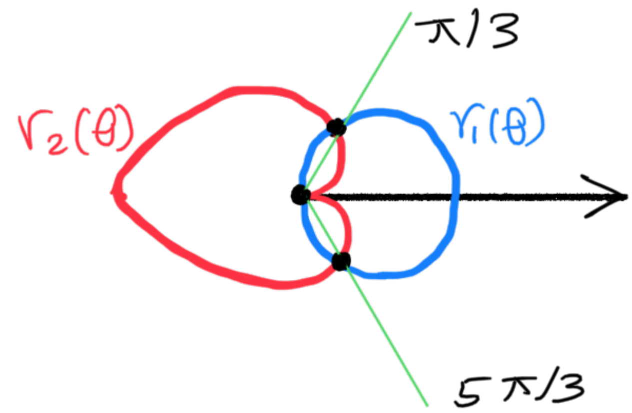
Areas in polar coordinates.

In this section, we study the area of a region whose boundary is given by polar curves. Let us first see how to find the points of intersection between two polar curves.

Ex. For each of the following polar curves, find their points of intersection, if any.

(i) $r_1 = \cos\theta$ and $r_2 = 1 - \cos\theta$.



Soln. $r_1(\theta) = r_2(\theta)$

$\implies \cos\theta = 1 - \cos\theta$

$\implies 2\cos\theta = 1$

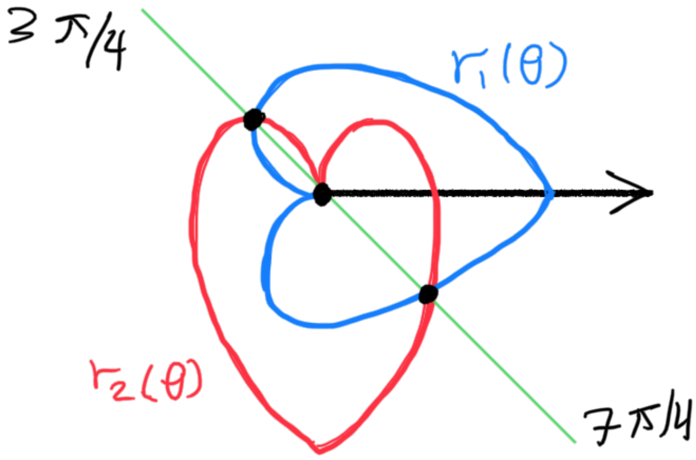
$\implies \cos\theta = 1/2$. So, $\theta = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$.

(2) $r_1 = 1 + \cos\theta$ and $r_2 = 1 - \sin\theta$.

<u>Soln.</u> $r_1 = r_2 \Longrightarrow \cancel{1} + \cos\theta = \cancel{1} - \sin\theta$

$\Longrightarrow \cos\theta = -\sin\theta \Longrightarrow \tan\theta = -1$.

So, $\theta = \frac{3\pi}{4} + 2n\pi$, $\frac{7\pi}{4} + 2n\pi$ (n is an integer).



It is recommended that you draw the graphs of both polar curves.

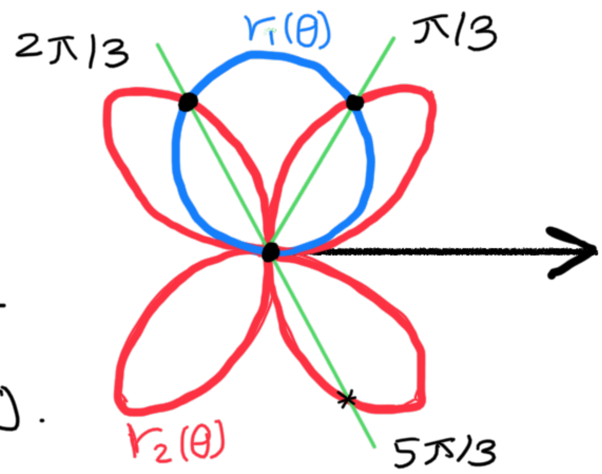
(3) $r_1(\theta) = 2\sin\theta$ and $r_2(\theta) = 2\sin 2\theta$.

