Alternating series

Def An alternating series is a series whose terms are alternatively positive or negative.

Ex. $\sum_{k=1}^{\infty} (-1)^k \frac{k}{k+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots$ is an alternating series.

Q: How could we test the convergence or divergence of an alternating series.

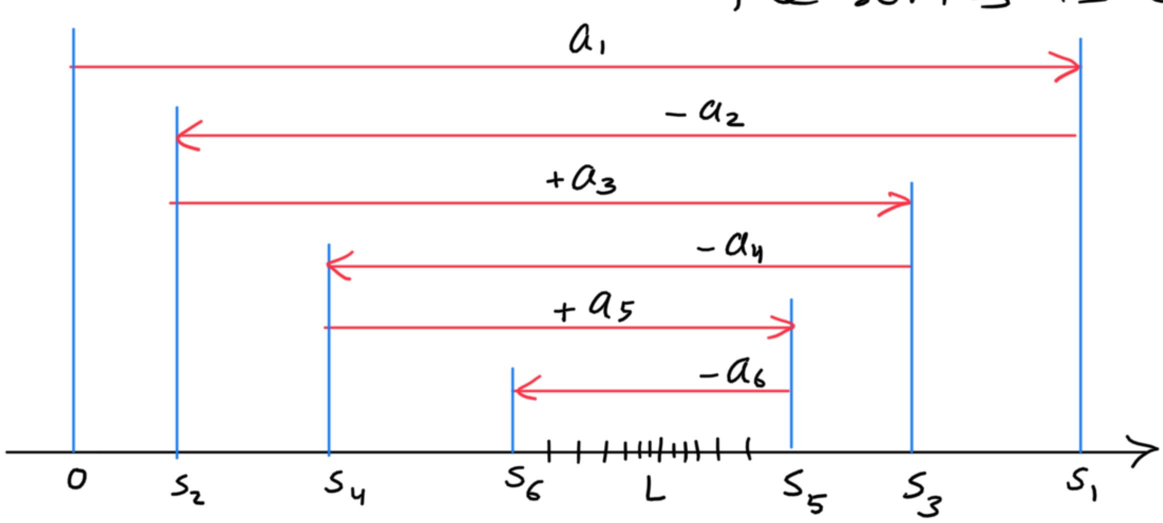
Answer: We use the following test.

The alternating series test (A.S.T.).

For the series $\boxed{\sum_{k=1}^{\infty} (-1)^k a_k}$

If $\lim_{k\to\infty} a_k \neq 0$ then the series is div. (using D.T.)

If $\lim_{k\to\infty} a_k = 0$ and a_k is decreasing on $[1, \infty)$, then the series is conv.



Ex. Test the series for convergence or divergence

(1) $\sum_{k=1}^{\infty} (-1)^k \underbrace{\frac{k}{k+1}}_{a_k}$.

Soln. $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k+1} = 1 \neq 0$.

By A.S.T., the series $\sum (-1)^k a_k$ is div.

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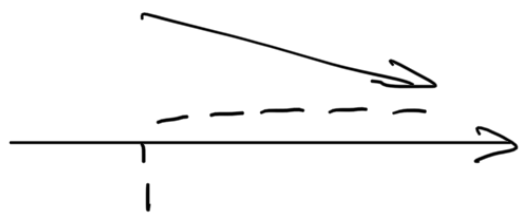
(2) $\sum_{k=1}^{\infty} (-1)^k \underbrace{\ln\left(1+\frac{1}{k}\right)}_{a_k}$.

Soln. $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \ln\left(1+\frac{1}{k}\right) = \ln\left(\lim_{k\to\infty}\left(1+\frac{1}{k}\right)\right)$

$= \ln(1) = \text{zero}$.

$(a_k)' = \frac{d}{dk} \ln\left(1+\frac{1}{k}\right) = \frac{\frac{-1}{k^2}}{1+\frac{1}{k}}$



So, $\{a_k\}$ is a decreasing seq.

By A.S.T., $\sum (-1)^k a_k$ is a conv. series.

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(3) $\sum_{k=1}^{\infty} (-1)^k \left(\frac{k}{k+1}\right)^k$.

Soln. $\lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^{-k}$

$= \left[\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k\right]^{-1}$

$= e^{-1} \neq 0$.

$\xrightarrow{\text{A.S.T.}}$ the series is div.

Absolute and conditional convergence.

