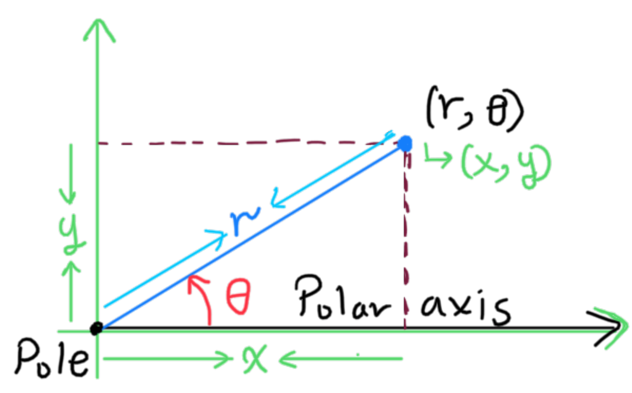
Polar coordinates.

- A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates.
- Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes.
- Now we describe another coordinate system called the polar coordinate system which more convenient for many purposes.
- In the polar coordinate system, each point is determined by a distance from a reference point and an angle from a reference direction.
- The reference point is called the pole, and the ray from the pole in the reference direction is called the polar axis.

Fact: For the Cartesian coordinate (x, y), the polar coordinate is the point (r, θ), where



$$\boxed{r=\sqrt{x^2+y^2}} \quad \text{———————} \quad (1)$$

and $$\boxed{\theta=\tan^{-1}(y/x)} \quad \text{———————} \quad (2)$$

Note that

$\cos\theta = \frac{x}{r}$, so $\boxed{x = r\cos\theta}$ ——————— (3)

$\sin\theta = \frac{y}{r}$, so $\boxed{y = r\sin\theta}$ ——————— (4)

Remarks:

(i) To convert a point from Cartesian to polar, we use Equations (1) and (2).

(ii) To convert a point from polar to Cartesian, we use Equations (3) and (4).

Ex. Convert the point from Cartesian to polar.

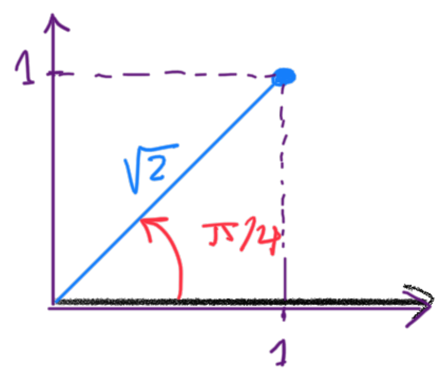
(1) $(x,y) = (1,1)$. (2) $(x,y) = (2\sqrt{3}, -2)$.

Soln.

(1) $r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$.

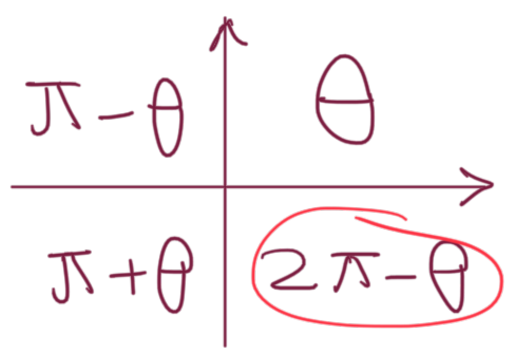
Then $(r,\theta) = (\sqrt{2}, \pi/4)$.



(2) $r = \sqrt{12+4} = 4$.

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Then $(r,\theta) = \left(4, \frac{11\pi}{6}\right)$.



Ex. Convert the point from polar to Cartesian.

(1) $(r,\theta) = (3, \frac{\pi}{2})$. (2) $(r,\theta) = (3, \frac{\pi}{3})$.

Soln.

(1) $x = r\cos\theta = 3\cos\frac{\pi}{2} = 3 * 0 = 0$.

$y = r\sin\theta = 3\sin\frac{\pi}{2} = 3 * 1 = 3$.

Then $(x,y) = (0,3)$.

