Sequences

A sequence (seq.) can be thought of as a list of real numbers: $\boxed{a_1}, a_2, a_3, \ldots, \boxed{a_n}, \ldots$.

The first term   The n^{th} term

Def. A seq. is a func. $a_n : \mathbb{N} \longrightarrow \mathbb{R}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$.

The natural numbers   The real numbers

Notation: The seq. $\{a_1, a_2, \ldots\}$ is also denoted by

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}.$$

func. $f(x)$ ⟶ $f(n)$ ⟶ $a(n)$ ⟶ a_n

Ex. Find the first three terms of the seq. $\{a_n\} = \left\{\frac{n}{n+2}\right\}$. ⟶ $f(x) = \frac{x}{x+2}$.

Soln. $a_1 = \frac{1}{3}$, $a_2 = \frac{2}{4}$ and $a_3 = \frac{3}{5}$.

Ex. If the seq. $\{a_n\}$ has the terms

$$\frac{-1}{4}, \frac{2}{9}, \frac{-3}{16}, \frac{4}{25}, \frac{-5}{36}, \ldots$$

Write an explicit formula for $\{a_n\}$.

Soln. $\{a_n\} = \left\{(-1)^n \frac{n}{(n+1)^2}\right\}$.

Boundedness

Def. We say that the seq. $\{a_n\}$ is:

(1) bounded above if $\exists$ a constant M s.t.

$$a_n \le M, \ \forall n.$$

(2) bounded below if $\exists$ a constant m s.t.

$$a_n \ge m, \ \forall n.$$

(3) bounded (bded) if it is bded above and below.

Ex. Determine the boundedness of the seq.

(1) $\{a_n\} = \{\frac{2}{n}\}$ is bded below by zero and above by 2. As for all n, we have $0 < \frac{1}{n} \le 1$. So $0 < \frac{2}{n} \le 2$.

(2) $\{a_n\} = \left\{\frac{n+(-1)^n}{n}\right\}$ is bded below by zero and above by $3/2$.

In fact, $\frac{(-1)^n}{n} = \begin{cases} -1/n, & n \text{ is odd}, \\ 1/n, & n \text{ is even}. \end{cases}$ So $-1 \le \frac{(-1)^n}{n} \le \frac{1}{2}$.

Hence, $a_n = \frac{n+(-1)^n}{n} = 1 + \frac{(-1)^n}{n}$. So $0 \le \underbrace{1 + \frac{(-1)^n}{n}}_{a_n} \le \frac{3}{2}$.

(3) $\{a_n\} = \{\sqrt{n^2+1}\}$ is bded below by $\sqrt{2}$ but it is not bded above.

Monotonicity

<u>Def.</u> We say that the seq. $\{a_n\}$ is:

(1) <u>increasing</u> if $a_1 < a_2 < a_3 < \cdots$

(2) <u>nondecreasing</u> if $a_1 \leq a_2 \leq a_3 \leq \cdots$

(3) <u>decreasing</u> if $a_1 > a_2 > a_3 > \cdots$

(4) <u>nonincreasing</u> if $a_1 \geq a_2 \geq a_3 \geq \cdots$

(5) <u>monotonic</u> if it is increasing, nondecreasing, decreasing, or nonincreasing.

<u>Ex.</u> Consider the seq. $\{a_n\} = \left\{\frac{n}{n+2}\right\}$.

Note that $\underset{\frac{1}{3}}{a_1} < \underset{\frac{2}{4}}{a_2} < \underset{\frac{3}{5}}{a_3} < \underset{\frac{4}{6}}{a_4} < \cdots$

So $\{a_n\}$ is an increasing seq.

<u>Remark:</u> To test monotonicity, we look for:

Increasing	v.s.	Decreasing
(1) $\frac{a_{n+1}}{a_n} > 1$		$\frac{a_{n+1}}{a_n} < 1$
(2) $a_{n+1} - a_n > 0$		$a_{n+1} - a_n < 0$
(3) $(a_n)' > 0$		$(a_n)' < 0$

Ex. Test the monotonicity for the given seq.