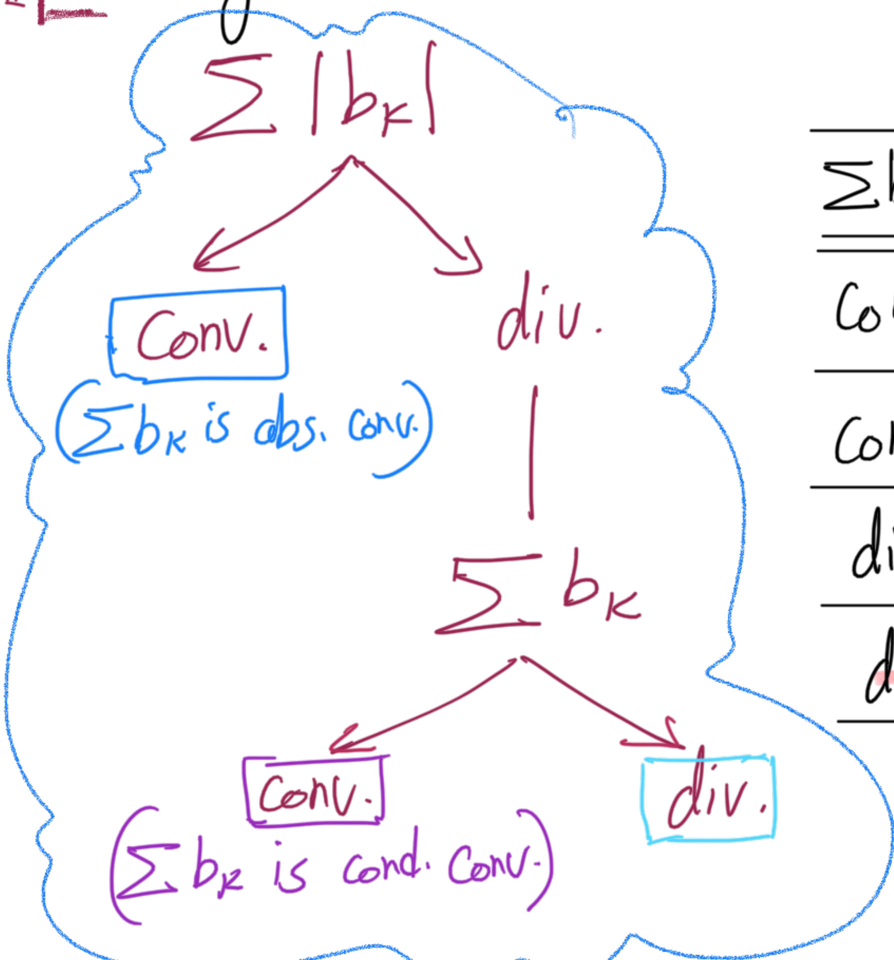
For the alternating series $\sum (-1)^k a_k$, let $b_k = (-1)^k a_k$.

FACT
- If $\sum |b_k|$ conv., then $\sum b_k$ conv.

DEFINITION
- If $\sum |b_k|$ conv., we say that the series $\sum b_k$ is absolutely convergent (abs. conv.).

DEFINITION
- If $\sum |b_k|$ div., but $\sum b_k$ conv., then we say that the series $\sum b_k$ is conditionally convergent (cond. conv.).

FACT
- Every abs. conv. series is conv.



$\sum b_k$	$\sum \lvert b_k \rvert$	Conclusion
conv.	conv.	abs. conv.
conv.	div.	cond. conv.
div.	div.	div.
~~div.~~	~~conv.~~	~~impossible~~

Ex. Test the following series for abs. conv., cond. conv., or divergence.

(1) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2^k}$.

Soln. $\sum \left| (-1)^{k+1} \frac{1}{2^k} \right| = \sum \left(\frac{1}{2}\right)^k$ which is conv.

by G.S. $(x = \frac{1}{2} < 1)$.

Then, $\sum (-1)^{k+1} \frac{1}{2^k}$ is abs. conv.

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(2) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$.

Soln. $\sum \left| (-1)^{k+1} \frac{1}{k} \right| = \sum \frac{1}{k}$ is div (harmonic series P.T. $(p=1)$).

let $a_k = \frac{1}{k}$, $\lim_{k \to \infty} \frac{1}{k} = 0$

and $(a_k)' = \frac{-1}{k^2} < 0$, and $\{a_k\}$ is a decreasing seq.

By A.S.T., $\sum a_k$ is conv. It follows that

$\sum (-1)^{k+1} a_k = \sum (-1)^{k+1} \frac{1}{k}$ is cond. conv.