<u>Soln.</u> $r_1(\theta) = r_2(\theta)$

$\Longrightarrow \cancel{2}\sin\theta = \cancel{2}\sin 2\theta$

$\Longrightarrow \sin\theta = 2\sin\theta\cos\theta$.

$\Longrightarrow \sin\theta(2\cos\theta - 1) = 0$.



It follows that $\sin\theta = 0$ or $2\cos\theta - 1 = 0$.

Then $\theta = n\pi$ or $\theta = \pi/3 + 2n\pi$, $5\pi/3 + 2n\pi$.

Both the points $(\sqrt{3}, \frac{2\pi}{3} + 2n\pi)$ and $(-\sqrt{3}, \frac{5\pi}{3} + 2n\pi)$ satisfy the circle r_1, while only the point $(-\sqrt{3}, \frac{5\pi}{3} + 2n\pi)$ satisfies the petal curve r_2.

This means that $(-\sqrt{3}, \frac{5\pi}{3})$ satisfies both r_1 and r_2 simultaneously. In summary, at $\theta = 5\pi/3$ both curves at the point A and they collide.

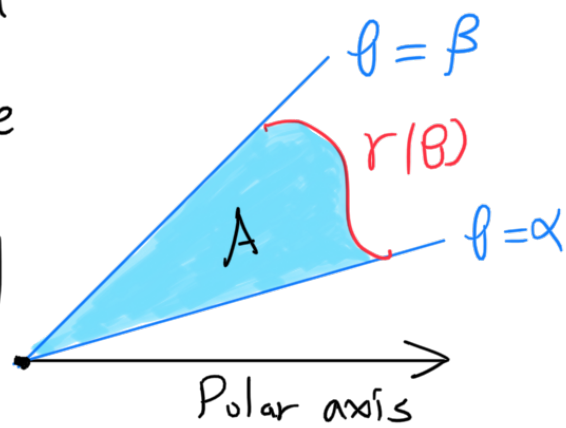
(4) $r_1(\theta) = \cos 2\theta$ and $r_2(\theta) = \frac{1}{2}$. Exc.

This Ex 3, Page 671, Stewart 8E.

Area enclosed by a polar curve.

The area of the polar region generated by the polar curve $r(\theta)$, $\alpha \le \theta \le \beta$ is given by

$$\boxed{A = \frac{1}{2}\int_\alpha^\beta [r(\theta)]^2 \, d\theta}$$



Ex. Calculate the area enclosed by each of the given polar curve.

(1) $r(\theta) = 2\sin 2\theta$ ← Rose.

Recall that: ROSE
$r = a\cos n\theta$,
$r = a\sin n\theta$.

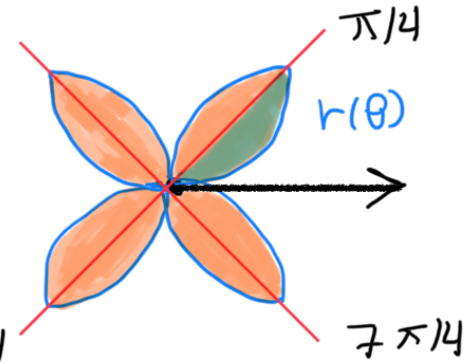
- If n is even then number of petals $= 2n$
- If n is odd then number of petals $= n$

Soln. Number of petals $= (2)(2) = 4$.

