Continuity

"Continuity" is the intuitive concept of a func. that varies with no jumps.

For example, the height of a growing flower at time t is a continuous func. In contrast, the amount of money in a bank account at a time t is a discontinuous func. (because it jumps each time when money is deposited or withdrawn.

Continuity at a point.

In the previous lecture, we observed that the limit of a func. $f(x)$ as $x \to a$ can be found simply by calculating $f(a)$. Functions with this property are called continuous at a.

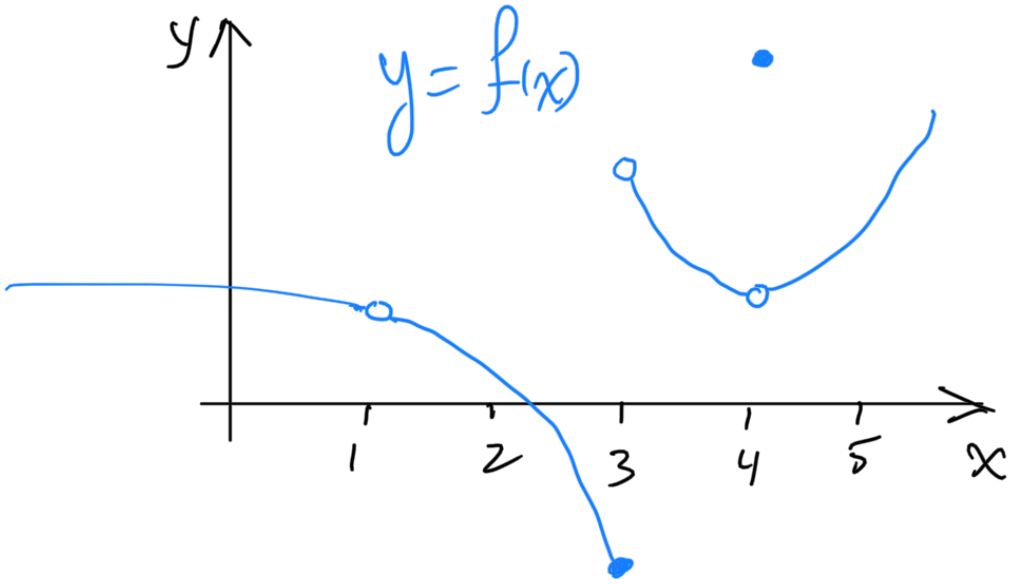
Def. A func. f is <u>continuous (cts) at a number a</u> if $\lim_{x \to a} f(x) = f(a)$.

So, if f is cts at a, we require 3 things :-

(1) $f(a)$ is defined. AND (2) $\lim_{x\to a} f(x)$ exists. AND (3) $\lim_{x\to a} f(x) = f(a)$.

Otherwise, we say that <u>f is discontinuous (discts) at a</u>

Ex: The figure below shows the graph of a func. f.
At which numbers is f discts? Why?

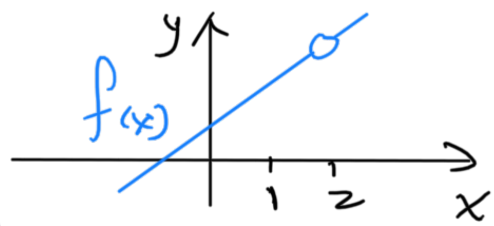


Soln. f is discts at the numbers :-

- 1 because $f(1)$ is undefined.
- 3 because $\lim\limits_{x \to 3} f(x)$ DNE.
- 5 because $\lim\limits_{x \to 5} f(x) \neq f(5)$.

Ex. Where are each of the following functions discts?

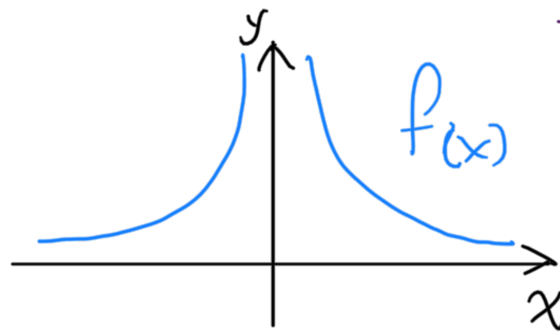
(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$.



Soln. $f(2)$ is undefined, so f is discts at 2.

(Later we will see that f is cts at all other numbers).

