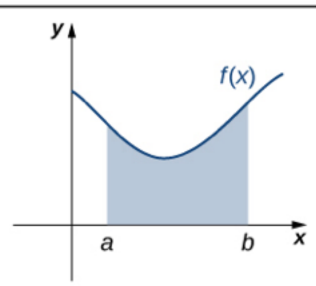
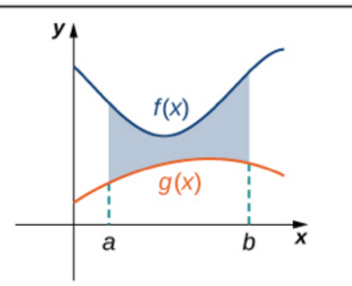
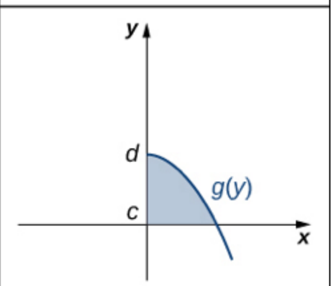
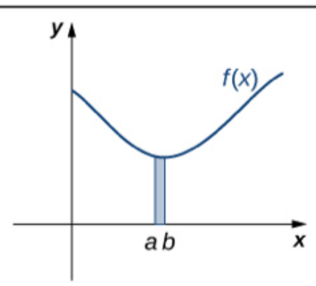
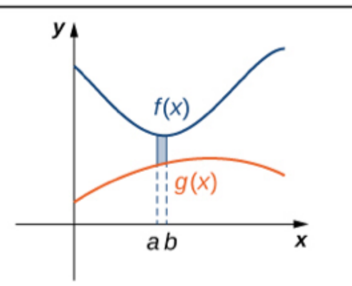
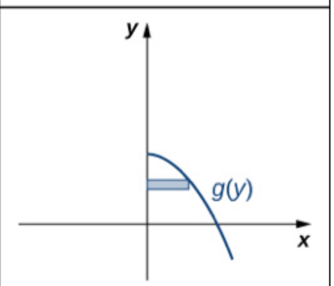
(a) $V = 2\pi \int_0^1 (2-y)\left[(2-y)-y\right] dy = \frac{10}{3}\pi$.

(b) $V = 2\pi \int_0^1 (y-(-1))\left[(2-y)-y\right] dy = \frac{8}{3}\pi$.

(c) $V = 2\pi \int_0^1 (3-x)(x-0)\, dx + 2\pi \int_1^2 (3-x)\left[(2-x)-0\right] dx = 4\pi$. ← SH.M.

or $V = \pi \int_0^1 \left[(3-y)^2 - \left[3-(2-y)\right]^2\right] dy = 4\pi$. ← W.M.

Comparing the Methods for Finding the Volume of a Solid Revolution around the x-axis

Compare	Disk Method	Washer Method	Shell Method
Volume formula	$V=\int_a^b \pi[f(x)]^2\,dx$	$V=\int_a^b \pi\left[(f(x))^2-(g(x))^2\right]dx$	$V=\int_c^d 2\pi y\, g(y)\,dy$
Solid	No cavity in the center	Cavity in the center	With or without a cavity in the center
Interval to partition	$[a, b]$ on x-axis	$[a, b]$ on x-axis	$[c, d]$ on y-axis
Rectangle	Vertical	Vertical	Horizontal
Typical region			
Typical element			

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This lecture: Volumes.
Next lecture: Arc length.

Searching keywords:

- Find the volume of the solid احسب حجم الجسم
- Cross sections, disk, washer, cylinderical shells
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

د. بهاء محمود الزالق
The University of Jordan
Dr. Baha Alzalg
baha2math@gmail.com