Improper Integrals

If we have a definite integral $I = \int_a^b f(x)\,dx$ such that $f(x)$ is not continuous at some point in $[a,b]$, then I is called an improper integral.

Examples:

continuous

(1) $\int_1^2 \frac{dx}{x-1}$ is improper; as $\frac{1}{x-1}$ is not cts at $1 \in [1,2]$.

(2) $\int_0^1 \ln x \, dx$ is improper; as $\ln x$ is not cts at $0 \in [0,1]$.

(3) $\int_{-1}^3 \frac{1}{x}\,dx$ is improper; as $\frac{1}{x}$ is not cts at $0 \in [-1,3]$.

(4) $\int_3^5 \frac{dx}{x-2}$ is ~~not~~ ~~im~~proper; here $\frac{1}{x-2}$ is not cts at $2 \notin [3,5]$.

Question: How can we deal with improper integrals?

Answer: We have the following cases:-

Case 1: If $f(x)$ is not cts at an endpoint (a or b), then

$$\int_a^b f(x)\,dx = \lim_{k \to a^+} \int_k^b f(x)\,dx \quad \text{and} \quad \int_a^b f(x)\,dx = \lim_{k \to b^-} \int_a^k f(x)\,dx$$

If the limit exists, then the improper integral is convergent.

If the limit does not exist, then the improper integ. is divergent.

Ex. Determine if each of the following integrals converges or division. If the integral converges determine its value.

(1) $\displaystyle\int_1^2 \frac{dx}{x-1} = \lim_{k \to 1^+} \int_k^2 \frac{dx}{x-1}$

$\displaystyle= \lim_{k \to 1^+} \Big[\ln|x-1|\Big]_k^2$

$\displaystyle= \lim_{k \to 1^+} \Big[\ln(1) - \ln|k-1|\Big]$

$= 0 + \infty$ DNE. So the integral is divergent

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(2) $\displaystyle\int_0^4 \frac{dx}{\sqrt{x}} = \lim_{k \to 0^+} \int_k^4 \frac{dx}{\sqrt{x}}$

$\displaystyle= \lim_{k \to 0^+} 2x^{1/2}\Big|_k^4$

$\displaystyle= \lim_{k \to 0^+} \left(4 - 2\sqrt{k}\right)$

$= 4$. So the integral is convergent to 4.

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(3) $I = \int_0^1 \ln x \, dx$. $y = \ln x$ is not cts at $x = 0$.

Soln. $\int \ln x \, dx = x \ln x - x + C$.

dv	u	
x	$\ln x$	$+$
1	$1/x$	$-$

$\displaystyle I = \lim_{k \to 0^+} \int_k^1 \ln x \, dx$

$\displaystyle= \lim_{k \to 0^+} \left(x \ln x - x\right)\Big|_k^1$

$\displaystyle= \lim_{k \to 0^+} \left(0 - 1 - k \ln k + k\right)$

$\displaystyle= -1 - \lim_{k \to 0^+} k \ln k$

$$= -1 - \lim_{k \to 0^+} \frac{\ln k}{1/k}$$

L.R.

$$= -1 - \lim_{k \to 0^+} \frac{-1/\cancel{k}}{-1/k^{\cancel{2}}}$$

$$= -1 - \lim_{k \to 0^+} k$$

$= -1$. So the integral is convergent to -1.

Case 2: If $f(x)$ is not ctis at some point $c \in (a,b)$, then

$$\underbrace{\int_a^b f(x)\,dx}_{I} = \underbrace{\int_a^c f(x)\,dx}_{I_1} + \underbrace{\int_c^b f(x)\,dx}_{I_2 \text{ (let)}}.$$

Note that I is convergent if and only if I_1 and I_2 are both convergent.

So, if I_1 or I_2 is divergent, then so is I.

Ex. Determine if the integral converges or diverges. If converges, determine its value.

(1) $$\underbrace{\int_0^2 \frac{x}{x^2-1}\,dx}_{I} = \underbrace{\int_0^1 \frac{x}{x^2-1}\,dx}_{I_1} + \underbrace{\int_1^2 \frac{x}{x^2-1}\,dx}_{I_2}.$$

$I = I_1 + I_2$.

$$I_1 = \lim_{k \to 1^-} \int_0^k \frac{x}{x^2-1}\,dx = \lim_{k \to 1^-} \frac{1}{2}\int_0^k \frac{2x}{x^2-1}\,dx = \lim_{k \to 1^-} \frac{1}{2}\ln|x^2-1| \Big|_0^k$$

$$= \frac{1}{2}\lim_{k \to 1^-} \left(\ln|k^2-1| - 0\right) \quad \text{DNE}.$$

I_1 is divergent, so is I.