$\sin 2\theta = \pm 1$, so $2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Due to symmetry, we only integrate on $[0, \pi/4]$ and multiply the result by 8.



$$\text{Area} = 8\left[\frac{1}{2}\int_0^{\pi/4} (2\sin 2\theta)^2\, d\theta\right] = 16\int_0^{\pi/4} \sin^2 2\theta\, d\theta$$

$$= 8\int_0^{\pi/4} (1 - \cos 4\theta)\, d\theta = 8\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4}$$

$$= 8\left(\frac{\pi}{4}\right) = 2\pi.$$

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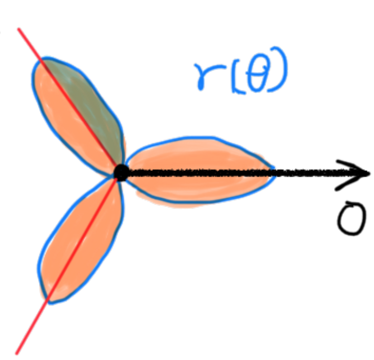
(2) $r(\theta) = 3\cos 3\theta$ ← Rose.

Soln. Number of petals $= 3$.

$\cos 3\theta = 1 \Rightarrow 3\theta = 0, 2\pi, 4\pi$.

So, $\theta = 0, 2\pi/3, 4\pi/3$.



Due to symmetry, we integrate on $[\pi/2, 2\pi/3]$ and multiply the result by 6.

$$\text{Area} = 6\left[\frac{1}{2}\int_{\pi/2}^{2\pi/3}(3\cos 3\theta)^2\,d\theta\right] = 27\int_{\pi/2}^{2\pi/3}\cos^2 3\theta\,d\theta$$

$$= \frac{27}{2}\int_{\pi/2}^{2\pi/3}(1+\cos 3\theta)\,d\theta = \underset{\text{Exc.}}{\underline{\quad\quad}}\,.$$

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(3) $r = 4\cos\theta + 2\sin\theta$.

Soln. This is a circle with center $(2, 1)$ and radius $\frac{1}{2}\sqrt{16+4} = \sqrt{20}/2 = \sqrt{5}$.

Recall that: $r = a\cos\theta + b\sin\theta$ is a circle with center $\left(\frac{a}{2}, \frac{b}{2}\right)$ and radius $\frac{1}{2}\sqrt{a^2+b^2}$

$$\text{Area} = \pi r^2 = \pi(\sqrt{5})^2 = 5\pi.$$

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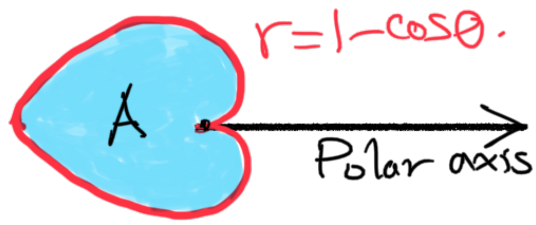
(4) $r = 1 - \cos\theta$.

Soln. $A = \frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$



$$= \frac{1}{2}\int_0^{2\pi}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta$$

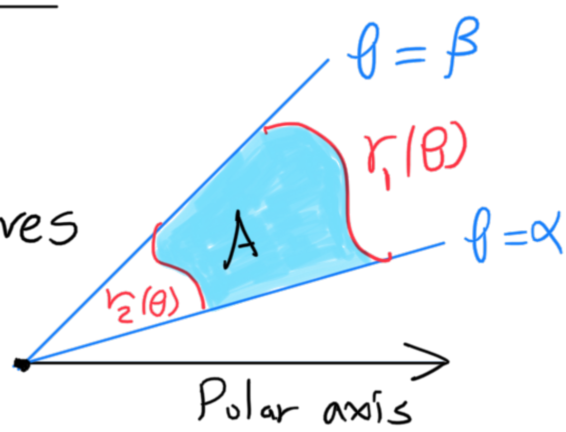
$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}\left[\int_0^{2\pi}\frac{3}{2}\,d\theta - 2\underbrace{\cancel{\int_0^{2\pi}\cos\theta\,d\theta}}_{0} + \frac{1}{2}\underbrace{\cancel{\int_0^{2\pi}\cos 2\theta\,d\theta}}_{0}\right]$$

$$= \frac{3}{4}\int_0^{2\pi}d\theta = \frac{3}{4}\,\theta\Big|_0^{2\pi} = \frac{3}{2}\pi.$$

Area enclosed by a polar curve.

