Power series.

A series of the form
$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
is called a <u>power series in x</u>.

Here, x is a variable and a_k's are constants called the coefficients of the series.

<u>Ex.</u> If $a_k = 1$ for all k, the power series becomes the geometric series (G.S.)
$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots$$

More generally, we have

<u>Def.</u> A series of the form $\sum_{k=0}^{\infty} a_k (x-c)^k$ is called a <u>power series in $(x-c)$</u>, or a <u>power series about c</u>.

Q: When we say a power series converges or diverges?

<u>Def.</u> A power series $\sum a_k (x-c)^k$ is said to converge:

(i) at L if $\sum a_k (L-c)^k$ converges,

(2) on a set S if $\sum a_k (x-c)^k$ converges for each $x \in S$.

<u>Ex</u>. The G.S. $\sum x^k$ conv. when $|x|<1$ and div. when $|x| \geq 1$.

<u>Def</u>. For a power series:

(1) The <u>interval of convergence (I.C.)</u> is the interval that consists all values of x for which the series converges.

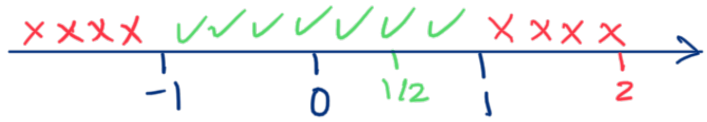
(2) The <u>radius of convergence (R.C.)</u> is the radius of the largest disk in which the series converges

Radius of convergence $\underline{\underline{r}}$; (there are only 3 possibilities)

- $\underbrace{(-\infty, \infty)}_{\hookrightarrow r=\infty}$.
- $\underbrace{[a,b], (a,b], [a,b), (a,b)}_{\hookrightarrow r=\frac{b-a}{2}}$.
- $\underbrace{\{a\}}_{\hookrightarrow r=0}$.

<u>Ex</u>. The I.C. of the G.S. $\sum_{k=0}^{\infty} x^k$ is $(-1,1)$.

The R.C. is $r=1$.



Putting $x=\frac{1}{2}$, $\sum \left(\frac{1}{2}\right)^k$ conv., putting $x=2$, $\sum 2^k$ div.

Q: How to find the I.C. for a power series?

Fact: To find the I.C. of the power series

$\sum_{k=1}^{\infty} a_k (x-c)^k$, we let $b_k = a_k (x-c)^k$. If

$$l = \lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| \begin{cases} l<1, & \text{then the series conv.} \\ l>1, & \text{then the series div.} \end{cases}$$

So, we use Ratio T. (or sometimes Root T.) to find I.C.

Ex. Find the I.C. and R.C. of the series.

(1) $\sum_{k=1}^{\infty} \frac{k}{3^{k+1}} x^k \leftarrow b_k$

Soln. let $b_k = \frac{k}{3^{k+1}} x^k$. Then

$$l = \lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty} \frac{(k+1)\,|x|^{k+1}}{3^{k+2}} \cdot \frac{3^{k+1}}{k\,|x|^k}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right) \frac{|x|}{3} = \frac{|x|}{3}.$$

The series is conv. if $l = \frac{|x|}{3} < 1$,

which is true when $|x|<3$, hence $-3<x<3$.

If $x=3$, then $\sum_{k=1}^{\infty} \frac{k}{3^{k+1}} 3^k = \sum_{k=1}^{\infty} \frac{k}{3}$

$\lim_{k\to\infty} \frac{k}{3} = \infty \neq 0$, so the series is div.

If $x=-3$, then $\sum_{k=1}^{\infty} \frac{k}{3^{\cancel{k}+1}} (-\cancel{3})^k = \sum_{k=1}^{\infty} (-1)^k \frac{k}{3}$

$\lim_{k\to\infty} \frac{k}{3} = \infty \neq 0$, so the series is div.

Thus, the I.C. is $(-3,3)$.

Now, the R.C. is $r = \frac{3-(-3)}{2} = \frac{6}{2} = 3$.

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(2) $\sum_{k=0}^{\infty} \frac{10^k}{k!}(x-1)^k \leftarrow b_k$ (let).