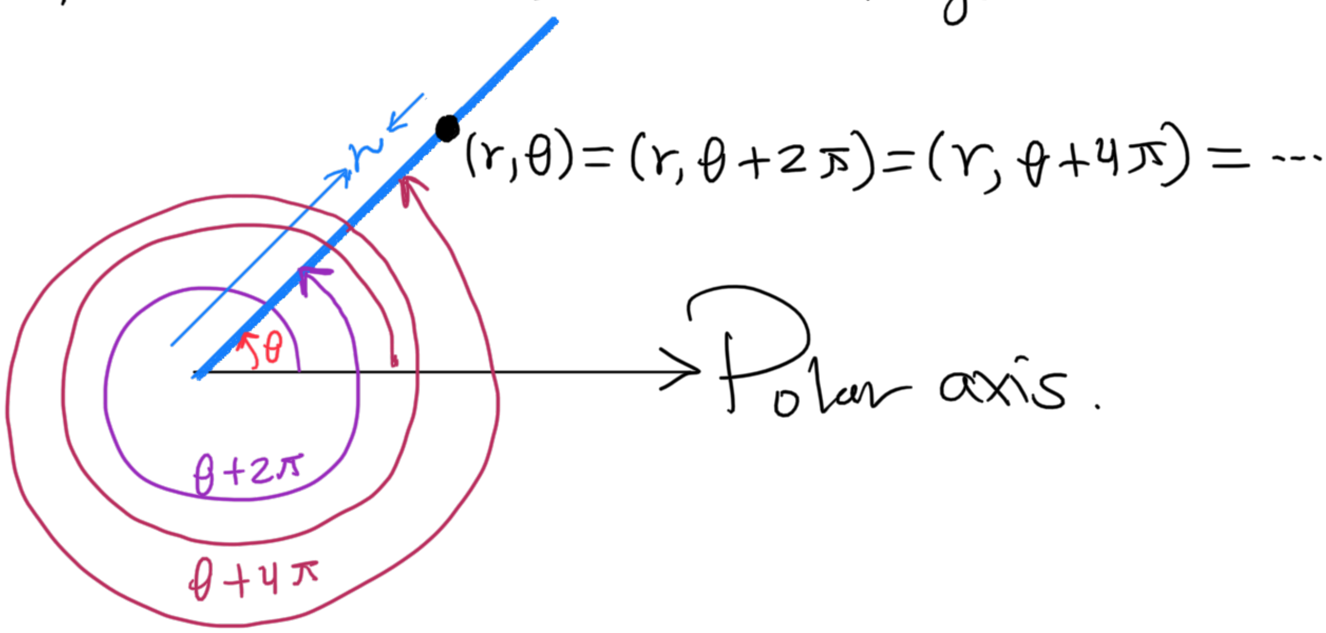
(2) $x = 3\cos(\pi/3) = 3(1/2) = 3/2$.

$y = 3\sin(\pi/3) = 3(\sqrt{3}/2) = 3\sqrt{3}/2$.

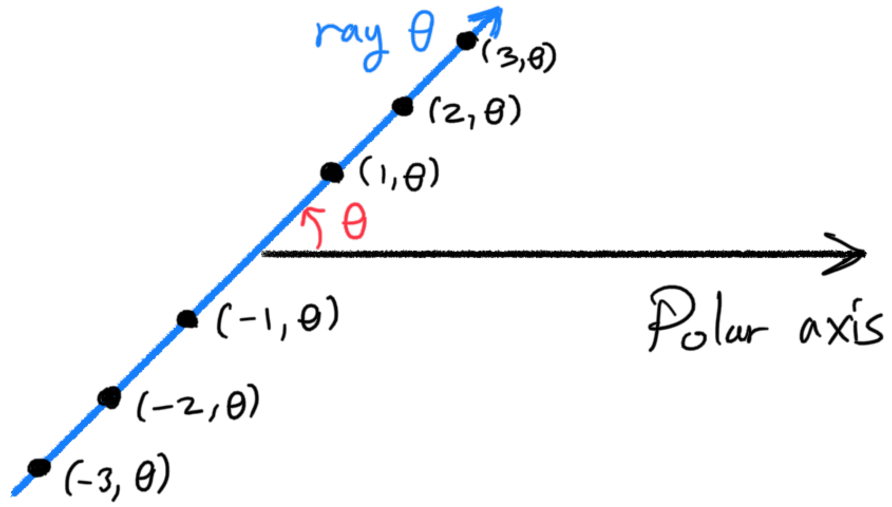
Then $(x,y) = (3/2, 3\sqrt{3}/2)$.

