Integration of trigonometric functions.

In this section, we take into account the following 4 remarks:

Remark (1): To find $\int \sin^n x \cos^m x \, dx$;

If n is odd
write $\sin^n x = \sin^{n-1} x \sin x$
use $\sin^2 x = 1 - \cos^2 x$
and let $u = \cos x$ (substit.)

If m is odd
write $\cos^m x = \cos^{m-1} x \cos x$
use $\cos^2 x = 1 - \sin^2 x$
and let $u = \sin x$ (substit.)

If n and m are both even
use $\sin x \cos x = \frac{1}{2} \sin 2x$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Ex. Evaluate

(1) $\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx = \int (\sin^2 x)^2 \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx$

Set $u = \cos x$, then $du = -\sin x \, dx$.

$= \int (1-u^2)^2 \cancel{\sin x} \frac{du}{-\cancel{\sin x}} = -\int (1 - 2u^2 + u^4) \, du$

$= -u + \frac{2}{3} u^3 - \frac{1}{5} u^5 + C = -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C.$

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(2) $\int \cos^5 x \, dx$. Exc.

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(3) $\int \sin^3 x \cos^2 x \, dx = \int \sin x \sin^2 x \cos^2 x \, dx$

$= \int \sin x (1 - \cos^2 x) \cos^2 x \, dx$

let $u = \cos x$, then $du = -\sin x \, dx$

$= \int \cancel{\sin x} (1 - u^2) u^2 \frac{du}{-\cancel{\sin x}}$

$= -\int (u^2 - u^4) \, du$

$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$

$= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C.$

(4) $\int \sin^4 x \cos^3 x \, dx$. Exc.

(5) $\int \sqrt{\sin x} \cos^3 x \, dx$. Exc

(6) $$\int \sin^4 x \, dx = \int (\sin^2 x)^2 dx$$
$$= \int \left(\tfrac{1}{2}(1-\cos 2x)\right)^2 dx$$
$$= \tfrac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx$$
$$= \tfrac{1}{4}\int \left(1 - 2\cos 2x + \tfrac{1}{2}[1+\cos 4x]\right) dx$$
$$= \tfrac{1}{4}\left(x - \sin 2x + \tfrac{1}{2}\left[x + \tfrac{1}{4}\sin 4x\right]\right) + C.$$
$$= \tfrac{3}{4}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{8}\sin 4x + C.$$

(7) $\int \cos^4 x \, dx$. Exc.

(8) $$I = \int \sin^2 x \cos^4 x \, dx$$
$$= \int \sin^2 x \cos^2 x \cos^2 x \, dx$$
$$= \int (\sin x \cos x)^2 \cos^2 x \, dx$$
$$= \int \left(\tfrac{1}{2}\sin 2x\right)^2 \left(\tfrac{1}{2}[1+\cos 2x]\right) dx$$
$$= \underbrace{\tfrac{1}{8}\int \sin^2 2x \, dx}_{II} + \tfrac{1}{8}\int \sin^2 2x \cos 2x \, dx$$

$$II = \tfrac{1}{8}\int \tfrac{1}{2}(1-\cos 4x)\, dx = \tfrac{1}{16}\left(x - \tfrac{1}{4}\sin 4x\right) + C_1 .$$

For III, let $u = \sin 2x$, then $du = 2\cos 2x \, dx$, and hence

$$III = \tfrac{1}{8}\int u^2 \cancel{\cos 2x} \frac{du}{2\cancel{\cos 2x}} = \tfrac{1}{16}\int u^2 du = \tfrac{1}{16}\tfrac{u^3}{3} + C_2$$

So, $III = \tfrac{1}{16}\left(\tfrac{1}{3}\sin^3(2x)\right) + C_2$.

Thus, $I = II + III = \tfrac{1}{16}\left[x - \tfrac{1}{4}\sin 4x + \tfrac{1}{3}\sin^3 2x\right] + C$.

Remark (2). To find

$\int \sin mx \cos nx \, dx$, $\int \sin mx \sin nx \, dx$, and $\int \cos mx \cos nx \, dx$:

If $n = m$

see Remark (1)

If $n \neq m$, we use

$\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$

$\cos\alpha \cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$

$\sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha-\beta) + \sin(\alpha+\beta)\right)$

Ex. $\int \cos 4x \cos 3x \, dx = \frac{1}{2}\int \left(\cos(4x-3x) + \cos(4x+3x)\right) dx$

$= \frac{1}{2}\int \left(\cos x + \cos 7x\right) dx$

$= \frac{1}{2}\left(\sin x + \frac{1}{7}\sin 7x\right) + C.$

Remark (3): To find $\int \tan^m x \sec^n x \, dx$

If $m = 0$

(Case 1)

If $n = 0$

write $\tan^m x = \tan^{m-2} x \tan^2 x$

use $\tan^2 x = \sec^2 x - 1$

and let $u = \tan x$ (subs.)