(1) $\{a_n\} = \{n - 2^n\}$.

Soln. $a_{n+1} - a_n = (\cancel{n}+1 - 2^{n+1}) - (\cancel{n} - 2^n)$

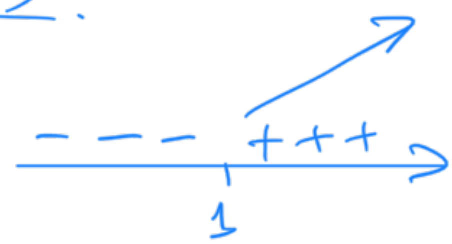
$$= 1 - 2^n \cdot 2 - 2^n$$
$$= 1 - 2^n(2-1)$$
$$= 1 - 2^n$$
$$< 0.$$

So $\{a_n\}$ is a decreasing seq.

(2) $\{a_n\} = \left\{\frac{e^n}{n}\right\}$.

Soln. $(a_n)' = \frac{ne^n - e^n}{n^2} = \frac{e^n(n-1)}{n^2}$.

Now, $e^n(n-1) > 0$ when $n > 1$



So, $\{a_n\}$ is increasing.

(3) $\{a_n\} = \left\{\frac{n!}{e^n}\right\}_{n=2}^{\infty}$.

$k! = k(k-1)(k-2)\cdots(3)(2)(1)$

$(k+1)! = (k+1)k!$

e.g., $4! = 4\underbrace{(3)(2)(1)}_{3!}$

Soln. $\frac{a_{n+1}}{a_n} = \frac{\overset{(n+1)}{\cancel{(n+1)!}}}{e\,\cancel{e^{n+1}}} \cdot \frac{\cancel{e^n}}{\cancel{n!}} = \frac{n+1}{e}$

which is greater than 1 when $n \geq 2$.

(4) $\{a_n\} = \left\{\dfrac{n^n}{n!}\right\}$.

Soln. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^{n+1}}{(n+1)!} \cdot \dfrac{n!}{n^n} = \dfrac{(n+1)^n \cancel{(n+1)}}{\cancel{(n+1)}\,\cancel{n!}} \cdot \dfrac{\cancel{n!}}{n^n} = \left(\dfrac{n+1}{n}\right)^n$

So, $\dfrac{a_{n+1}}{a_n} = \left(1+\dfrac{1}{n}\right)^n$, and hence $\{a_n\}$ is increasing.

(5) $\{a_n\} = \{5^n 2^{-n^2}\} = \left\{\dfrac{5^n}{2^{n^2}}\right\}$.

Soln. $\dfrac{a_{n+1}}{a_n} = \dfrac{5^{n+1}}{2^{(n+1)^2}} \cdot \dfrac{2^{n^2}}{5^n} = \dfrac{\cancel{5^n}\, 5}{2^{\cancel{n^2}+2n+1}} \cdot \dfrac{\cancel{2^{n^2}}}{\cancel{5^n}} = \dfrac{5}{2^{2n+1}} < 1$.

So, $\{a_n\}$ is decreasing.

(6) $\{a_n\} = \{\tan^{-1}(n)\}$.

Soln. $(a_n)' = \dfrac{d}{dn}(\tan^{-1} n) = \dfrac{1}{1+n^2} > 0$. Increasing.

(7) $\{a_n\} = \left\{\dfrac{\ln(n+2)}{n+2}\right\}$.

Soln. $(a_n)' = \dfrac{(n+2)\cdot\frac{1}{n+2} - \ln(n+2)\cdot(1)}{(n+2)^2}$

$$= \frac{1-\ln(n+2)}{(n+2)^2} < 0.$$

Thus, $\{a_n\}$ is decreasing.