Remarks:

(1) $(r,\theta) = (r, \theta + 2n\pi)$ for all integers n



(2) The point (r, θ) lies at a distance $|r|$ from the pole along the ray θ if $r \geq 0$, or along the ray $\theta + \pi$ if $r < 0$.



(3) $(r, \theta + \pi) = (-r, \theta)$.

Polar curves.

We can also use Equations (1)-(4) above to convert equations from Cartesian to polar and conversely.

Ex: Write the equation in polar coordinates.

(1) $y = \sqrt{3}\, x$.

Soln. $\cancel{r} \sin\theta = \sqrt{3}\, \cancel{r} \cos\theta$, then $\tan\theta = \frac{\sin\theta}{\cos\theta} = \sqrt{3}$.

Thus, the eq. in polar is $\tan\theta = \sqrt{3}$.

(2) $(x^2 + y^2)^2 = 2xy$.

Soln $(r^2)^{\cancel{2}} = 2\, \cancel{r} \cos\theta\, \cancel{r} \sin\theta$. Then $r^2 = \sin 2\theta$.

(3) $(x^2+y^2)^2 = x^2-y^2$.

Soln. $(r^2)^{\cancel{2}} = \cancel{r^2}\cos^2\theta - \cancel{r^2}\sin^2\theta$. Then $r^2 = \cos 2\theta$.

(4) $x^2+(y-2)^2=4$.

Soln. $x^2+y^2-4y+\cancel{4}^{0} = \cancel{4}$, then $x^2+y^2=4y$.

It follows that $r^2\cos^2\theta + r^2\sin^2\theta = 4r\sin\theta$.

$\boxed{\cos^2\theta+\sin^2\theta=1} \Rightarrow r^{\cancel{2}} = 4\cancel{r}\sin\theta$. Thus, $r = 4\sin\theta$.

Ex. Write the equation in the Cartesian coordinates.

(1) $(r = \cos\theta) * r$.

Soln. $r^2 = r\cos\theta$, then $x^2+y^2=x$.

(2) $(r^2 = \sin 2\theta = 2\sin\theta\cos\theta) * r^2$.

Soln. $r^4 = 2r\sin\theta \; r\cos\theta$.

Then $(x^2+y^2)^2 = 2xy$.

(3) $r^2 = \theta$.

Soln. $x^2+y^2 = \tan^{-1}\left(\frac{y}{x}\right)$, then $\tan(x^2+y^2) = y/x$.

Thus, $y = x\tan(x^2+y^2)$.

(4) $r = 4\tan\theta\sec\theta$.

Soln. $r = 4\,\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta} = \frac{4\sin\theta}{\cos^2\theta}$.

Then $(r\cos^2\theta = 4\sin\theta) * r$, hence $r^2\cos^2\theta = 4r\sin\theta$.

Thus, $x^2 = 4y$, or $y = x^2/4$.

(5) $\tan\theta = 2$.

Soln. $\frac{y}{x} = \tan\theta = 2$. Then $y = 2x$.

(6) $(r = a\cos\theta + b\sin\theta) * r$.

Soln. $r^2 = ar\cos\theta + br\sin\theta$, then $x^2+y^2 = ax+by$.

We have $x^2 - ax + y^2 - by = 0$.

Completing the square $\Rightarrow$ $x^2 - ax + \frac{a^2}{4} + y^2 - by + \frac{b^2}{4} = \frac{a^2+b^2}{4}$.

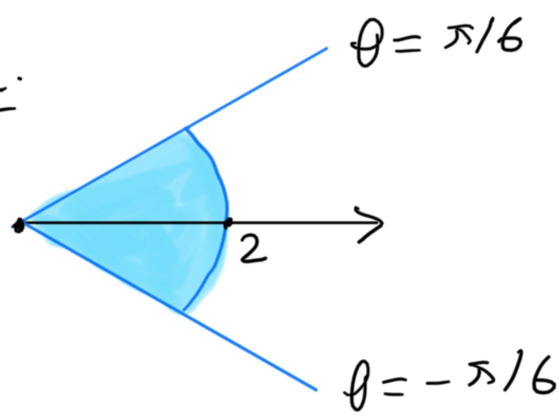
Thus, we get $\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \left(\frac{1}{2}\sqrt{a^2+b^2}\right)^2$.