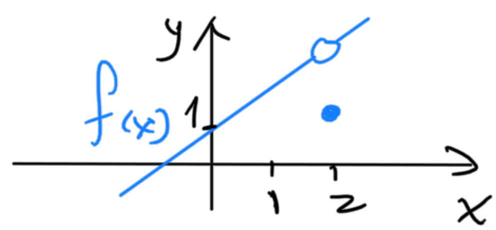
(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$



Soln. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{1}{x^2}$ DNE.

So, f is discts at 0.

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2, \\ 1 & \text{if } x = 2. \end{cases}$



Soln. $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{x-2}}$

$$= \lim_{x\to 2}(x+1) = 3.$$

But $f(2) = 1 \neq \lim_{x\to 2} f(x)$.

So, f is discts at 2.

Theorem. If f and g are cts at a and c is a constant, then the following functions are also cts at a:-

1. $f \pm g$
2. cf
3. fg
4. $\frac{f}{g}$ if $g(a) \neq 0$.

Theorem: If g is cts at a and f is cts at $g(a)$, then $f\circ g$ is cts at a.

The composition func. given by $(f\circ g)(x) = f(g(x))$.

Ex: Prove that $h(x) = \ln(1+\cos x)$ is cts at $x=0$.

Proof. Note that $g(x) \overset{\text{def.}}{=} \cos x$ is cts at $x=0$, and $f(x) \overset{\text{def.}}{=} \ln(1+x)$ is cts at $x=1$.

Using the above theorem, $h(x) = f(g(x)) = \ln(1+\cos x)$ is cts at $x = 0$.

<u>Ex</u>. Find the value of A that makes the func.

$$f(x) = \begin{cases} A^2x^2 & , x<2 ; \\ (1-A)x & , x \geq 2 , \end{cases} \quad \text{cts at } x = 2.$$

<u>Soln</u>. f is cts at $x=2 \Rightarrow 2-2A = f(2) = \lim\limits_{x \to 2^-} f(x) = 4A^2$.

$\Rightarrow 2A^2 + A - 1 = 0 \Rightarrow (A+1)(2A-1) = 0 \Rightarrow A = -1, \frac{1}{2}$.

Types of discontinuities

f is discts at x –

- $\lim\limits_{x \to a} f(x)$ DNE
- $\lim\limits_{x \to a} f(x)$ exists, but $f(a)$ is undefined

OR

- $\lim\limits_{x \to a} f(x) \neq f(a)$

  exists   defined

f is discts at $x=a$ if:

$\lim_{x \to a} f(x)$ DNE

Nonremovable discontinuity.

$\lim_{x \to a} f(x)$ exists, but $f(a)$ is undefined

f has a removable discontinuity at a.

OR

$\lim_{x \to a} f(x) \neq f(a)$

exists defined

f has a removable discontinuity at a

$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$

f has a jump discontinuity at a.

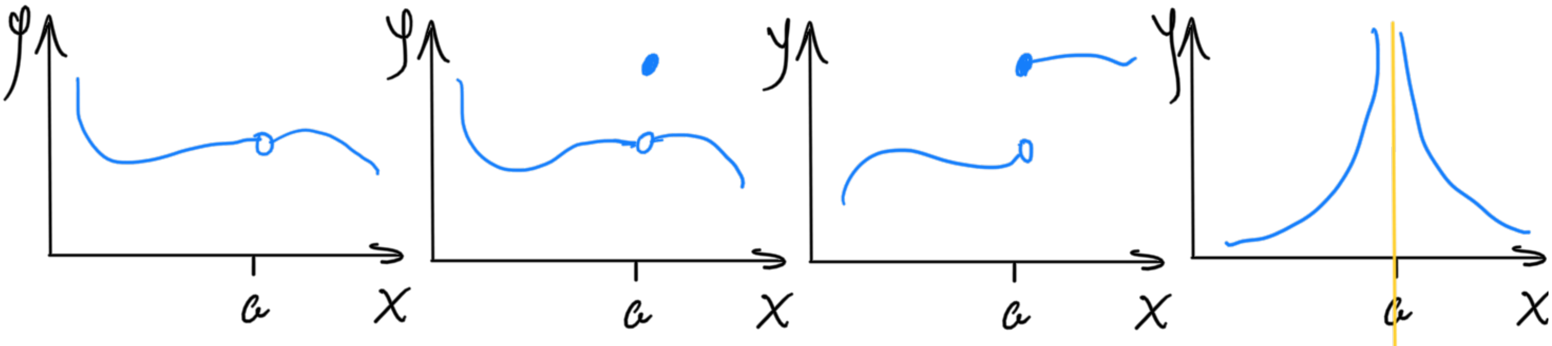
$\lim f(x) = \pm\infty$
as $x \to a$
or $x \to a^-$
or $x \to a^+$

f has an infinite discontinuity at a.

