Indefinite integrals

Recall that the definite integral $\int_a^b f(x)\,dx$ is a number, whereas an indefinite integral $\int f(x)\,dx$ is a func. (or family of functions).

Def. Let f be a cts func. on $[a,b]$. The <u>indefinite integral of f</u> is given by $\boxed{\int f(x)\,dx = F(x) + C}$, where $F'(x) = f(x)$, and C is the constant of integration.

- To connect this to the definite integral, we recall that

$$\int_a^b f(x)\,dx = \Big[F(x) + C\Big]_a^b = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

- linearity of integration:

$$\int \left[\alpha f(x) \pm \beta g(x)\right] dx = \alpha \int f(x)\,dx \pm \beta \int g(x)\,dx.$$

Some integration formulas.

1) $\int k\,dx = kx + C$.
2) $\int (ax+b)^r dx = \frac{1}{a(r+1)}(ax+b)^{r+1} + C \;;\; r \neq -1.$

3) $\int \frac{1}{x}\,dx = \ln|x| + C$; $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$.

4) $\int e^x\,dx = e^x + C$; $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$.

5) $\int b^x\,dx = \frac{b^x}{\ln b} + C$.

6) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$.

7) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$.

8) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$.

9) $\int \csc^2(ax+b)\,dx = \frac{-1}{a}\cot(ax+b) + C$.

10) $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$.

11) $\int \csc(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\csc(ax+b) + C$.

12) $\int \frac{dx}{x^2+1} = \tan^{-1}x + C$; $\int \frac{f'(x)}{(f(x))^2+1}\,dx = \tan^{-1}(f(x)) + C$.

13) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$; $\int \frac{f'(x)}{\sqrt{1-(f(x))^2}} = \sin^{-1}(f(x)) + C$.

14) $\int \sinh(ax+b)\,dx = \frac{1}{a}\cosh(ax+b) + C$.

15) $\int \cosh(ax+b)\,dx = \frac{1}{a}\sinh(ax+b) + C$.

Examples:

1) $\int \left(\frac{1}{\sqrt[3]{x}} + 1\right) dx = \int \left(x^{-\frac{1}{3}} + 1\right) dx = \frac{3}{2} x^{2/3} + x + C.$

2) $\int e^{(3x+1)} dx = \frac{1}{3} e^{3x+1} + C.$

3) $\int e^{ex} dx = \frac{1}{e} e^{ex} + C = e^{-1} e^{ex} + C = e^{ex-1} + C.$

4) $\int \frac{x+1}{3x^2+6x+1} dx = \frac{1}{6} \int \frac{6x+6}{3x^2+6x+1} dx = \frac{1}{6} \ln |3x^2+6x+1| + C.$

5) $\int \frac{\cos\theta}{\sin^2\theta} d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) d\theta = \int \csc\theta \cot\theta \, d\theta = -\csc\theta + C.$

6) $\int \cot x \, dx = \int \frac{\cos x \leftarrow f'}{\sin x \leftarrow f} dx = \ln |\sin x| + C.$ $\quad \swarrow f$

7) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x| + C.$

8) $\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} dx \quad \begin{matrix} \leftarrow f' \\ \leftarrow f \end{matrix}$$

$$= \ln | \tan x + \sec x | + C.$$

9) $\int \csc x \, dx = \int \csc x \left(\frac{\csc x + \cot x}{\csc x + \cot x} \right) dx$

$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx$$

$$= -\int \frac{-\csc^2 x - \csc x \cot x}{\csc x + \cot x} \, dx$$

$$= -\ln \left| \csc x + \cot x \right| + C.$$

10) $\int \left(2x^3 - 6x + \frac{3}{x^2+1} \right) dx = \frac{\cancel{2}^{1}}{\cancel{4}_{2}} x^4 - \frac{\cancel{6}^{3}}{\cancel{2}} x^2 + 3 \tan^{-1} x + C.$

11) $\int \frac{2t^2 + t^2\sqrt{t} - 1}{t^2} \, dt = \int \left(2 + t^{1/2} - t^{-2} \right) dt = 2t + \frac{2}{3} t^{3/2} + \frac{1}{t} + C$

12) $\int \left[\left(2 - \sqrt{(2x+1)^2 + 4} \right) \left(2 + \sqrt{(2x+1)^2 + 4} \right) \right] dx$

$$= \int \left[\cancel{4} - \left((2x+1)^2 + \cancel{4} \right) \right] dx$$

$$= -\int (2x+1)^2 \, dx$$

$$= -\frac{(2x+1)^3}{(2)(3)} + C$$

$$= \frac{-1}{6} (2x+1)^3 + C.$$

13) $\int x^{2/3} \left(x^{-4/3} + 3 \right) dx.$ Exc.

14) $\int \frac{4}{(4x+1)^2} \, dx.$ Exc.

Ex. Let $I = \int \frac{d}{dx}[f(x)]\,dx$ and $J = \frac{d}{dx}\left[\int f(x)\,dx\right]$.

Then only one of the following is true;

A) $I = J$ B) $J = I + C$ C) $I = J + C$.

Ex. Find f from the information given

(1) $f'(x) = 3 - 4x$, $f(1) = 6$.

Soln. $f(x) = \int f'(x)\,dx = \int (3-4x)\,dx = 3x - 2x^2 + C$.

But $f(1) = 6$, so $3 - 2 + C = 6$, hence $C = 5$.

Thus, $f(x) = 3x - 2x^2 + 5$.