Notes: (1) $\{a_n\}_{n=1}^{\infty}$ is increasing $\Rightarrow a_n \geq a_1, \forall n$

$\Rightarrow \{a_n\}$ is bded below by a_1.

(2) $\{a_n\}_{n=1}^{\infty}$ is decreasing $\Rightarrow a_n \leq a_1, \forall n$

$\Rightarrow a_n$ is bded above by a_1.

Limit of a sequence

Def. A seq. $\{a_n\}_{n=1}^{\infty}$ is said to be convergent (conv.) if $\lim_{n \to \infty} a_n$ exists. Otherwise, the seq. is divergent (div.).

Ex. Decide whether the seq. $\{a_n\} = \{\frac{2}{n}\}$ is convergent or divergent.

Soln. $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2}{n} = 0$ exists.

Thus, $\{a_n\}$ is conv. to zero. ([illegible])

Notation: We write $a_n \to L$ to mean that $\{a_n\}$ conv. to L.

Fact: If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} = \begin{cases} \text{zero} & , \deg(p) < \deg(q), \\ \infty & , \deg(p) > \deg(q), \\ \dfrac{\text{Coef. of term of largest degree of } p(x)}{\text{Coef. of term of largest degree of } q(x)} & , \deg(p) = \deg(q). \end{cases}$$

Ex. Test the convergence for the seq.

(1) $\{a_n\} = \left\{ \dfrac{2n^2-4}{n^3+3} \right\}$.

Soln. $\lim\limits_{n\to\infty} \dfrac{2n^2-4}{n^3+3} = 0$. Thus $a_n \longrightarrow 0$.

i.e. a_n is conv. to zero.

(2) $\{b_n\} = \left\{ \dfrac{5n^3-4}{3-n^3} \right\}$.

Soln. $\lim\limits_{n\to\infty} \dfrac{5n^3-4}{3-n^3} = \dfrac{5}{-1} = -5$.

$\therefore \{b_n\}$ is conv. to -5.

(3) $\{c_n\} = \left\{ \dfrac{n^3}{n^2+1} \right\}$.

Soln $\lim\limits_{n\to\infty} \dfrac{n^3}{n^2+1} = \infty$. Thus, $\{c_n\}$ is div.

L'Hospitals rule

Suppose that a_n and b_n (as funcs of n) are diff. with $(b_n)' \neq 0$ and that $\lim\limits_{n\to\infty} a_n/b_n$ has the indeterminate from $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

If $\lim\limits_{n\to\infty} \dfrac{a_n'}{b_n'} = L$ or $\pm\infty$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{a_n'}{b_n'}$.

Ex. Test the seq. for convergence.

(1) $\{a_n\} = \left\{ \dfrac{e^{2/n} - 1}{1/n} \right\}$. $\left(\frac{0}{0}\right)$

Soln. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{e^{2/n} - 1}{1/n}$

$$\overset{L.R.}{=} \lim_{n\to\infty} \frac{(\cancel{-}2/\cancel{n^2})\, e^{2/n}}{\cancel{-}1/\cancel{n^2}}$$

$$= 2 \lim_{n\to\infty} e^{2/n}$$

$$= 2.$$

$\therefore \{a_n\}$ conv. to TWO!

(2) $\{a_n\} = \{2^n/n^2\}$. $\left(\frac{\infty}{\infty}\right)$

Recall that $(r^n)' = r^n \ln r$, $r > 0$.

Soln. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2^n}{n^2}$

$$\overset{L.R.}{=} \lim_{n\to\infty} \frac{2^n \ln 2}{2n}$$

$$\overset{L.R.}{=} \lim_{n\to\infty} \frac{2^n (\ln 2)^2}{2}$$

$= \infty$. $\therefore \{a_n\}$ div.

Note: The L.R. can be used with five additional forms

Indeterminate forms	Determinate forms
$0/0$	$\infty + \infty = \infty$
$\pm\infty/\pm\infty$	$-\infty - \infty = -\infty$
$\infty - \infty$	$\infty \cdot \infty = \infty$
$0 \cdot \infty$	$0^{\infty} = 0$
0^0	$0^{-\infty} = \infty$
1^{∞}	
∞^0	
Use the L.R.	Do not use the L.R.