This means that $r(\theta)$ is a circle with center $\left(\frac{a}{2}, \frac{b}{2}\right)$ and radius $\frac{1}{2}\sqrt{a^2+b^2}$.

Ex. Sketch the region.

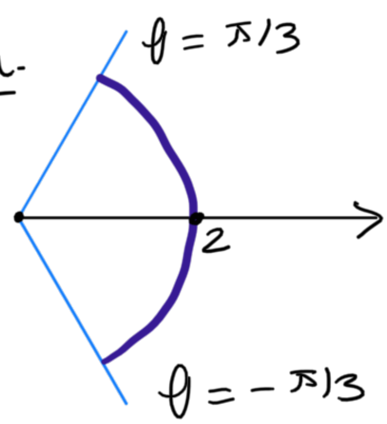
(1) $r \le 2$, $|\theta| \le \pi/6$.

Soln.



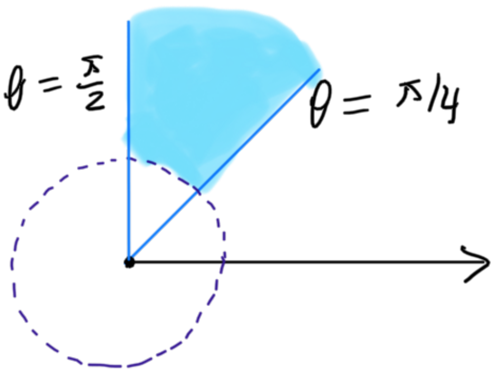
(2) $r = 2$, $|\theta| \le \pi/3$.

Soln.



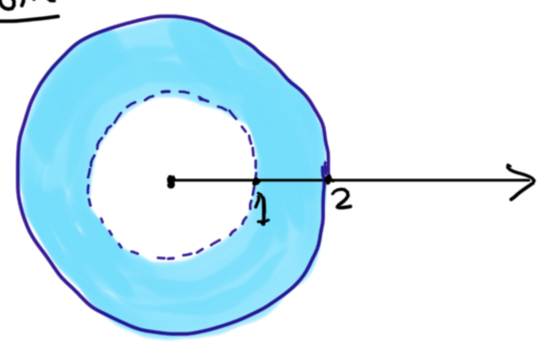
(3) $r > 1$, $\pi/4 < \theta < \pi/2$

Soln.



(4) $1 < r \leq 2$.

Soln.



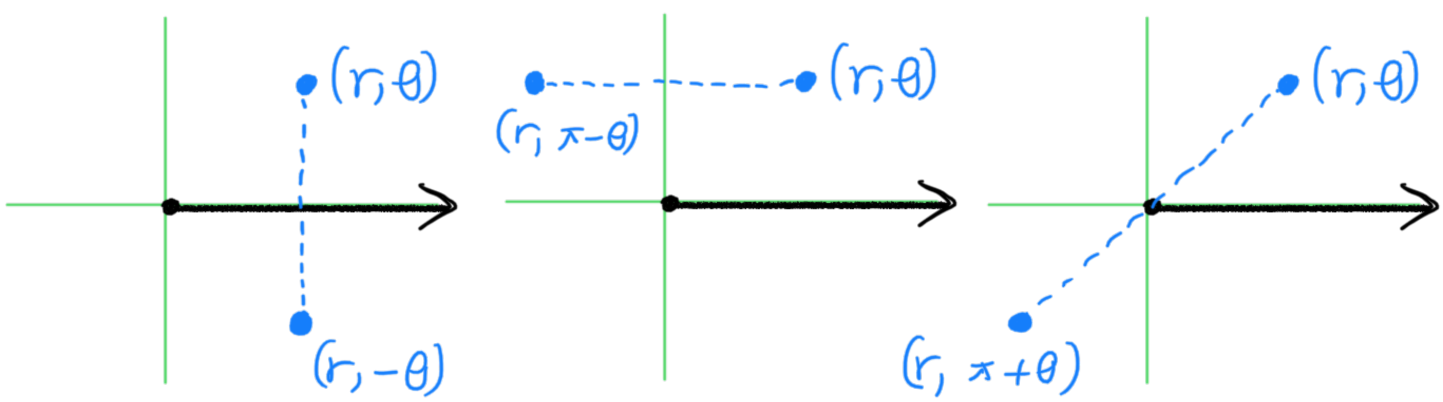
Symmetry.