The area of the polar region generated by the 2 polar curves $r_1(\theta)$ and $r_2(\theta)$ with $\alpha \le \theta \le \beta$ is given by



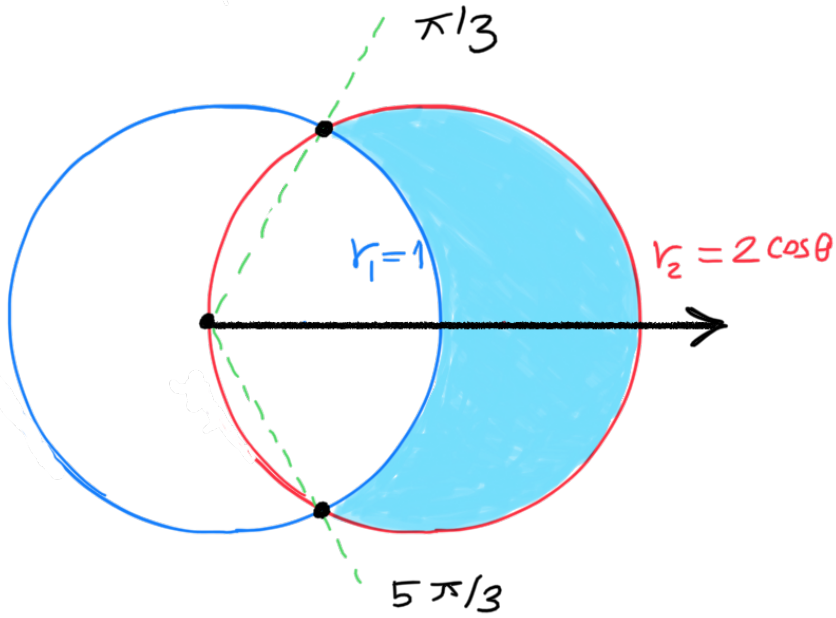
$$\boxed{A = \frac{1}{2}\int_\alpha^\beta \left([r_1(\theta)]^2 - [r_2(\theta)]^2 \right) d\theta}.$$

Ex. Find the area outside $r_1 = 1$ and inside $r_2 = 2\cos\theta$.

Soln. $2\cos\theta = 1$

$\Rightarrow \cos\theta = 1/2$

$\Rightarrow \theta = \pi/3,\ 5\pi/3.$



$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[(2\cos\theta)^2 - (1)^2 \right] d\theta$$

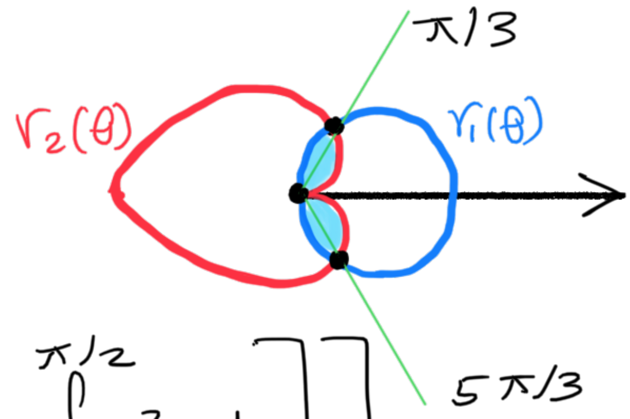
$$= \int_0^{\pi/3} (4\cos\theta - 1)\, d\theta = \underset{\text{Exc.}}{\ldots} = \frac{\pi}{3} + \frac{1}{\sqrt{2}} \simeq 1.91.$$

Ex. Find the area of common region between $r_1 = \cos\theta$ and $r_2 = 1 - \cos\theta$.

Soln. $1-\cos\theta=\cos\theta$

$\Rightarrow \cos\theta = 1/2$.