Soln. $L = \lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = \lim_{k\to\infty} \frac{10^{\cancel{k}+1}}{\underbrace{\cancel{(k+1)!}}_{(k+1)}} |x-1|^{\cancel{k}+1} \cdot \frac{\cancel{k!}}{\cancel{10^k}\,\cancel{|x-1|^k}}$

$$= \lim_{k\to\infty} \frac{10\,|x-1|}{k+1} = 0 < 1.$$

Thus, $L<1$ always (regardless of the value of x).

So, I.C. is $(-\infty, \infty)$ and R.C. is $r=\infty$.

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(3) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k\,4^k} \leftarrow b_k$.

Soln. $\displaystyle l = \lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty} \frac{|x|^{\cancel{k}+1}}{(k+1)4^{\cancel{k}+1}} \cdot \frac{k\,\cancel{4^k}}{\cancel{|x|^k}}$

$\displaystyle = \frac{|x|}{4} \lim_{k\to\infty} \frac{k}{k+1} = \frac{|x|}{4}$.

Then $l < 1$ when $|x| < 4$, i.e. $-4 < x < 4$.

If $x = 4$, then $\displaystyle\sum \frac{\cancel{4^k}}{k\,\cancel{4^k}} = \sum \frac{1}{k}$ div. ((P=1) test).

If $x = -4$, then $\displaystyle\sum (-1)^k \frac{\cancel{4^k}}{k\,\cancel{4^k}} = \sum \frac{(-1)^k}{k}$,

$\displaystyle\lim_{k\to\infty} \frac{1}{k} = 0$ and $(1/k)' = -1/k^2 < 0$, so the seq. $\{\frac{1}{k}\}$ is decreasing, hence $\displaystyle\sum \frac{(-1)^k}{k}$ is conv.

In summary, the I.C. is $[-4, 4)$.

The R.C. is $r = 4$.

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(4) $\displaystyle\sum_{k=0}^{\infty} k!\,(x-5)^k \leftarrow b_k$.

Soln. $\displaystyle l = \lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty} \frac{(k+1)!\;|x-5|^{k+1}}{k!\;|x-5|^k}$

$\displaystyle = \lim_{k\to\infty} (k+1)|x-5|$ $\rightarrow l = 0$ if $x = 5$, $\rightarrow l = \infty$ if $x \neq 5$.

Thus, the I.C. is $\{5\}$ and hence the R.C. is $r = 0$.

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(5) $\displaystyle\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2} \leftarrow b_k$.

Soln. $\displaystyle \mathcal{L} = \lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = \lim_{k\to\infty} \frac{|x-5|^{\cancel{k}+1}}{(k+1)^2} \cdot \frac{k^2}{\cancel{|x-5|^k}}$

$\displaystyle = |x-5| \lim_{k\to\infty} \frac{k^2}{(k+1)^2}$

$\displaystyle = |x-5| \left(\lim_{k\to\infty} \frac{k}{k+1} \right)^2$

$= |x-5|$.

Then $\mathcal{L} < 1$ when $|x-5| < 1$,

$-1 < x-5 < 1$,

$4 < x < 6$.

If $x = 6$, then the series becomes $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ which is conv. by P.T.

If $x = 4$, then the series becomes $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$.

$\displaystyle\sum_{k=1}^{\infty} \left| (-1)^k \frac{1}{k^2} \right| = \sum \frac{1}{k^2}$ which converges by P.T.

So, $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$ is abs. conv., hence it is conv.

Therefore, the I.C. is $[4, 6)$

The R.C. is $r = \frac{6-4}{2} = 1$ (Jupb)

(6) $\sum \frac{(-1)^k}{k^2 3^k}(x+2)^k \leftarrow b_k$.

Soln. $L = \lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty}\frac{|x+2|^{\cancel{k}+1}}{(k+1)^2 3^{\cancel{k}+1}}\cdot\frac{k^2 \cancel{3^k}}{\cancel{|x+2|^k}}$

$= \frac{|x+2|}{3}\left(\lim_{k\to\infty}\frac{k}{k+1}\right)^2 = \frac{|x+2|}{3}$.

Then $L<1$ when $\frac{|x+2|}{3}<1$,

$|x+2|<3$,

$-3<x+2<3$,

$-5<x<1$.