Def. We say that the graph of a polar curve $r = r(\theta)$ is

(1) symmetric about the polar axis if $r(-\theta) = r(\theta)$.

(2) symmetric about the line $\theta = \frac{\pi}{2}$ if $r(\pi - \theta) = r(\theta)$.

(3) symmetric about the pole if $r(\pi + \theta) = r(\theta)$.



Ex. Test the polar curve for symmetry.

(1) $r = r(\theta) = 1 - 2\cos\theta$.

Soln. $r(-\theta) = 1 - 2\cos(-\theta) = 1 - 2\cos\theta = r(\theta)$.

Therefore, this curve is symmetric about the polar axis.

It can be shown that the curve is not symmetric about the vertical line $\theta = \frac{\pi}{2}$, nor about the pole.

(2) $r = r(\theta) = \cos 2\theta$.

Soln. $r(-\theta) = \cos(2(-\theta)) = \cos(2\theta) = r(\theta)$.

Then the curve is symmetric about the polar axis.

$$r(\theta - \pi) = \cos(2(\theta-\pi)) = \cos(2\pi - 2\theta)$$
$$= \cos(-2\theta) = \cos(2\theta) = r(\theta).$$

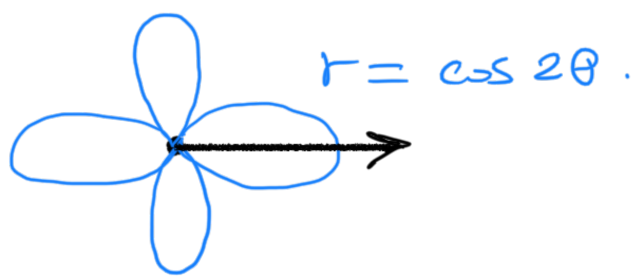
Then the curve is symmetric about the line $\theta = \frac{\pi}{2}$.

Being symmetric about both lines ($\theta = 0$ and $\theta = \frac{\pi}{2}$), the curve must be also symmetric about the pole.

We can see this by noting that

$$r(\theta + \pi) = \cos(2\theta + 2\pi) = \cos(2\theta) = r(\theta).$$

A sketch of the curve which is called a rose shows this symmetry.



Graphing in polar coordinates.

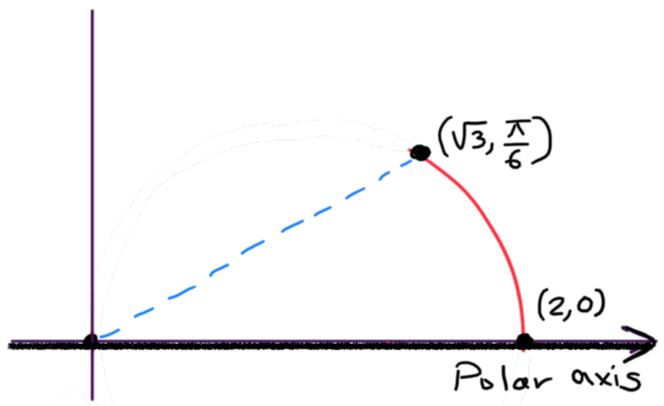
The graph of a polar equation $r = r(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

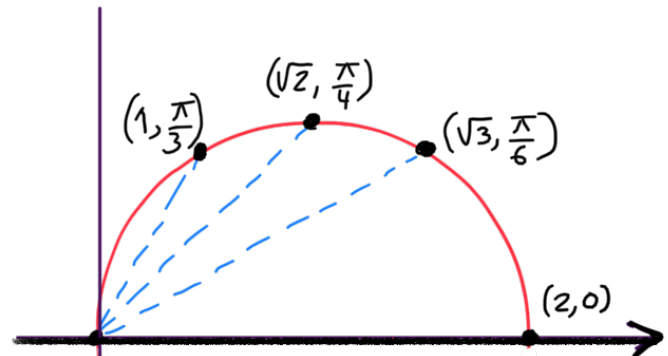
Ex. Sketch the polar curve.

