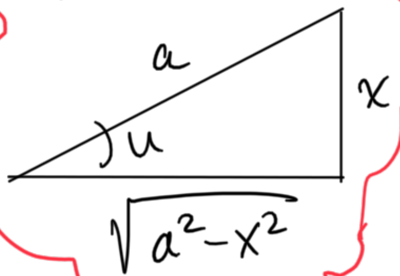
Trigonometric Substitutions.

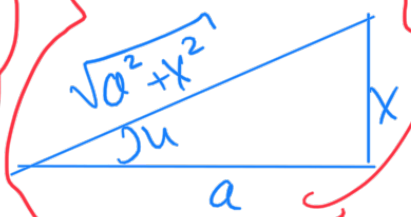
To evaluate the integrals that contain

Set $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$

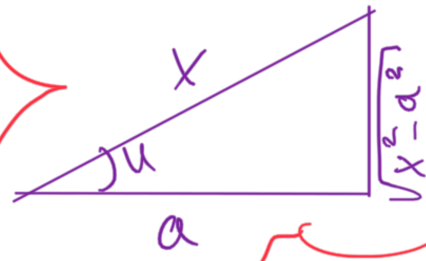
$x = a\sin u$



$x = a\tan u$



$x = a\sec u$

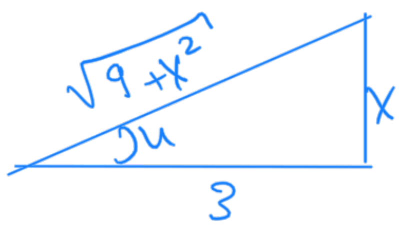


Ex Evaluate the given integrals.

(1) $I = \int \frac{dx}{(9+x^2)^{3/2}} = \int \frac{dx}{\left(\sqrt{9+x^2}\right)^3}$.

let $x = 3\tan u$, then $dx = 3\sec^2 u\, du$

$$= \int \frac{3\sec^2 u\, du}{\left(\sqrt{9+9\tan^2 u}\right)^3} = \int \frac{\cancel{3}\,\cancel{\sec^2 u}}{\cancel{27}_{\,9}\ \sec^{\cancel{3}} u}\, du$$



$$= \frac{1}{9}\int \frac{du}{\sec u} = \frac{1}{9}\int \cos u\, du = \frac{1}{9}\sin u + C$$

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$$= \frac{1}{9}\,\frac{x}{\sqrt{x^2+9}} + C.$$

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(2) $\int \frac{dx}{\sqrt{9+x^2}} \overset{\text{Exc}}{=} \ln\left|\frac{\sqrt{x^2+9}+x}{3}\right| + C.$

Fact: $\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2 \ln\left(x+\sqrt{a^2+x^2}\right) + C.$

Proof. Exc

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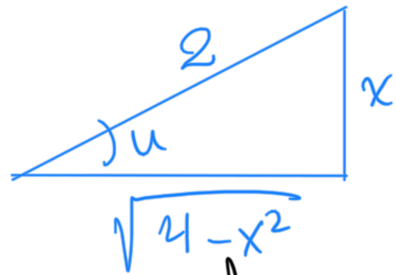
(3) $I = \int \frac{dx}{x^2\sqrt{4-x^2}}$. Here $-2 < x < 2$.



Soln. Let $x = 2\sin u$, then $dx = 2\cos u\,du$.

Then $I = \int \frac{2\cos u\,du}{4\sin^2 u\sqrt{4-4\sin^2 u}} = \int \frac{\cancel{2\cos u}\,du}{(4\sin^2 u)(\cancel{2\cos u})}$

$= \frac{1}{4}\int \csc^2 u\,du = -\frac{1}{4}\cot u + C = \frac{-\sqrt{4-x^2}}{4x} + C.$

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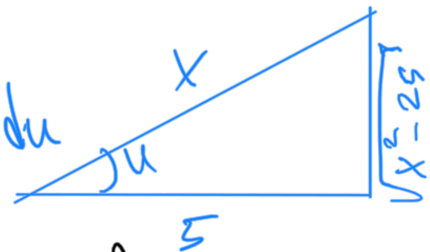
(4) $I = \int \frac{\sqrt{x^2-25}}{x}\,dx$. Here $x > 5$.