(2) $f''(x) = 1 - x$, $f'(2) = 1$, $f(2) = 0$.

Soln. $f'(x) = \int (1-x)\,dx = x - \frac{1}{2}x^2 + C_1$.

But $f'(2) = 1$, then $2 - \frac{4}{2} + C_1 = 1$, hence $C_1 = 1$.

Thus, $f'(x) = x - \frac{x^2}{2} + 1$.

Next, $f(x) = \int \left(x - \frac{x^2}{2} + 1\right) dx = \frac{x^2}{2} - \frac{x^3}{(2)(3)} + x + C_2$.

But $f(2) = 0$, then $\frac{4}{2} - \frac{8}{6} + 2 + C_2 = 0$,

hence $C_2 = -4 + \frac{4}{3} = \frac{-8}{3}$.

Thus, $f(x) = \frac{x^2}{2} - \frac{x^3}{6} + x - \frac{8}{3}$.

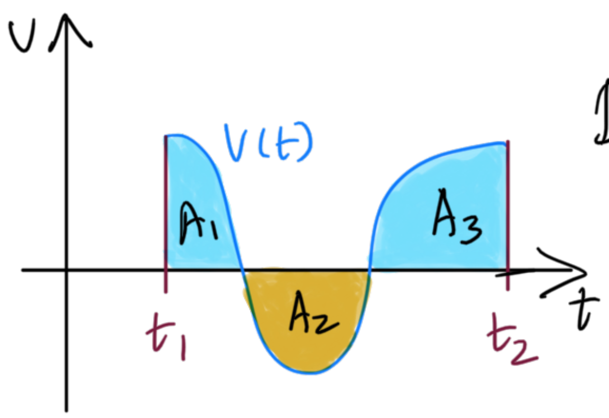
Some motion examples.

Let $s(t)$ be the position func. of an object, $v(t)$ be its velocity, and $a(t)$ be its acceleration.

* It is known that $v(t) = s'(t)$, so

$$\boxed{s(t) = \int v(t)\,dt.}$$

- The number $\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$ is called the displacement of the object from time t_1 to t_2.
- The number $\int_{t_1}^{t_2} |v(t)|\,dt$ is called the distance traveled during the time period $[t_1, t_2]$.



$$\text{Displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3.$$

$$\text{Distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3.$$

* It is also known that $a(t) = s''(t)$, so

$$\boxed{v(t) = \int a(t)\, dt.}$$

Ex. An object moves along a coordinate line with velocity $v(t) = 2 - 3t + t^2$ unit per second. Its initial position (position at time $t=0$) is 2 units to the right of origin. Find the position of the object 4 seconds later.

Soln. Note that $v(t) = 2 - 3t + t^2$ and $s(0) = 2$.

Want: $s(4)$.

$s(t) = \int v(t)\, dt = \int (2 - 3t + t^2)\, dt = 2t - \frac{3}{2}t^2 + \frac{1}{3}t^3 + C.$

But $s(0) = 2$, then $C = 2$.

So $s(t) = 2t - \frac{3}{2}t^2 + \frac{1}{3}t^3 + 2.$

Thus, $x(4) = 2(4) - \frac{3}{2}(4)^2 + \frac{1}{2}(4)^3 + 2 = 7\frac{1}{3}.$

Ex. An object moves along the x-axis with acceleration $a(t) = 2t - 2$ units per $(\text{second})^2$. Its initial position is 5 units to the right of the origin. One second later the object is moving left at the rate of 4 units per second.

1) Find the position of the object at time $t=4$ seconds?

2) How far does the object travel during these 4 seconds?

Soln. Note that $a(t) = 2t-2$, $v(1) = -4$ and $s(0) = 5$.

$v(t) = \int (2t-2)\,dt = t^2 - 2t + C_1$.

But $v(1) = -4$, then $1 - 2 + C_1 = -4$, so $C_1 = -3$.

Thus, $v(t) = t^2 - 2t - 3$.

$s(t) = \int (t^2 - 2t - 3)\,dt = \frac{t^3}{3} - t^2 - 3t + C_2$.

But $s(0) = 5$, then $C_2 = 5$.

Thus, $s(t) = \frac{1}{3}t^3 - t^2 - 3t + 5$.

1) $s(4) = \frac{1}{3}(4)^3 - (4)^2 - 3(4) + 5$

$= -5/3$ units to left of the origin.

2) The distance from time $t=0$ to $t=4$ is

$$s=\int_0^4 |v(t)|\,dt=\int_0^4 |t^2-2t-3|\,dt$$

$$=\int_0^3 -(t^2-2t-3)\,dt+\int_3^4 (t^2-2t-3)\,dt$$

$$=\int_0^3 (3+2t-t^2)\,dt+\int_3^4 (t^2-2t-3)\,dt$$

$$=\left[3t+t^2-\frac{t^3}{3}\right]_0^3+\left[\frac{t^3}{3}-t^2-3t\right]_3^4$$

$$=34/3 \text{ units.}$$

Exc. A particle moves along a line so that its velocity at time t is $v(t)=t^2-t-6$ meters/second.

1) Find the displacement of the particle during the time period $1\le t\le 4$.

2) Find the distance traveled during this time period.

FINAL ANS. (1) $-9/2$ (2) $61/6\approx 10.17$ m.

Searching keywords:

- Indefinite integral التكامل غير المحدود
- Position, velocity, acceleration
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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