(2) $I = \int_0^{\pi} \sec^2 x \, dx.$

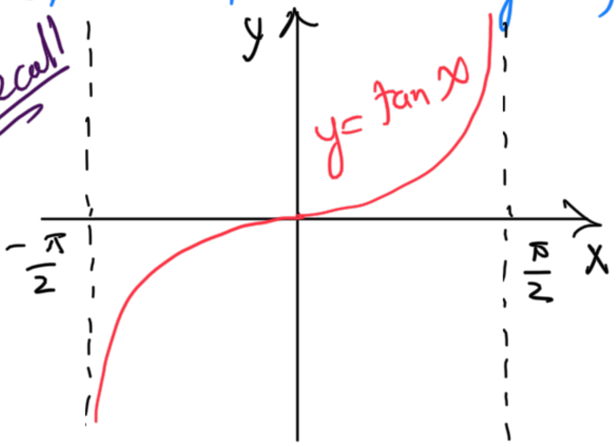
$\sec^2 x = \frac{1}{\cos^2 x}$, $\cos x = 0$ when $x = \frac{\pi}{2}$.

$$I = \underbrace{\int_0^{\pi/2} \sec^2 x \, dx}_{I_1} + \underbrace{\int_{\pi/2}^{\pi} \sec^2 x \, dx}_{I_2}.$$

$$I_1 = \lim_{k \to \frac{\pi}{2}^-} \int_0^k \sec^2 x \, dx = \lim_{k \to \frac{\pi}{2}^-} \tan x \Big]_0^k = \lim_{k \to \frac{\pi}{2}^-} \tan k - 0 = \infty.$$

Since I_1 is divergent, I is divergent.

Recall



$\tan(0) = 0.$

$\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty.$

$\lim_{x \to -\frac{\pi}{2}^+} \tan x = -\infty.$

$\tan^{-1}(0) = 0.$

$\lim_{x \to \infty} \tan x = \frac{\pi}{2}.$

$\lim_{x \to -\infty} \tan x = -\frac{\pi}{2}.$

Case 3: If one of the endpoints is ∞ or $-\infty$, we have

$$\int_a^{\infty} f(x)\,dx = \lim_{k \to \infty} \int_a^k f(x)\,dx \quad \text{and} \quad \int_{-\infty}^{b} f(x)\,dx = \lim_{k \to -\infty} \int_k^b f(x)\,dx.$$

Ex: Determine if the integral converges or diverges. If converges, determine its value.

(1) $I = \int_0^{\infty} \cos x \, e^{-\sin x} \, dx = \lim_{k \to \infty} \int_0^k \cos x \, e^{-\sin x} \, dx.$

Let $u = \sin x$, then $du = \cos x \, dx$. Then

$$I = \lim_{k \to \infty} \int_0^k \cancel{\cos x} \, e^{-u} \frac{du}{\cancel{\cos x}} = \lim_{k \to \infty} \int_0^{\sin k} e^{-u} \, du = \lim_{k \to \infty} -e^{-u} \Big|_0^{\sin k}$$

$$= \lim_{k \to \infty} \left(-e^{-\sin k} + 1 \right)$$ DNE as $\lim_{x \to \infty} \sin x$ DNE. So the integral diverges.

(2) $I=\int_0^\infty (x-1)e^{-x}\,dx = \lim_{k\to\infty}\int_0^k (x-1)e^{-x}\,dx.$

Now $\int (x-1)e^{-x}\,dx = -(x-1)e^{-x} - e^{-x} + C$

$= e^{-x}[-(x-1)-1]+C$

$= -xe^{-x}+C.$

$$\begin{array}{cc} dv & u \\ e^{-x} & x-1 \quad \oplus \\ -e^{-x} & 1 \quad \ominus \end{array}$$

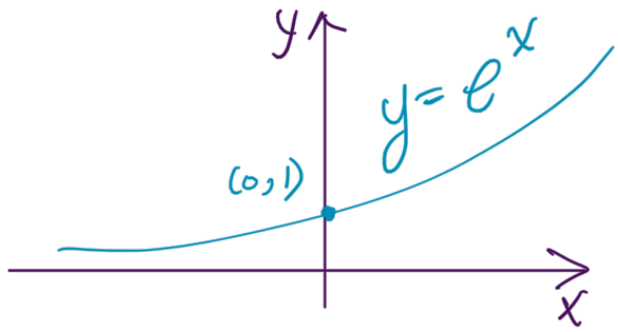
Then $I = \lim_{k\to\infty} -xe^{-x}\Big|_0^k$

$= \lim_{k\to\infty} -ke^{-k}+0$

$= -\lim_{k\to\infty}\frac{k}{e^k}$

$\overset{L.R.}{=} -\lim_{k\to\infty}\frac{1}{e^k}$

$= 0.$ So the integral is convergent to 0.