Neither

Example
The Dirichlet function

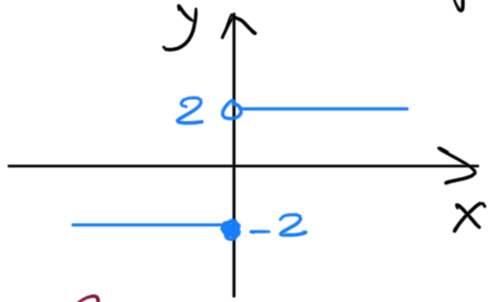
$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$



Removable | Removable | Nonremovable jump | Nonremovable infinite

Ex. Determine whether the discontinuity is a removable, a jump, or an infinite discontinuity, at the indicated point:

1) $f(x) = \begin{cases} -2, & x \le 0. \\ 2, & x > 0. \end{cases}$ at $x = 0$



Soln. $-2 = \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) = 2.$

$\therefore$ f has a jump discont. at $x = 0$.

2) $f(x) = \begin{cases} x^2+4, & x < 2. \\ 5, & x = 2. \\ x^3, & x > 2. \end{cases}$ at $x = 2$.

Soln. $8 = \lim_{x \to 2} f(x) \neq f(2) = 5.$

$\therefore$ f has a removable discont. at $x = 2$.

(3) $f(x) = \tan x$, at $x = \frac{\pi}{2}$.

Soln $\lim_{x \to \frac{\pi}{2}^-} \tan x = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \infty.$

$\therefore$ f has an infinite discont. at $x = \frac{\pi}{2}$.

Ex. Determine the discontinuity types of the func.

$$f(x)=\begin{cases} x^3, & x<-1. \\ x^2-2, & 1<x<0. \\ 3-x, & 0<x<2. \\ \dfrac{4x-1}{x-2}, & 2<x<4. \\ \dfrac{15}{10-2x}, & 4<x<5. \\ 1, & 5\le x. \end{cases}$$

Soln.

a	$f(a)$	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	Result
-1	-1	-1	-1	cts
0	3	-2	3	jump discont.
2	1	1	DNE	infinite discont.
4	undefined	7.5	7.5	removable discont.
5	1	DNE	1	infinite discont.

One-sided continuity.

Def. A func. f is <u>cts from the right</u> at a if

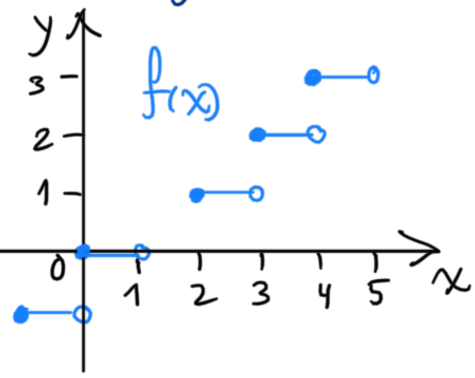
$$\lim_{x\to a^+} f(x) = f(a)$$

and f is <u>cts from the left</u> at a if

$$\lim_{x\to a^-} f(x) = f(a).$$

Ex. Let $f(x) = \lceil x \rceil$; the greatest integer func.

Note that f has discontinuities at all of the integers because $\lim\limits_{x \to n} \lceil x \rceil$ DNE if n is an integer.



Now, at each integer n, $f(x) = \lceil x \rceil$ is cts from right because $\lim\limits_{x \to n^+} \lceil x \rceil = n = f(n)$, and discts from left because $\lim\limits_{x \to n^-} \lceil x \rceil = n-1 \neq f(n)$.

Continuity on intervals.

Definitions. Let f be a func. and $a, b \in \mathbb{R}$ $(a < b)$. Then

(1) f is cts on (a,b) if f is cts at every point on (a,b).

(2) f is cts on $[a,b]$ if i) f is cts on (a,b).

ii) f is cts from right at a,

and iii) f is cts from left at b.