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(3) $\sum_{k=1}^{\infty} \frac{\cos^k \pi}{k^5} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^5}$.

Soln. $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k^5} \right| = \sum_{k=1}^{\infty} \frac{1}{k^5}$ conv. (P.T. $(p = 5 \geq 1)$).

Hence $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^5}$ is abs. conv.

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(4) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^3+1}$.

Soln. $\sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{k}{k^3+1} \right| = \sum \frac{k}{k^3+1}$.

$\frac{k}{k^3+1} \le \frac{k}{k^3} = \frac{1}{k^2}$ and $\sum \frac{1}{k^2}$ conv. by P.T. ($p=2$).

Then $\sum \frac{k}{k^3+1}$ conv. by C.T.

Thus, $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^3+1}$ is abs. conv.

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(5) $\sum_{k=2}^{\infty} (-1)^{k} \frac{1}{\ln k}$.

Soln. $\sum_{k=2}^{\infty} \left| (-1)^{k} \frac{1}{\ln k} \right| = \sum_{k=2}^{\infty} \frac{1}{\ln k}$ div. by C.T.. In fact,

$\frac{1}{\ln k} > \frac{1}{k}$ (as $\ln k < k$) and $\sum \frac{1}{k}$ div. (harmonic series).

Now, let $a_k = \frac{1}{\ln k}$, then $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{\ln k} = 0$,

and $(a_k)' = \frac{-1/k}{(\ln k)^2} < 0$, so $\{a_k\}$ is decreasing.

This implies that $\sum (-1)^k \frac{1}{\ln k}$ is conv.,

while $\sum \left| (-1)^k \frac{1}{\ln k} \right|$ is div.

Therefore, $\sum (-1)^k \frac{1}{\ln k}$ is cond. conv.

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(6) $\sum_{k=1}^{\infty} (-1)^k \frac{k}{2^k}$.

Soln. $\sum_{k=1}^{\infty} \left| (-1)^k \frac{k}{2^k} \right| = \sum_{k=1}^{\infty} \frac{k}{2^k}$ conv. ?

In the previous lecture, we used the Ratio T. and proved that this series is conv.

(In fact, $\lim_{k\to\infty} \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} = \frac{1}{2} \lim_{k\to\infty} \frac{k+1}{k} = \frac{1}{2} < 1$.)

Hence, $\sum (-1)^k \frac{k}{2^k}$ is abs. conv.

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(7) $\sum_{k=1}^{\infty} (-1)^k \frac{k^3}{e^k}$.

Soln. $\sum_{k=1}^{\infty} \left| (-1)^k \frac{k^3}{e^k} \right| = \sum_{k=1}^{\infty} \frac{k^3}{e^k}$ is conv. by

Ratio T.: $\lim_{k\to\infty} \left(\frac{k^3}{e^k}\right)^{1/k} = \lim_{k\to\infty} \frac{k^{3/k}}{e} = \frac{1}{e} < 1$.

Thus, the alternating series $\sum (-1)^k \frac{k^3}{e^k}$ is abs. conv.

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(8) $\sum_{k=2}^{\infty} (-1)^{k+1} \frac{\sqrt{k}}{k+1}$.

Soln. $\sum_{k=2}^{\infty} \left| (-1)^{k+1} \frac{\sqrt{k}}{k+1} \right| = \sum_{k=2}^{\infty} \frac{\sqrt{k}}{k+1}$ a_k.

let $b_k = \frac{\sqrt{k}}{k} = \frac{1}{\sqrt{k}}$, then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt{k}}{k+1} \cdot \frac{\sqrt{k}}{1} = \lim_{k\to\infty} \frac{k}{k+1} = 1 > 0.$$

$\sum b_k = \sum \frac{1}{k^{1/2}}$ conv. by P.T. $(P=\frac{1}{2}<1)$ $\xrightarrow{\boxed{\text{L.C.T.}}}$ $\sum a_k = \sum \frac{\sqrt{k}}{k+1}$ div.

Next, $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\sqrt{k}}{k+1} = 0$, and

$(a_k)' \leq 0$ which implies that $\{a_k\}$ is a decreasing seq..

Then $\sum a_k = \sum \frac{\sqrt{k}}{k+1}$ conv. by A.S.T.

Thus, $\sum (-1)^{k+1} \frac{\sqrt{k}}{k+1}$ is cond. conv.

This lecture: Alternating series.

Next lecture: Power series.

Searching keywords:

- Test the series for convergence or divergence افحص المتتالية للتقارب أو التباعد
- Alternating series, absolutely convergence, conditionally convergence
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

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