$e^0=1$

$\lim_{x\to\infty} e^x=\infty$

$\lim_{x\to-\infty} e^{-x}=0$

$\ln(1)=0$

$\lim_{x\to\infty}\ln x=\infty$

$\lim_{x\to 0^+}\ln x=-\infty$

Fact: $\int_1^\infty \frac{dx}{x^p}$ → converges and equals $\frac{1}{p-1}$, if $p>1$; → diverges, if $p\le 1$.

Case 4: If the endpoints are $-\infty$ and ∞, then

$$\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{I} = \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{I_1} + \underbrace{\int_{0}^{\infty} f(x)\,dx}_{I_2}.$$

The integral I is convergent iff I_1 and I_2 are both conv.

Note: In general, $\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{K\to\infty} \int_{-K}^{K} f(x)\,dx$.

For example, take $f(x) = x$.

Now $\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx = \cdots = \infty - \infty$,

while $\lim_{K\to\infty} \int_{-K}^{K} x\,dx = \lim_{K\to\infty} \frac{x^2}{2}\Big|_{-K}^{K} = \lim_{K\to\infty}\left(\frac{K^2}{2} - \frac{K^2}{2}\right) = \text{Zero}$.

Ex: Determine if the integral converges or diverges. If converges, determine its value.

(1) $$\underbrace{\int_{-\infty}^{\infty} \frac{dx}{x^2+1}}_{I} = \underbrace{\int_{-\infty}^{0} \frac{dx}{x^2+1}}_{I_1} + \underbrace{\int_{0}^{\infty} \frac{dx}{x^2+1}}_{I_2}.$$

$$I_1 = \int_{-\infty}^{0} \frac{dx}{x^2+1} = \lim_{K\to-\infty} \int_{K}^{0} \frac{dx}{x^2+1}$$

$$= \lim_{K\to-\infty} \tan^{-1}x \Big|_{K}^{0}$$

goes to $-\pi/2$ when K goes to $-\infty$.

$$= \lim_{K\to-\infty} \left(0 - \tan^{-1}K\right)$$

$$= \pi/2.$$

$$I_2 = \int_{0}^{\infty} \frac{dx}{x^2+1} = \lim_{K\to\infty} \int_{0}^{K} \frac{dx}{x^2+1}$$

$$= \lim_{K\to\infty} \tan^{-1}x \Big|_{0}^{K}$$

goes to $\pi/2$ when K goes to ∞.

$$= \lim_{K\to\infty} \left(\tan^{-1}K - 0\right)$$

$$= \pi/2.$$

Thus, $I = I_1 + I_2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

(2) $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$. Exc.

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(3) $\int_{-\infty}^{\infty} \frac{dx}{x^2-1}$.

Soln. $\underbrace{\int_{-\infty}^{\infty} \frac{dx}{x^2-1}}_{I} = \int_{-\infty}^{0} \frac{dx}{x^2-1} + \int_{0}^{\infty} \frac{dx}{x^2-1}$

$= \int_{-\infty}^{-1} \frac{dx}{x^2-1} + \int_{-1}^{0} \frac{dx}{x^2-1} + \int_{0}^{1} \frac{dx}{x^2-1} + \int_{1}^{\infty} \frac{dx}{x^2-1}$

$= \underbrace{\int_{-\infty}^{-2} \frac{dx}{x^2-1}}_{I_1} + \underbrace{\int_{-2}^{-1} \frac{dx}{x^2-1}}_{I_2} + \underbrace{\int_{-1}^{0} \frac{dx}{x^2-1}}_{I_3} + \underbrace{\int_{0}^{1} \frac{dx}{x^2-1}}_{I_4} + \underbrace{\int_{1}^{2} \frac{dx}{x^2-1}}_{I_5} + \underbrace{\int_{2}^{\infty} \frac{dx}{x^2-1}}_{I_6}$

I is convergent if I_j is convergent for each $j = 1, 2, \dots, 6$.

Note that $\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$; here $A = -1/2$, $B = 1/2$.

Now $\int \frac{dx}{x^2-1} = \int \left(\frac{-1/2}{x-1} + \frac{1/2}{x+1} \right) dx$

$= -\frac{1}{2} \ln|x-1| + \frac{1}{2} \ln|x+1| + C$

$= \frac{1}{2} \ln\left| \frac{x+1}{x-1} \right| + C$.

Next, we find I_j for $j = 1, 2, \dots, 6$. Exc ...

This lecture: Improper integrals.

Next lecture: Areas.

Searching keywords:

- Evaluate the integral احسب التكامل
- Determine if the integral converges or diverges.
- Improper integrals التكاملات المعتلة
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

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