(1) $r = 2\cos\theta$. ← (Ex 5, Page 661, Stewart 8E)

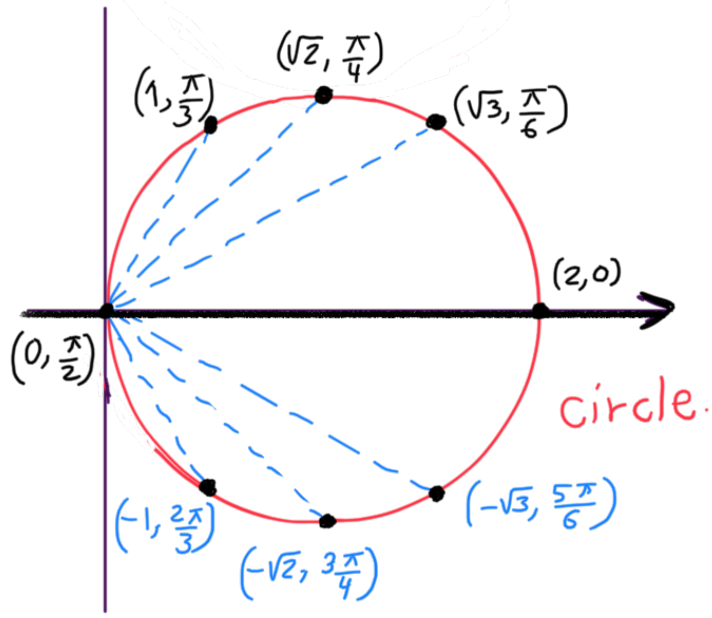
Soln. $\cos\theta$ is periodic with period 2π, so the curve $r = 2\cos\theta$ is a closed curve.

The curve is symmetric about the polar axis, because $r(-\theta) = 2\cos(-\theta) = 2\cos\theta = r(\theta)$.





θ	$\cos\theta$	$r = 2\cos\theta$
0	1	2
$\pi/6$	$\sqrt{3}/2$	$\sqrt{3}$
$\pi/4$	$1/\sqrt{2}$	$\sqrt{2}$
$\pi/3$	$1/2$	1
$\pi/2$	0	0
$2\pi/3$	$-1/2$	-1
$3\pi/4$	$-1/\sqrt{2}$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}/2$	$-\sqrt{3}$
π	-1	-2

