Integration by partial fractions

Let f(x) and g(x) be functions. To evaluate $\int \frac{f(x)}{g(x)}\,dx$, we have the following cases:

Case 1. If $f(x)=g'(x)$, then

$$\int \frac{f(x)}{g(x)}\,dx = \int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)|+C.$$

Ex. Evaluate

(1) $\int \frac{x^2}{2x^3+5}\,dx = \frac{1}{6}\int \frac{6x^2}{2x^3+5} = \frac{1}{6}\ln\left|2x^3+5\right|+C.$

(2) $\int \frac{\sin x+\cos x}{\sin x-\cos x}\,dx = \ln\left|\sin x-\cos x\right|+C.$

Case 2. If $g'(x) \neq f(x)$. Then if f(x) and g(x) are polynomials, we have the following subcases:

Subcase 2I. If $\deg(f) \geq \deg(g)$, then we find the division $f(x)/g(x)$.

Ex. Evaluate.

(1) $I=\int \frac{x^2}{x-1}\,dx$

$$=\int\left(x+1+\frac{1}{x-1}\right)dx$$

$$=\frac{x^2}{2}+x+\ln|x-1|+C.$$

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & \cancel{x^2} \\ & \cancel{\mp x^2} \pm x \\ \hline & \cancel{x} \\ & \cancel{\mp x} \pm 1 \\ \hline & 1 \end{array}$$

(2) $I = \int \frac{3}{\sqrt{x}-1}\,dx$.

Soln. Let $u = \sqrt{x}$, then $u^2 = x$ and $2u\,du = dx$

Then $I = \int \frac{3}{u-1}(2u\,du)$

$$= 6\int \frac{u}{u-1}\,du$$

$$= 6\int \left(1 + \frac{1}{u-1}\right) du$$

$$= 6\left(u + \ln|u-1|\right) + C$$

$$= 6\left(\sqrt{x} + \ln|\sqrt{x}-1|\right) + C.$$

$$\begin{array}{r} 1 \\ u-1 \overline{\smash{)}\ \cancel{u}\qquad} \\ \underline{\cancel{\mp u} \pm 1} \\ 1 \end{array}$$

Subcase 2 II. If $\deg(f) \lneq \deg(g)$, we have the following sub-subcases:

Sub-subcase 2 II A. If $g(x)$ can be written as a product of distinct linear functions.

Function $g(x)$	A product of distinct linear? (Yes or No)
$(x-1)(x-2)(x-3)$	Yes.
$(x-1)^2(x-3)$	No; $(x-1)(x-1)(x-3)$.
$(x^2+1)(x-3)$	No; Not linear.
$(x^2-1)(x-3)$	Yes; $(x-1)(x+1)(x-3)$.
$(x^2-3x+2)(x-3)$	Yes; $(x-1)(x-2)(x-3)$.

Rule: Terms in partial fraction decomposition are as follows

$$\frac{1}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$

$A_1, A_2, \ldots, A_k$ are constants whose values are to be calculated.

Ex. Evaluate the given integrals

(1) $I = \int \frac{5}{x^2-4}\,dx.$

Soln. $\frac{5}{x^2-4} = \frac{5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2)+B(x-2)}{(x-2)(x+2)}.$

Then $5 = A(x+2) + B(x-2)$. To find A and B, take

$x=-2$, then $5 = b(-4)$, so $b = -5/4$.

$x = 2$, then $5 = 4a$, so $a = 5/4$.

It follows $I = \int \left(\frac{5/4}{x-2} + \frac{-5/4}{x+2} \right) dx$

$$= \frac{5}{4}\ln|x-2| - \frac{5}{4}\ln|x+2| + C$$

$$= \ln\left| \left(\frac{x-2}{x+2} \right)^{5/4} \right| + C.$$

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(2) $I = \int \frac{\sqrt{x}}{x-9}\,dx.$

Soln. Let $u = \sqrt{x}$, then $u^2 = x$ and $2u\,du = dx$.

Then $I = \int \frac{u}{u^2-9}(2u\,du)$

$$= \int \frac{2u^2}{u^2-9}\,du$$

$$= \int \left(2 + \frac{18}{u^2-9} \right) du$$

$$\begin{array}{r|l} & 2 \\ \hline u^2-9 & \cancel{2u^2} \\ & \mp\cancel{2u^2} \pm 18 \\ \hline & 18 \end{array}$$

$$= \int \left(2 + \frac{3}{u-3} - \frac{3}{u+3}\right) du$$

$$= 2u + 3 \ln|u-3| - 3 \ln|u+3| + C$$

$$= 2\sqrt{x} + 3 \ln|\sqrt{x}-3| - 3 \ln|\sqrt{x}+3| + C.$$

Sub-subcase 2 II B. If $g(x)$ is a product of repeated linear functions (not distinct).