Ex. Test the seq. for convergence.

(1) $\{a_n\} = \{e^n - n\}$. $(\infty - \infty)$

Soln. $\lim\limits_{x \to \infty} (e^x - x)$

$$= \lim_{x \to \infty} x \left(\frac{e^x}{x} - 1 \right)$$

$$= \left(\lim_{x \to \infty} x \right) \left[\left(\lim_{x \to \infty} \frac{e^x}{x} \right) - 1 \right]$$

$$\overset{L.R.}{=} \left(\lim_{x \to \infty} x \right) \left[\left(\lim_{x \to \infty} \frac{e^x}{1} \right) - 1 \right]$$

$$= \infty.$$

$\therefore \{a_n\}$ div.

(1) $\{a_n\} = \{(n+1)^{2/n}\}$. (∞^0).

Soln. $\ln(a_n) = \ln(n+1)^{2/n} = \frac{2}{n}\ln(n+1)$.

It follows that $\lim\limits_{n\to\infty} \ln(a_n) = \lim\limits_{n\to\infty} \frac{2\ln(n+1)}{n}$

$$\overset{L.R.}{=} \lim_{n\to\infty} \frac{2/(n+1)}{1}$$

$$= 0.$$

Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} e^{\ln(a_n)} = e^0 = 1.$

(2) $\{a_n\} = \{(3^n+4^n)^{\frac{1}{n}}\}$. Exc.

Soln outline $\ln a_n \overset{L.R.}{=} \lim\limits_{n\to\infty} \frac{3^n\ln 3 + 4^n\ln 4}{3^n+4^n}$

$$= \lim_{n\to\infty} \frac{(3/4)^n \ln 3 + \ln 4}{(3/4)^n + 1}$$

$$= \ln 4.$$

so $a_n \longrightarrow 4$.

Ex. Test for convergence the seq. $\left\{\sqrt{n^2+n}-n\right\}$.

Soln. $\displaystyle\lim_{n\to\infty}\sqrt{n^2+n}-n=\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)\cdot\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}$

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$$=\lim_{n\to\infty}\frac{\cancel{n^2}+n-\cancel{n^2}}{\sqrt{n^2+n}+n}$$

$$=\lim_{n\to\infty}\frac{n/n}{n/n+\sqrt{\frac{n^2+n}{n^2}}}$$

$$=\lim_{n\to\infty}\frac{1}{1+\sqrt{1+\frac{n}{n^2}}}=\frac{1}{2}.$$

$\therefore \left\{\sqrt{n^2+n}-n\right\}\longrightarrow\frac{1}{2}$.

The squeeze theorem

If $a_n\le b_n\le c_n$ such that $a_n\to L$ and $c_n\to L$ then $b_n\longrightarrow L$.

Ex. Test the seq. for convergence

(1) $\{a_n\}=\left\{\dfrac{\sin^2(n)}{n}\right\}$.

Soln. Note that $-1\le\sin n\le 1$,

then $0\le\sin^2 n\le 1$,

and hence $0\le\dfrac{\sin^2 n}{n}\le\dfrac{1}{n}$.

$$\underset{0}{\downarrow}\qquad \therefore\underset{0}{\downarrow}\qquad \underset{0}{\downarrow}$$

By squeeze thm

Thus $\lim_{n\to\infty} \frac{\sin^2 n}{n} = 0$, and therefore a_n is conv. to 0.

(2) $\{b_n\} = \left\{\sqrt{9+\left(\frac{1}{n}\right)^2}\right\}$.