Soln. Let $x = 5\sec u$, then $dx = 5\sec u\tan u\,du$

$I = \int \frac{\sqrt{25\sec^2 u - 25}}{\cancel{5\sec u}} \cdot \cancel{5\sec u}\tan u\,du$.

$\sqrt{\sec^2 u - 1}$

$= 5\int \sqrt{\sec^2 u - 1}\,\tan u\,du$

$= 5\int (\sec^2 u - 1)\,du$

$= 5(\tan u - u) + C$

$= 5\left(\frac{\sqrt{x^2-25}}{5} - \sec^{-1}\left(\frac{x}{5}\right)\right) + C.$

Note that $x = 5\sec u$
then $\frac{x}{5} = \sec u$
So $\sec^{-1}\left(\frac{x}{5}\right) = \sec^{-1}(\sec u) = u$

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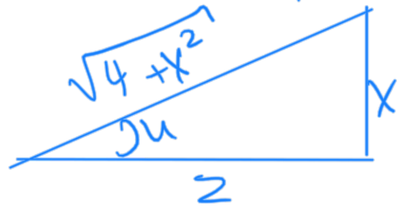
(5) $I=\int \frac{dx}{(x^2+4)^2} = \int \frac{dx}{\left(\sqrt{x^2+4}\right)^4}$.

Soln. let $2\tan u = x$, then $2\sec^2 u\,du = dx$. Then



$$I=\int \frac{2\sec^2 u}{(4\tan^2 u+4)^2}\,du = \frac{1}{8}\int \cos^2 u\,du$$

$$= \frac{1}{16}\int (1+\cos 2u)\,du = \frac{1}{16}u + \frac{1}{32}\sin 2u + C$$

$$= \frac{1}{16}u + \frac{1}{16}\sin u\cos u + C$$

$$= \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{1}{16}\left(\frac{x}{\sqrt{x^2+4}}\right)\left(\frac{2}{\sqrt{x^2+4}}\right) + C.$$

$$= \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{1}{8}\left(\frac{x}{x^2+4}\right) + C.$$

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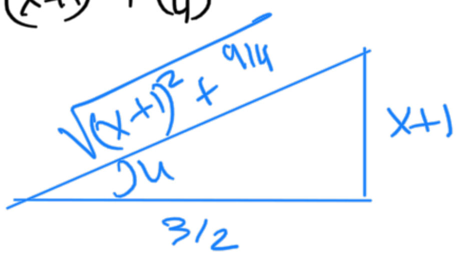
(6) $I=\int \frac{dx}{(x+1)\sqrt{4x^2+8x+13}}$.

Soln Note that $4x^2+8x+13 = 4x^2+8x+4+9 = 4(x+1)^2+9$

Then $I=\int \frac{dx}{(x+1)\sqrt{4(x+1)^2+9}} = \frac{1}{2}\int \frac{dx}{(x+1)\sqrt{(x+1)^2+\left(\frac{9}{4}\right)}}$.

Let $x+1 = \frac{3}{2}\tan u$, then $dx = \frac{3}{2}\sec^2 u\,du$



It follows that

$$I=\frac{1}{2}\int \frac{\frac{3}{2}\sec^2 u\,du}{\left(\frac{3}{2}\tan u\right)\sqrt{\frac{9}{4}\tan^2 u+\frac{9}{4}}} = \frac{1}{3}\int \frac{\sec u}{\tan u}\,du$$

$$= \frac{1}{3}\int \csc u\,du = \frac{1}{3}\ln\left|\csc u - \cot u\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{\sqrt{(x+1)^2+9/4}}{x+1} - \frac{3/2}{x+1}\right| + C.$$

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(7) $\int \frac{dx}{\sqrt{x^2+2x-3}}$. Exc.

This lecture: Trigonometric substitutions.
Next lecture: Integration by partial fractions.

Searching keywords:

- Evaluate the integral احسب التكامل
- Trigonometric substitutions التكامل بالتعويض باستخدام اقترانات مثلثية
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

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