Rule: Terms in partial fraction decomposition are as follows

$$\frac{1}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

For instance,

$$\frac{*}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

$$\frac{*}{(x-1)^2(x+2)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3}.$$

We also use $\int (ax+b)^n dx = \frac{1}{n+1} \cdot \frac{1}{a} (ax+b)^{n+1} + C;\ n \neq -1.$

Ex. Evaluate

(i) $I = \int \frac{dx}{(x-1)^2(x+2)}.$

Soln. $\frac{1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} = \frac{A(x-1)(x+2) + B(x+2) + C(x-1)^2}{(x-1)^2(x+2)}.$

Then $1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2.$

Take $x=1$, then $1 = 3b$, so $b = 1/3$

$x=2$, then $1 = 9c$, so $c = 1/9$

$x=0$, then $1 = -2a + 2b + c$, so $a = -1/9.$

Then $I = \int \left[\frac{-1/9}{x-1} + \frac{1/3}{(x-1)^2} + \frac{1/9}{x+2} \right] dx$

$= \frac{-1}{9} \ln|x-1| - \frac{1}{3} (x-1)^{-1} + \frac{1}{9} \ln|x+2| + C.$

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(2) $\int \frac{2x^2+3}{x(x-1)} dx.$ Exc.

Sub-sub case 2 II C. If $g(x)$ is a product of functions such that at least one of them is irreducible quadratic function.

For instance,

$$\frac{*}{(x+3)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}.$$

$$\frac{*}{(x+2)^2(x^2+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+1}.$$

$$\frac{*}{(x+2)(x^2+1)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Ex. Evaluate the given integrals.

(1) $I = \int \frac{x^2+x-2}{(3x-1)(x^2+1)} dx.$

Soln. $\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(3x-1)}{(3x-1)(x^2+1)}.$

Then $x^2+x-2 = A(x^2+1) + (Bx+C)(3x-1)$

$= (A+3B)x^2 + (3C-B)x + (A-C).$

Coef. of x^2: $1 = A + 3B$

Coef. of x: $1 = -B + 3C$

Coef. of x^0: $-2 = A - C$

Solving 3 equations with 3 variables, we get $A = \frac{-7}{5}$, $B = \frac{4}{5}$ and $C = \frac{3}{5}$.

Then
$$I = \int \left(\frac{-\frac{7}{5}}{3x-1} + \frac{\frac{4}{5}x + \frac{3}{5}}{x^2+1} \right) dx$$

$$= \frac{-7}{15} \int \frac{3\,dx}{3x-1} + \frac{\overset{2}{\cancel{4}}}{5} \int \frac{2x}{x^2+1}\,dx + \frac{3}{5} \int \frac{dx}{x^2+1}$$

$$= \frac{-7}{15} \ln|3x-1| + \frac{2}{5} \ln(x^2+1) + \frac{3}{5} \tan^{-1} x + C.$$

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(2) $\displaystyle\int \frac{5x^2+6x+2}{(x+2)(x^2+2x+5)}\,dx$. Exc.

Final answer: $\int = 2\ln|x+2| + \frac{3}{2}\ln(x^2+2x+5) - \frac{7}{5}\tan^{-1}\left(\frac{x+1}{2}\right) + C$

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(3) $\displaystyle I = \int \frac{x^5+2}{x^2-1}\,dx$.

Soln.
$$\frac{x^5+2}{x^2-1} = x^3 + x + \frac{x+2}{x^2-1}$$

$$= x^3 + x + \frac{\overset{-1/2}{\boxed{A}}}{x+1} + \frac{\overset{3/2}{\boxed{B}}}{x-1}$$

$$= x^3 + x - \frac{1}{2(x+1)} + \frac{3}{2(x-1)}.$$

$$\begin{array}{r} x^3 + x \\ x^2-1 \overline{\smash{)}\, \cancel{x^5} + 2} \\ \cancel{\mp x^5} \pm x^3 \\ \hline \cancel{x^3} + 2 \\ \cancel{\mp x^3} \pm x \\ \hline x + 2 \end{array}$$

Then
$$I = \int \left(x^3 + x - \frac{1}{2(x+1)} + \frac{3}{2(x-1)} \right) dx$$

$$= \frac{1}{4}x^4 + \frac{1}{2}x^2 - \frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x-1| + C.$$

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(4) $\displaystyle I = \int \frac{dx}{\sqrt[3]{x} + \sqrt{x}}$.