<u>Soln.</u> $3 = \sqrt{9} \le \sqrt{9+\left(\frac{1}{n}\right)^2} \le \sqrt{9+\frac{6}{n}+\left(\frac{1}{n}\right)^2} = \sqrt{\left(3+\frac{1}{n}\right)^2} = 3+\frac{1}{n}$

$\downarrow$ 3 $\quad$ " $\quad$ $\downarrow$ 3 by squeeze thm. $\quad$ $\downarrow$ 3

$\therefore$ b_n is conv. to 3.

<u>Fact:</u> If $|a_n| \longrightarrow 0$, then $a_n \longrightarrow 0$.

<u>Proof.</u> By squeeze thm, $-|a_n| \le a_n \le |a_n|$.

$\swarrow 0 \quad \downarrow 0 \quad \searrow 0$

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<u>Ex.</u> Test the seq. for convergence.

(1) $\{a_n\} = \left\{\frac{(-1)^n}{n}\right\}$.

<u>Soln.</u> $|a_n| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \longrightarrow 0$

$\xrightarrow{\text{Fact}}$ $a_n \longrightarrow 0$. i.e, $\{a_n\}$ is conv. to zero.

(2) $\{b_n\} = \left\{\frac{\cos^n \pi}{n^2+3}\right\}$.

<u>Soln.</u> $|a_n| = \left|\frac{\cos^n \pi}{n^2+3}\right| = \left|\frac{(\cos\pi)^n}{n^2+3}\right| = \left|\frac{(-1)^n}{n^2+3}\right| = \frac{1}{n^2+3} \longrightarrow 0$

$\xrightarrow{\text{Fact}}$ $a_n \longrightarrow 0$.

Fact: $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a,\ a\in\mathbb{R}.$

Ex- Test the seq. for convergence.

(1) $\{a_n\} = \left\{\left(1-\frac{2}{n}\right)^n\right\}_{n=1}^{\infty}.$

Soln. $\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n = e^{-2}.$ $\therefore a_n \to e^{-2}.$

(2) $\{b_n\} = \left\{\left(\frac{n}{n+1}\right)^n\right\}_{n=1}^{\infty}.$

Soln. $$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^n\right]^{-1}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n}{n}+\frac{1}{n}\right)^n\right]^{-1}$$

$$= \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^{-1}$$

$$= e^{-1}.$$

$\therefore b_n \longrightarrow 1/e.$

(3) $\{c_n\} = \left\{\left(\frac{n-1}{n+1}\right)^n\right\}_{n=1}^{\infty}.$

Soln. $\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{\frac{n-1}{n}}{\frac{n+1}{n}}\right)^n$

$$= \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n \Big/ \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n \Big/ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= e^{-1}/e = 1/e^2.$$

$\therefore\ C_n \longrightarrow e^{-2}$.

(4) $\{d_n\} = \left\{\left(1-\frac{4}{n^2}\right)^n\right\}$.

Soln. $\lim_{n\to\infty}\left(1-\frac{4}{n^2}\right)^n = \lim_{n\to\infty}\left[\left(1-\frac{2}{n}\right)^n\left(1+\frac{2}{n}\right)^n\right]$

$$= \lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$$

$$= e^{-2}\cdot e^2$$

$$= 1.$$

Thm. Assume that $a_n \longrightarrow L$ and that $a_n \in \mathrm{Dom}(f)$, $\forall n$. If f is continuous at L, then $f(a_n) \longrightarrow f(L)$.

Ex. Since $\frac{\pi}{n} \longrightarrow 0$ and $f(x) = \cos x$ is cts at 0, we conclude that $\cos\left(\frac{\pi}{n}\right) \longrightarrow \cos(0) = 1$.

