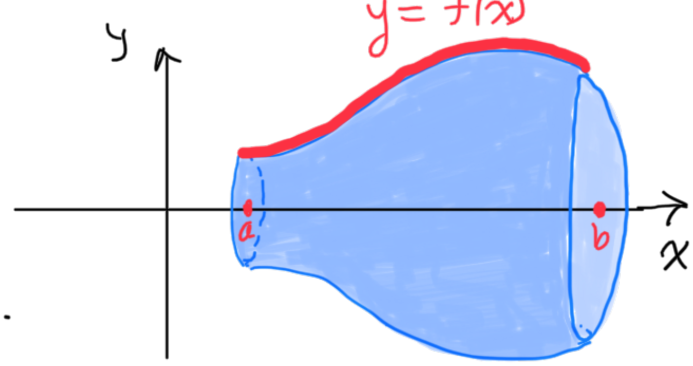
Area of a surface of revolution

A surface of revolution is formed when a curve is rotated about a line.



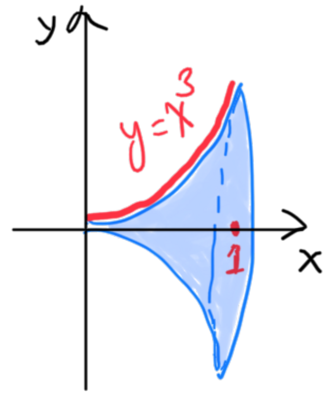
<u>Fact</u>: The area of the surface generated by revolving the graph of $y=f(x)$, $a \le x \le b$, about the x-axis is given by

$$A = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi \int_a^b y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

<u>Note</u>: If the curve is described by $x = g(y)$, $c \le y \le d$, then A becomes $A = 2\pi \int_c^d y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dy$.

<u>Ex</u>. Find the area of the surface generated by revolving the graph of $f(x) = x^3$, $0 \le x \le 1$, about the x-axis.

<u>Soln</u>. $A = 2\pi \int_0^1 x^3 \sqrt{1+(3x^2)^2}\,dx$



$$= 2\pi \int_0^1 x^3 \sqrt{1+9x^4}\, dx$$

let $u = 1+9x^4$, then $du = 36x^3$, and $x = 0 \longrightarrow u=1$
$x = 1 \longrightarrow u = 10$.

$$= 2\pi \int_1^{10} \cancel{x^3} \sqrt{u} \; \frac{du}{36\cancel{x^3}}$$

$$= \frac{\pi}{18} \int_1^{10} u^{1/2}\, du$$

$$= \frac{\pi}{18} \left(\frac{2}{3}\right) u^{3/2} \Big|_1^{10}$$

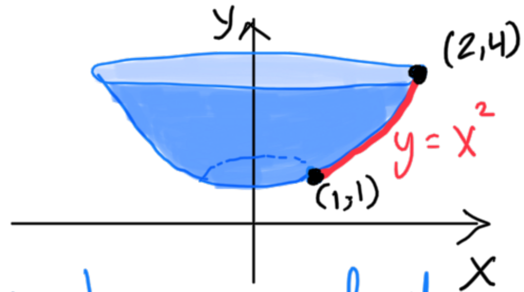
$$= \frac{\pi}{27} \left(10^{3/2} - 1\right).$$

Fact: The area of the surface generated by revolving the graph of $x = g(y)$, $c \le y \le d$, about the y-axis is given by

$$A = 2\pi \int_c^d g(y) \sqrt{1 + (g'(y))^2}\, dy = 2\pi \int_c^d x(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Note: If the curve is described by $y = f(x)$, $a \le x \le b$, then A becomes $A = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

Ex. The arc of the parabola $y=x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis. Find the area of the resulting surface.



Soln. $y=x^2 \Rightarrow x=\sqrt{y}=g(y)$ and $g'(y)=\frac{1}{2\sqrt{y}}$

$$A = 2\pi\int_1^4 g(y)\sqrt{1+(g'(y))^2}\,dy$$

$$= 2\pi\int_1^4 \sqrt{y}\sqrt{1+\frac{1}{4y}}\,dy$$

$$= \pi\int_1^4 \sqrt{4y+1}\,dy$$

Let $u=4y+1$
then $du=4\,dy$

$$= \pi\int_5^{17}\sqrt{u}\,du$$

$$= \frac{\pi}{6}\left[17\sqrt{17}-5\sqrt{5}\right].$$

Another solution. $f(x)=x^2$, so $f'(x)=2x$. Then

$$A = 2\pi\int_1^2 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_1^2 x\sqrt{1+(2x)^2}\,dx$$

$$= 2\pi\int_1^2 x\sqrt{1+4x^2}\,dx$$

Let $u=1+4x^2$
then $du=8x\,dx$
$x=1 \longrightarrow u=5$
$x=2 \longrightarrow u=17$

$$= 2\pi\int_5^{17}\cancel{x}\sqrt{u}\,\frac{du}{8\cancel{x}}$$

$$= \frac{\pi}{4}\int_{5}^{17} u^{\frac{1}{2}}\,du$$

$$= \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_{5}^{17}$$

$$= \frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right].$$

This lecture: Area of a surface of revolution.
Next lecture: Sequences.
Searching keywords:

- Find the area of the surface generated احسب مساحة سطح الجسم الناتج عن دوران
- The University of Jordan الجامعة الأردنية
- Calculus II 2 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/102.aspx

For any comments or concerns, please use my email to contact me.

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