If $x=-5$, then $\sum \frac{\cancel{(-1)^k}\cancel{(-3)^k}}{k^2 \cancel{3^k}} = \sum \frac{1}{k^2}$ conv. (P.T.)

If $x=1$, then $\sum \frac{(-1)^k \cancel{3^k}}{k^2 \cancel{3^k}} = \sum \frac{(-1)^k}{k}$ conv.

as $\sum \left|\frac{(-1)^k}{k^2}\right| = \sum \frac{1}{k^2}$ conv. by P.T.

Thus, the I.C. is $[-5,1]$.

The R.C. is $r = \frac{1-(-5)}{2} = \frac{6}{2} = 3$.

(7) $\sum_{k=1}^{\infty} \frac{2^k}{k^2}(x+2)^k$. Exc. Final Ans. $\left[\frac{-5}{2}, \frac{-3}{2}\right]$.

(8) $\sum \frac{(2k)!}{(3k)!} x^k$. Exc. Final Ans. $(-\infty, \infty)$.

Representation of functions as power series.

Recall that the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.
This series converges for $|x|<1$.

Ex: Express the given func as a power series and find the interval of convergence.

(1) $f(x) = \frac{1}{1+x}$.

Soln. The G.S. says that $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, $|x|<1$.
Replace x with $-x$ to get

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum (-x)^k = \sum (-1)^k x^k.$$

This series converges for $|-x|<1$, that is, for $|x|<1$
Therefore, the I.C. is $(-1, 1)$.

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(2) $f(x) = \frac{1}{1+x^2}$.

Soln. From item (1), we have $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$.

Replace x by x^2 to obtain

$$\frac{1}{1+x^2} = \sum (-1)^k (x^2)^k = \sum (-1)^k x^{2k}.$$

This series conv. for $|x^2|<1$, that is, for $|x|<1$.

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(3) $f(x) = \dfrac{2x}{1-x^3}$.

Soln. The G.S. states that $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, $|x|<1$.

Replacing x with x^3, we have

$$\frac{1}{1-x^3} = \sum (x^3)^k = \sum x^{3k}.$$

Multiplying both sides with $2x$, we get

$$\frac{2x}{1-x^3} = 2x \sum x^{3k} = \sum 2x^{3k+1}.$$

This series converges for $|x^3|<1$, that is, for $|x|<1$.

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(4) $f(x) = \dfrac{x^3}{x+2}$.

Soln. Note that $\dfrac{1}{x+2} = \dfrac{1}{2\left(1+\frac{x}{2}\right)} = \dfrac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]}$.

Now, the G.S. states that

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x|<1.$$

Replace x by $\frac{-x}{2}$ to get

$$\frac{1}{1-\left(\frac{-x}{2}\right)} = \sum \left(\frac{-x}{2}\right)^k = \sum \frac{(-1)^k}{2^k} x^k.$$

Multiply both sides with $\frac{x^3}{2}$ to obtain

$$\frac{x^3}{x+2} = \frac{x^3}{2} \frac{1}{1-\left(\frac{-x}{2}\right)} = \frac{x^3}{2} \sum \frac{(-1)^k}{2^k} x^k = \sum \frac{(-1)^k}{2^{k+1}} x^{k+3}.$$

This series converges for $\left|\frac{-x}{2}\right| < 1$, that is, for $|x| < 2$.

Therefore, the I.C. is $(-2, 2)$.

Differentiation and integration of power series.

Thm. If $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ converges with R.C. r, then $f(x)$ is differentiable (and hence continuous) on $(c-r, c+r)$, and

(1) $f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k (x-c)^k\right] = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1}$

converges on $(c-r, c+r)$.

(2) $\int f(x)\,dx = \sum_{k=0}^{\infty} \int \left[a_k (x-c)^k\right] dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-c)^{k+1} + \text{constant}$

converges on $(c-r, c+r)$.

Note that the R.C.s of the power series in (1) and (2) are both r.

Ex. The following power series

$$\sum_{k=0}^{\infty} x^k, \quad \sum_{k=1}^{\infty} k x^{k-1}, \quad \sum_{k=2}^{\infty} k(k-1) x^{k-2}, \ldots$$

$$\sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1}, \quad \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} x^{k+2}, \ldots$$

are all converge on $(-1,1)$ with the R.C. $r=1$.