Ex. Since $\left(1+\frac{2}{n}\right)^n \longrightarrow e^2$ and $f(x) = \ln x$ is cts at e^2, we conclude that

$$\ln\left(1+\frac{2}{n}\right)^n \longrightarrow \ln e^2 = 2.$$

Ex. Since $\frac{\pi^2 n^3 - 5}{16n^3} \longrightarrow \frac{\pi^2}{16}$ and $f(x) = \tan\sqrt{x}$ is cts at $\pi^2/16$, we conclude that

$$\tan\sqrt{\frac{\pi^2 n^3 - 5}{16 n^3}} \longrightarrow \tan\sqrt{\frac{\pi^2}{16}} = \tan\frac{\pi}{4} = 1$$

Fact: let $r \in \mathbb{R}$, then $\lim_{n\to\infty} r^n = \begin{cases} 0 & , \text{if } -1 < r < 1, \\ 1 & , \text{if } r = 1, \\ \text{DNE} & , \text{if } r \leq -1 \text{ or } r > 1. \end{cases}$

Thus, the seq. $\{r^n\}$ is conv. if $-1 < r \leq 1$ and div. otherwise.

Ex. Test $\{5^{n+1}/4^{2n-1}\}$ for convergence.

Soln. $\lim_{n\to\infty} \frac{5^{n+1}}{4^{2n-1}} = \lim_{n\to\infty} \frac{5 \cdot 5^n}{4^{-1} \cdot 4^{2n}}$

$$= (5)(4) \lim_{n\to\infty} \frac{5^n}{(4^2)^n}$$

$$= (20) \lim_{n\to\infty} \left(\frac{5}{16}\right)^n$$

$$= 0.$$

$-1 < r < 1.$

Therefore, the seq. $\{5^{n+1}/4^{2n-1}\}$ is conv. to zero.

<u>Facts</u>: (1) If $r>0$, then $r^{\frac{1}{n}} \longrightarrow 1$.

(2) If $r>0$, then $\frac{1}{n^r} \longrightarrow 0$.

(3) For each $r \in \mathbb{R}$, we have $\frac{r^n}{n!} \longrightarrow 0$.

<u>Ex</u>. Test the seq. for convergence.

(1) $\{a_n\} = \{5^{3/n}\}$.

<u>Soln</u>. $\{a_n\}$ is conv. to 1, because

$$\lim_{n\to\infty} 5^{3/n} = \left(\lim_{n\to\infty} 5^{1/n}\right)^3 = (1)^3 = 1.$$

r

(2) $\{b_n\} = \left\{ \frac{3^{\frac{n}{100}}}{n!} \right\}$.

<u>Soln</u>. The seq. $\{b_n\}$ is conv. to zero, because

$$\begin{array}{ccccc} 0 & \le & \frac{3^{\frac{n}{100}}}{n!} & \le & \frac{3^n}{n!} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

By squeeze thm.

Facts: (1) $\frac{\ln n}{n} \longrightarrow 0$.

(2) $\frac{n}{e^n} \longrightarrow 0$.

(3) $n^{1/n} \longrightarrow 1$.

Thm. If $a_n \longrightarrow L$ and $b_n \longrightarrow M$, then

(1) $a_n \pm b_n \longrightarrow L \pm M$.

(2) $\alpha a_n \longrightarrow \alpha L$ (α is any real number).

(3) $a_n b_n \longrightarrow LM$.

(4) $a_n / b_n \longrightarrow L/M$, provided that $M \neq 0$.

(5) $a_n^p \longrightarrow L^p$, provided that $p>0$ and $a_n>0$

Ex. The seq. $\left\{\left(\frac{2}{n}\right)^n\right\}$ is conv. to zero, as

$$\lim_{n\to\infty}\left(\frac{2}{n}\right)^n = \lim_{n\to\infty} \underbrace{\left(\frac{2}{n}\right)\left(\frac{2}{n}\right)\cdots\left(\frac{2}{n}\right)}_{n\text{-times}}$$

$$= \left(\lim_{n\to\infty}\frac{2}{n}\right)\left(\lim_{n\to\infty}\frac{2}{n}\right)\cdots\left(\lim_{n\to\infty}\frac{2}{n}\right)$$

$$= 0 \cdot 0 \cdots 0$$

$$= 0.$$

Monotonic sequence theorem (MST)

Every bounded, monotonic seq. is convergent.