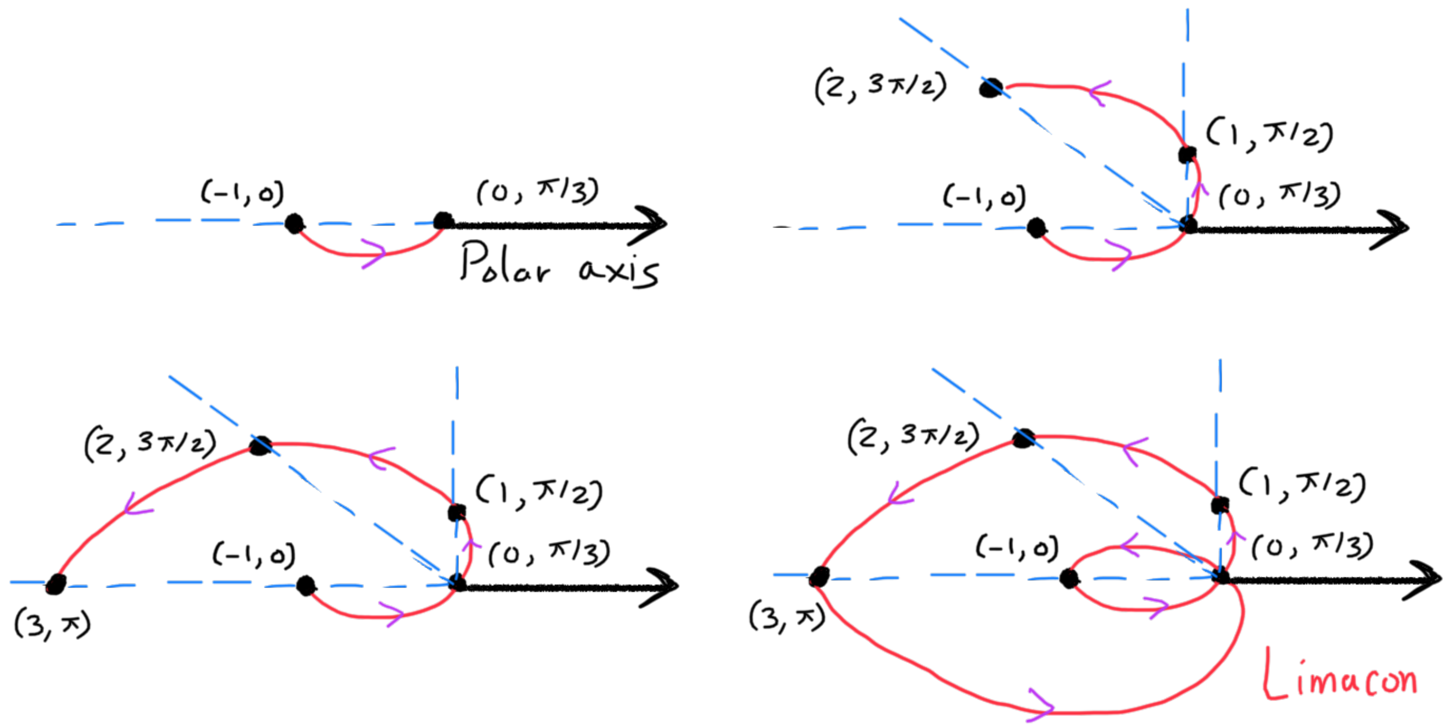


Due to the symmetry about the polar axis, there is no need to find $r(\theta)$ for $\theta > \pi$.

(2) $r = 1 - 2\cos\theta$.

Soln. $\cos\theta$ is periodic with period 2π, so r is a closed curve. This curve is symmetric about the polar axis as we showed in the above example.

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$\cos\theta$	1	$1/\sqrt{2}$	$1/2$	0	$-1/2$	$-1/\sqrt{2}$	-1
$r(\theta)$	-1	$1-\sqrt{2} \simeq -0.4$	0	1	2	$1+\sqrt{2} \simeq 2.4$	3



(3) $r = 3\cos 2\theta$

Soln. Number of petals $= (2)(2) = 4$.

$|r| = |3\cos 2\theta| \le 3$.

If $r = 3$, then $\cos 2\theta = \pm 1$.

Rose: $r = a\cos n\theta$,
$r = a\sin n\theta$.

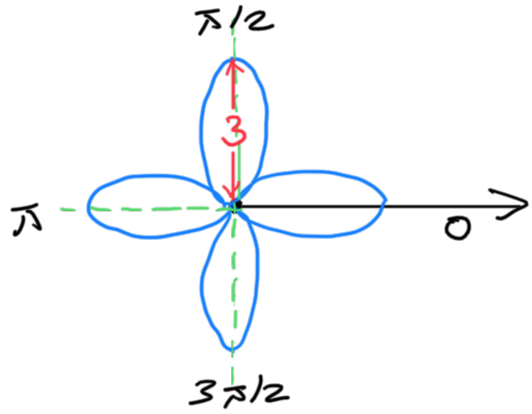
If n is even then number of petals $= 2n$