Ex. Express the given func. as a power series.

(1) $f(x) = \dfrac{1}{(1+x)^2}$.

Recall that:

1) $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \; |x|<1$.

2) $\dfrac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, \; |x|<1$.

Soln. We know that

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, \quad |x|<1.$$

Diff. both sides to get

$$\frac{-1}{(1+x)^2} = \frac{d}{dx}\left(\frac{1}{1+x}\right) = \sum (-1)^k \frac{d}{dx}(x^k) = \sum (-1)^k k x^k.$$

Thus, $$\frac{1}{(1+x)^2} = \sum_{k=1}^{\infty} (-1)^k k x^k, \quad |x|<1.$$

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(2) $f(x) = \tan^{-1} x = \int \dfrac{dx}{1+x^2}$.

Soln. $$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, \quad |x|<1.$$

Replace x with x^2 to get

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k (x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}, \quad |x|<1$$

Integ. both sides to obtain

$$\tan^{-1}x = \int \frac{dx}{1+x^2} = \sum (-1)^k \int x^{2k} dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + C, \quad |x|<1.$$

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(3) $f(x) = \ln(1-2x) \longleftarrow = -2\int \frac{dx}{1-2x}$.

Soln. $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x|<1.$

Replacing x with $2x$, we get $\frac{1}{1-2x} = \sum (2x)^k = \sum 2^k x^k$.

Integrating both sides, we get $\int \frac{dx}{1-2x} = \sum \frac{2^k}{k+1} x^{k+1}$.

Then $\ln(1-2x) = -2\int \frac{dx}{1-2x} = -2 \sum \frac{2^k}{k+1} x^{k+1}$.

Thus, $\ln(1-2x) = \sum_{k=0}^{\infty} \frac{-1}{k+1} (2x)^{k+1} + C$.

This series conv. for $|2x|<1$, that is, for $|x|<\frac{1}{2}$.

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(4) $f(x) = \ln(1+x)$. Exc.

Final Ans. $\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} + C$, $|x|<1$.

To find the value of C we put $x=0$ and get $C = \ln(1+0) = 0$.

Ex. Evaluate $\int \frac{1}{1+x^7} dx$ as a power series.

This is Example 8, Page 756 (Stewart 8E).

$\underline{\text{Soln.}}$ $\dfrac{1}{1+x^7} = \dfrac{1}{1-(-x^7)} \overset{G.S.}{=} \sum_{k=0}^{\infty} (-x^7)^k = \sum_{k=0}^{\infty} (-1)^k x^{7k}.$

Integrate both sides to get

$$\int \frac{dx}{1+x^7} = \sum_{k=0}^{\infty} \frac{(-1)^k}{7k+1} x^{7k+1} + C.$$

This series converges for $|-x^7| < 1$, that is, for $|x| < 1$.

Ex. Sum the series $\sum_{k=0}^{\infty} \frac{x^k}{k}$ for all x in $(-1,1)$.

$\underline{\text{Soln.}}$ let $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k}$ (f is wanted).

then $f'(x) = \sum_{k=0}^{\infty} \frac{k x^{k-1}}{k} = \sum_{k=1}^{\infty} x^{k-1} = \sum_{n=0}^{\infty} x^n \overset{G.S.}{=} \frac{1}{1-x}.$

Since $f'(x) = \frac{1}{1-x}$, $f(x) = \int \frac{dx}{1-x} = -\ln(1-x) + C.$

But $f(0) = 0$. Then $f(x) = -\ln(1-x) = \ln\left(\frac{1}{1-x}\right).$

Thus, $\sum_{k=1}^{\infty} \frac{x^k}{k} = \ln\left(\frac{1}{1-x}\right)$, for all $x \in (-1,1)$.

This lecture: Power series.

Next lecture: Taylor series.

Searching keywords:

- Test the series for convergence or divergence افحص المتتالية للتقارب أو التباعد
- Power series, interval of convergence, radius of convergence
- Representation of functions as power series
- Differentiation and integration of power series
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

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