Ex. Test the seq. $\{a_n\} = \left\{\frac{2^n}{n!}\right\}$ for convergence without finding its limit.

Soln.

- $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}}{(n+1)!} \cdot \dfrac{n!}{2^n} = \dfrac{2\cdot 2^n}{(n+1)n!} \cdot \dfrac{n!}{2^n} = \dfrac{2}{n+1} \leq 1.$

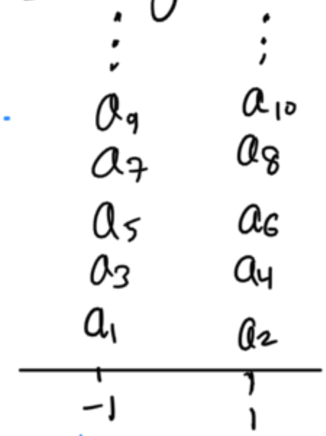
  Thus, $\{a_n\}$ is decreasing, i.e. it is a monotonic seq.

- $|a_n| = \dfrac{2^n}{n!} \leq \dfrac{2^1}{1!} = 2$. So, $-2 \leq a_n \leq 2$.

  Thus $\{a_n\}$ is bded.

Then by MST, $\{a_n\}$ is conv.

Remarks: (1) Bounded $\not\Rightarrow$ convergent, in general.



Ex. Consider the seq. $\{a_n\} = \{(-1)^n\}$.

$|a_n| = 1$, so $\{a_n\}$ is bded,

$\lim_{n\to\infty} (-1)^n$ DNE, so $\{a_n\}$ is div.

Thus, $\{a_n\}$ is bded but it is not conv.

(2) Monotonic $\not\Rightarrow$ Convergent, in general.

Ex. Consider the seq. $\{a_n\} = \{n\}$.

$(a_n)' = 1$, so $\{a_n\}$ is monotonic (increasing).

$\lim_{n\to\infty} n = \infty$, so $\{a_n\}$ is not conv. (div.).

Facts: (1) Every conv. seq. is bded.

(2) Every unbded seq. is div.

Ex. (1) Test $\{a_n\} = \left\{\frac{3-4n^2}{n^2+1}\right\}$ for boundedness.

(2) Test $\{b_n\} = \{n \ln n\}$ for convergence.

Soln. (1) $\lim_{n\to\infty} \frac{3-4n^2}{n^2+1} = \frac{-4}{1} = -4$.

$\{a_n\}$ is conv. $\Longrightarrow$ $\{a_n\}$ is bded.

(2) $\{b_n\} = \{n \ln n\}$ is unbded $\Longrightarrow$ $\{b_n\}$ is div.

Fact: Arithmetic Sum $\boxed{1+2+3+\cdots+(n-1)+n = \frac{n(n+1)}{2}}$.

Ex. Write an explicit formula for the seq. defined by the recurrence relation $a_n = a_{n-1} + n$ with $a_0 = 4$.

Soln. $a_0 = 4$

$a_1 = a_0 + 1 = 4+1$.

$a_2 = a_1 + 2 = 4+1+2$.

$a_3 = a_2 + 3 = 4+1+2+3$.

$a_4 = a_3 + 4 = 4+1+2+3+4$.

$a_5 = a_4 + 5 = 4+1+2+3+4+5$.

In general $a_n = 4 + 1 + 2 + 3 + \cdots + (n-1) + n$

$\therefore$ $a_n = 4 + \frac{n(n+1)}{2}$. Done!

This lecture: Sequences.
Next lecture: Infinite series.
Searching keywords:

- Sequence, bounded, increasing, decreasing, monotonic, convergence, divergence, test the sequence for المتسلسلات، تقارب، تباعد، تزايد، تناقص
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx
For any comments or concerns, please use my email to contact me.

د. بهاء محمود الزالق
The University of Jordan
Dr. Baha Alzalg
baha2math@gmail.com