If n is odd then number of petals $= n$

In this case,

$2\theta = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$

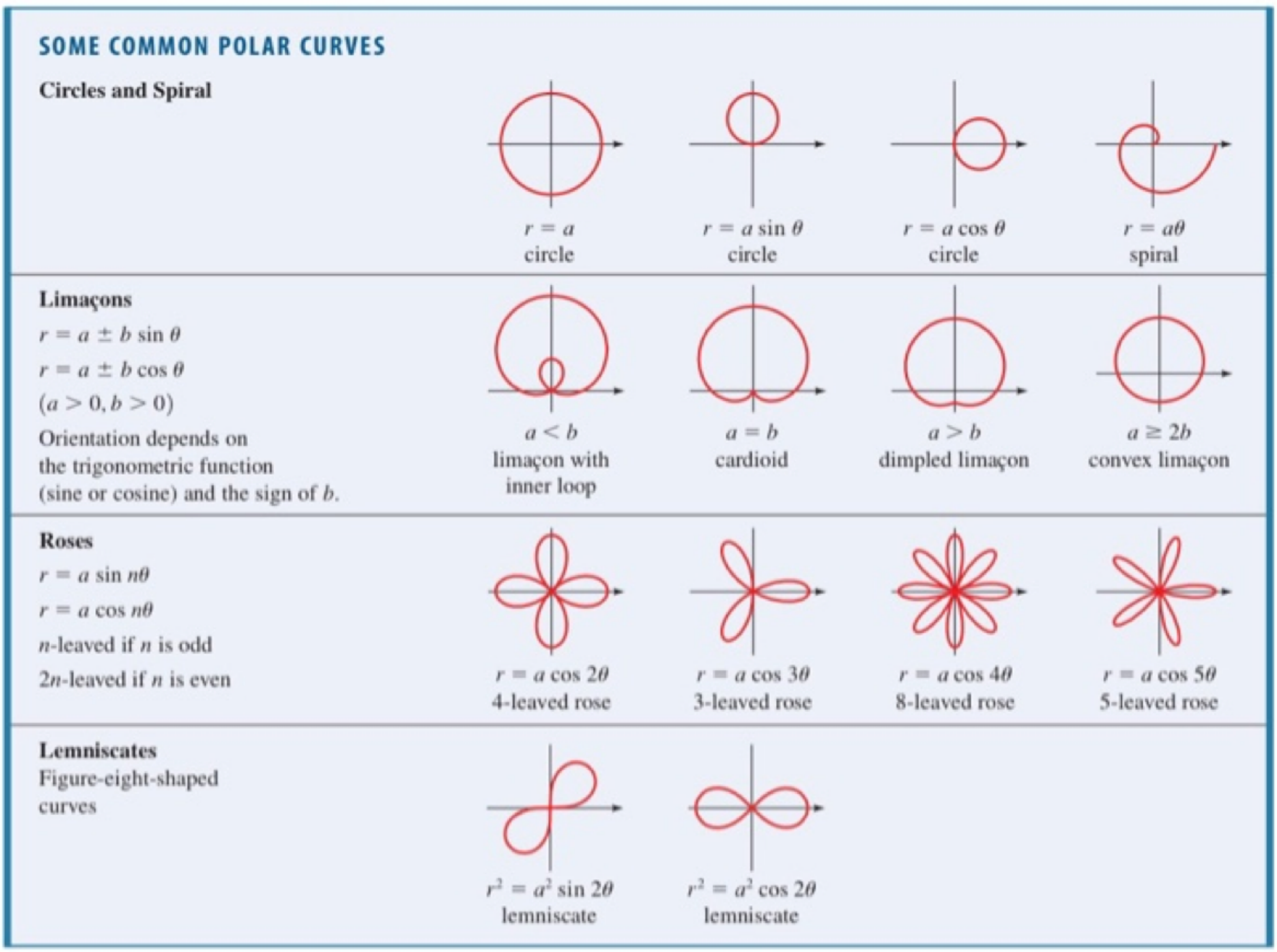
So, $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \ldots$



(4) $r(\theta) = 2\cos 3\theta$. Exc. ← Rose (3 petals).

(5) $r = \theta$, $\theta \geq 0$. Exc. ← Nonending spiral.

Basic polar curves.



Reference: This image was taken from

https://www.chegg.com/homework-help/questions-and-answers/

skills-3-8match-polar-equation-graphs-labeled-vi-use-table-page-552-help--3-r-3-cos-0-5-r--q37764269

This lecture: Polar coordinates.
Next lecture: Areas in polar coordinates.

Searching keywords:

- Polar coordinates الاحداثيات القطبية
- Polar curves, polar axis, pole, symmetry, graphing
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

د. بهاء محمود الزالق
The University of Jordan
Dr. Baha Alzalg
baha2math@gmail.com