(Case 2A)

If n is even

write $\sec^n x = \sec^{n-2} x \sec^2 x$

use $\sec^2 x = 1 + \tan^2 x$

and let $u = \tan x$

(Case 2B)

If n is odd

write $\sec^n x = \sec^{n-2} x \sec^2 x$

use integ. by parts by

Letting $dv = \sec^2 x$, $u = \sec^{n-2} x$

If $n \neq 0$ and $m \neq 0$

(Case 3A)

If n is even

write $\sec^n x = \sec^{n-2} x \sec^2 x$

use $\sec^2 x = 1 + \tan^2 x$

and let $u = \tan x$

(Case 3B)

If m is odd

write $\tan^m x \sec^n x = \tan^{m-1} x \sec^{n-1} x \tan x \sec x$

use $\tan^2 x = \sec^2 x - 1$

and let $u = \sec x$

(Case 3C)

If n is odd and m is even

Use $\tan^2 x = \sec^2 x - 1$

and write the product

as a sum of odd

powers of the secant.

Ex. Evaluate

(1) $\int \tan^4 x\,dx = \int \tan^2 x \tan^2 x\,dx$

$= \int \tan^2 x(\sec^2 x - 1)\,dx$

$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$

$= \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx$

Let $u = \tan x$, then $du = \sec^2 x\,dx$.

$= \int u^2 \cancel{\sec^2 x} \frac{du}{\cancel{\sec^2 x}} - \int \sec^2 x\,dx + \int dx$

$= \frac{1}{3}u^3 - \tan x + x + C$

$= \frac{1}{3}\tan^3 x - \tan x + x + C.$

(2) $\int \tan^6 x\,dx$ Exc.

(3) $\int \sec^4 x\,dx = \int \sec^2 x \sec^2 x\,dx$

$= \int (1+\tan^2 x)\sec^2 x\,dx$

Let $u = \tan x$, then $du = \sec^2 x\,dx$

$= \int (1+u^2)\cancel{\sec^2 x} \frac{du}{\cancel{\sec^2 x}}$

$= u + \frac{1}{3}u^3 + C$

$= \tan x + \frac{1}{3}\tan^3 x + C.$

(4) $\int \sec^6 x\,dx$. Exc.

(5) $I = \int \sec^3 x\,dx$

$= \int \sec^2 x \sec x\,dx$

$= \sec x \tan x - \int \tan^2 x \sec x\,dx$

dv	u
$\sec^2 x$	$\sec x$ (+)
$\tan x$	$\sec x \tan x$ (−)

$= \sec x \tan x - \int (\sec^2 x - 1)\sec x \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$\underbrace{\int \sec^3 x \, dx}_{I}$

Then $2I = \sec x \tan x + \int \sec x \, dx$

$= \sec x \tan x + \ln|\sec x + \tan x| + C.$

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(6) $\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x \sec^2 x \sec^2 x \, dx$

$= \int \tan^4 x (1 + \tan^2 x) \sec^2 x \, dx$

Let $u = \tan x$, then $du = \sec^2 x \, dx$

$= \int u^4 (1+u^2) \cancel{\sec^2 x} \frac{du}{\cancel{\sec^2 x}}$

$= \int (u^4 + u^6) \, du$

$= \frac{1}{5}u^5 + \frac{1}{7}u^7 + C$

$= \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C.$

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(7) $\int \tan^5 x \sec^4 x \, dx$. Exc.

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(8) $\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x \tan x \sec x \, dx$

$= \int (\sec^2 x - 1)\sec^2 x \tan x \sec x \, dx$

let $u = \sec x$, then $du = \sec x \tan x \, dx$.

$= \int (u^2 - 1) u^2 \cancel{\tan x \sec x} \frac{du}{\cancel{\sec x \tan x}}$

$= \int (u^4 - u^2) \, du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$

$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.$

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(9) $\int \tan^5 x \sec^3 x \, dx$. Exc.

(10) $\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \, dx$

$= \underbrace{\int \sec^3 x \, dx}_{II} - \underbrace{\int \sec x \, dx}_{III}$

From Ex (5), we have

$II = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C_1$.

On the other hand, $III = \ln|\sec x + \tan x| + C_2$.

Thus, $I = II - III = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C$.

Remark (4). To find $\int \cot^m x \csc^n dx$, use a similar technique in Remark (3).

This lecture: Integration of trigonometric functions.
Next lecture: Trigonometric substitutions.

Searching keywords:

- Evaluate the integral احسب التكامل
- Integration of trigonometric functions تكامل اقترانات مثلثية
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

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