$\Rightarrow \theta = \pi/3,\ 5\pi/3$.



$$A = 2\left[\frac{1}{2}\left[\int_0^{\pi/3}(1-\cos\theta)^2\,d\theta+\int_{\pi/3}^{\pi/2}\cos^2\theta\,d\theta\right]\right]$$

$$=\int_0^{\pi/3}\left(1-2\cos\theta+\cos^2\theta\right)d\theta+\int_{\pi/3}^{\pi/2}\cos^2\theta\,d\theta$$

$$=\int_0^{\pi/3}(1-2\cos\theta)\,d\theta+\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=\int_0^{\pi/3}(1-2\cos\theta)\,d\theta+\frac{1}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

Exc.

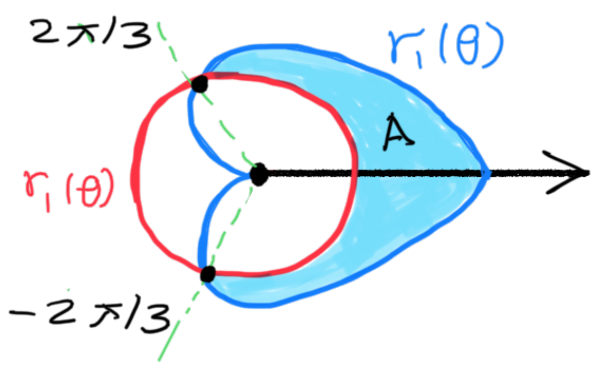
$= \cdots$.

Ex. Find the area inside the limacon $r_1 = 3+2\cos\theta$ and outside the circle $r_2 = 2$.

Soln. $3+2\cos\theta=2$

$\Rightarrow \cos\theta = -1/2$.

$\Rightarrow \theta = 2\pi/3,\ -2\pi/3$



$$A=\frac{1}{2}\int_{-2\pi/3}^{2\pi/3}\left[(3+2\cos\theta)^2-(2)^2\right]d\theta$$

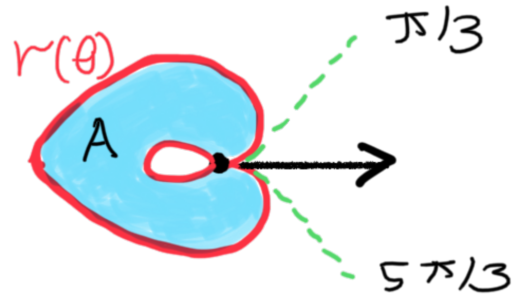
$$=\int_0^{2\pi/3}\left[(3+2\cos\theta)^2-4\right]d\theta = \cdots .$$

Exc

Ex: Find the area A of the region between the inner and outer loops of the limacon $r = 1 - 2\cos\theta$.

Soln. A = area within outer loop − area within inner loop

$= A_1 - A_2$.



Note that $r = 0$ when $\theta = \pi/3$ or $\theta = 5\pi/3$.

The outer loop is formed by having θ increase from $\pi/3$ to $5\pi/3$. Then

$$A_1 = \frac{1}{2}\int_{\pi/3}^{5\pi/3} (1 - 2\cos\theta)^2 \, d\theta = \overset{\text{Exc.}}{\cdots} = 2\pi + \frac{3\sqrt{3}}{2}.$$

The lower half of the inner loop is formed when θ increases from 0 to $\pi/3$, and the upper half when θ increases from $5\pi/3$ to 2π. Then

$$A_2 = \frac{1}{2}\left[\int_0^{\pi/3} (1 - 2\cos\theta)^2 \, d\theta + \int_{5\pi/3}^{2\pi} (1 - 2\cos\theta)^2 \, d\theta\right]$$

$$= \overset{\text{Exc.}}{\cdots} = \pi - \frac{3}{2}\sqrt{3}.$$

Thus, $A = A_1 - A_2 = \pi + 3\sqrt{3} \doteq 8.34$.

This lecture: Areas in polar coordinates.
Next lecture: Final exam.

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- Areas in polar coordinates المساحات في الاحداثيات القطبية
- Intersection of polar curves
- Find the area of the
- The University of Jordan الجامعة الأردنية
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References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

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