(3) f is cts if f is cts everywhere, i.e., f is cts on $(-\infty, \infty)$.

Ex. Prove that the func. $f(x) = 1 - \sqrt{1-x^2}$ is cts on the interval $[-1,1]$.

Soln. If $-1 < a < 1$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(1-\sqrt{1-x^2}\right)$$
$$= 1 - \lim_{x\to a}\sqrt{1-x^2}$$
$$= 1 - \sqrt{\lim_{x\to a}(1-x^2)}$$
$$= 1 - \sqrt{1-a^2}$$
$$= f(a).$$

Thus, f is cts on $(-1,1)$.

Note also that

$$\lim_{x\to -1^+} f(x) = 1 = f(-1) \text{ and } \lim_{x\to 1^-} f(x) = 1 = f(1).$$

So, f is cts from the right at -1, and cts from left at 1.

Therefore, f is cts on $[-1,1]$. Done!

Theorem.

(a) Any polynomial is cts everywhere, that is, it is cts on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational func. is cts whenever it is defined.

Ex. (1) The poly. $P(x) = x^3 + 2x^2 - 1$ is cts on $\mathbb{R}$.

(2) The rational func $R(x) = \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$ is cts on its domain: $\{x \in \mathbb{R} : x \neq \frac{5}{3}\} = (-\infty, \frac{5}{3}) \cup (\frac{5}{3}, \infty)$.

Theorem: The following types of functions are cts at every number in their domains:

- Polynomials.
- Rational functions.
- Root functions. e.g., $\sqrt[n]{x}$ is cts → everywhere if n is odd, → for $x > 0$ if n is even.
- Trigonometric functions. e.g., $\sin x$ and $\cos x$ are cts on $\mathbb{R}$.
- Inverse trigonometric funcs. e.g., $\tan^{-1} x$ is cts on $\mathbb{R}$.
- Exponential functions. e.g., e^x is cts on $\mathbb{R}$.
- Logarithmic functions. e.g., $\ln x$ is cts for $x > 0$.

Ex. Determine where is the given func. cts ?

1) $f(x) = \cos \frac{1}{x}$. Soln. f is cts on $\mathbb{R} - \{0\}$.

2) $f(x) = \frac{x^2}{5 - \sqrt{x^2+9}}$. Soln. f is cts on $\mathbb{R} - \{\pm 4\}$.

f is discts when $5 - \sqrt{x^2+9} = 0$,

$5 = \sqrt{x^2+9}$,

$x^2 + 9 = 25$,

$x^2 = 16$,

so $x = \pm 4$

3) $f(x) = \sqrt{\frac{x^2+1}{x-9}}$. Soln. f is cts on $(9, \infty)$.

When $x = 9$, the func. $\frac{x^2+1}{x-9}$ is undefined. positive!

Now, f is discts when $x = 9$ or $\frac{x^2+1}{x-9} < 0$,

So $\underline{x = 9}$ or $x - 9 < 0$, i.e., $\underline{x < 9}$, i.e, $x \in (-\infty, 9]$.

4) $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$. <u>Sln.</u> f is cts on $(0,1) \cup (1,\infty)$.

A

$\ln x$ is cts on $(0,\infty)$

A_1

$\tan^{-1} x$ is cts on $(-\infty,\infty)$

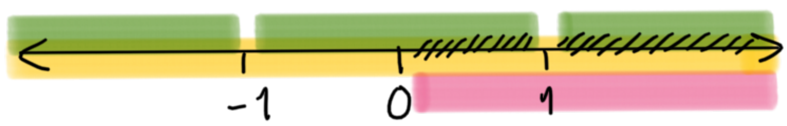
A_2

$x^2 - 1$ is cts on $\mathbb{R} - \{\pm 1\}$

$= (-\infty,-1) \cup (-1,1) \cup (1,\infty)$

A_3

The func. f is cts on $A_1 \cap A_2 \cap A_2 = A$.



<u>Theorem</u>. If f is cts at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

Ex. Evaluate $\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$.

Soln. Because $\sin^{-1}x$ is a cts func., we have

$$\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) = \sin^{-1}\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right)$$

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{\cancel{1-\sqrt{x}}}{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}\right)$$

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{1}{1+\sqrt{x}}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{6}.$$

Searching keywords:

- Continuity الاتصال
- Removable, jump, infinite discontinuities
- Where are each of the following functions discontinuous/ continuous.
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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