Soln. Let $u^6 = x$, then $6u^5\,du = dx$, $u^2 = x^{1/3}$ and $u^3 = x^{1/2}$.

Then
$$\begin{aligned} I &= \int \frac{6u^5}{u^2+u^3}\,du \\ &= 6\int \frac{u^3}{1+u}\,du \\ &= 6\int \left(u^2 - u + 1 - \frac{1}{u+1}\right) du \\ &= 6\left(\tfrac{1}{3}u^3 - \tfrac{1}{2}u^2 + u - \ln|u+1|\right) + C \\ &= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\left|1+\sqrt[6]{x}\right| + C. \end{aligned}$$

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(5) $I = \int \sqrt{1-e^x}\,dx$.

Soln. Let $u^2 = 1 - e^x$, then $\ln(1-u^2) = x$,

so $\frac{-2u}{1-u^2} = dx$.

$$I = \int u\left(\frac{-2u}{1-u^2}\right) du = \int \frac{2u^2}{u^2-1}\,du = \int \left(2 + \frac{2}{u^2-1}\right) du$$

$$= \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right) du = 2u + \ln|u-1| - \ln|u+1| + C$$

$$= 2u + \ln\left|\frac{u-1}{u+1}\right| + C = 2\sqrt{1-e^x} + \ln\left|\frac{\sqrt{1-e^x}-1}{\sqrt{1-e^x}+1}\right| + C$$

Ex. Show that $\int \frac{dx}{1+\cos x - \sin x} = -\ln|1 - \tan(\frac{x}{2})| + C$.

Proof. To convert a ratio expression in sine and cosine to a rational func. in u, we use the substitution

$$u = \tan\left(\frac{x}{2}\right), \quad -\pi < x < \pi.$$

Then $\cos\left(\frac{x}{2}\right) = \frac{1}{\sec\left(\frac{x}{2}\right)} = \frac{1}{\sqrt{1+\tan^2\left(\frac{x}{2}\right)}} = \frac{1}{\sqrt{1+u^2}}$, and $\sin\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right) \tan\left(\frac{x}{2}\right) = \frac{u}{\sqrt{1+u^2}}$.

It follows that $\sin x = 2\sin(x/2)\cos(x/2) = \frac{2u}{1+u^2}$,

and $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{4u^2}{(1+u^2)^2}} = \sqrt{\frac{1+2u^2+u^4-4u^4}{(1+u^2)^2}}$

$$= \sqrt{\frac{1-2u^2+u^4}{(1+u^2)^2}} = \sqrt{\frac{(1-u^2)^2}{(1+u^2)^2}} = \frac{1-u^2}{1+u^2}.$$

Or $\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \cdots = \frac{1-u^2}{1+u^2}$.

Now $1+\cos x - \sin x = 1 + \frac{1-u^2}{1+u^2} - \frac{2u}{1+u^2}$

$$= \frac{1+\cancel{u^2}+1-\cancel{u^2}-2u}{1+u^2}$$

$$= \frac{2(1-u)}{1+u^2}.$$

Also, $\tan\left(\frac{x}{2}\right) = u$, then $x = 2\tan^{-1} u$, and $dx = \frac{2\,du}{1+u^2}$.

Then $$\int \frac{dx}{1+\cos x-\sin x} = \int \frac{\frac{\cancel{2}}{\cancel{1+u^2}}}{\frac{\cancel{2}(1-u)}{\cancel{1+u^2}}}\,du$$

$$= \int \frac{du}{1-u}$$

$$= -\ln|1-u| + C$$

$$= -\ln\left|1-\tan\left(\frac{x}{2}\right)\right| + C.$$

Exc. Find $\int \frac{dx}{3\sin x - 4\cos x}$.

Hint: Let $u = \tan\left(\frac{x}{2}\right)$, then $\int = \int \frac{du}{2u^2+3u-2}$ and use partial fractions.

This lecture: Integration by partial fractions.
Next lecture: Improper integrals.

Searching keywords:

- Evaluate the integral احسب التكامل
- Integration by partial fractions التكامل بالكسور الجزئية
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

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