Integration.

Differentiation and integration are used in calculus to study change!
While differential calculus finds derivatives, the integral calculus finds antiderivatives.

The definite integral

The definite integral of a cts func.

Def. Let f be a cts func. on $[a,b]$ and $P=\{x_0, x_1, \ldots, x_{n-1}, x_n\}$ be a partition of $[a,b]$. On each subinterval $[x_{i-1}, x_i]$ we have $\Delta x_i = x_i - x_{i-1}$. Let

$$M_i = \max_{x\in[x_{i-1},x_i]} f(x) \text{ and } m_i = \min_{x\in[x_{i-1},x_i]} f(x). \text{ Then}$$

1) $U_f(P) = \sum_{i=1}^{n} M_i \Delta x_i = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n \Delta x_n$

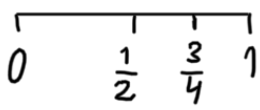
is called the P-upper sum for f.

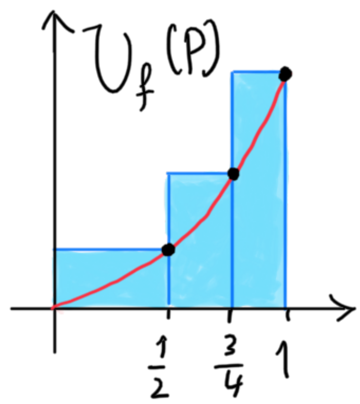
2) $\mathcal{L}_f(P) = \sum_{i=1}^{n} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots m_n \Delta x_n$

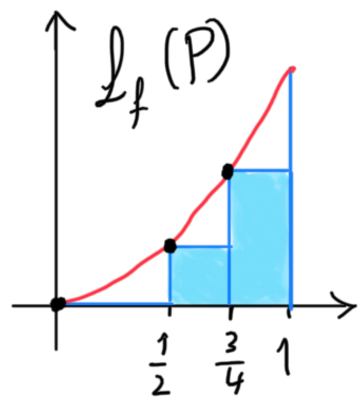
is called <u>the P-lower sum</u> for f.

Ex. Let $f(x) = x^2$, $x \in [0,1]$.

Let $P = \{0, \frac{1}{2}, \frac{3}{4}, 1\}$ be a partition of $[0,1]$.







We have 3 subintervals:

$[x_0, x_1] = [0, \frac{1}{2}]$,
$\Delta x_1 = \frac{1}{2} - 0 = \frac{1}{2}$,
$M_1 = f(\frac{1}{2}) = \frac{1}{4}$,
$m_1 = f(0) = 0$,

$[x_0, x_1] = [\frac{1}{2}, \frac{3}{4}]$,
$\Delta x_1 = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$,
$M_2 = f(\frac{3}{4}) = \frac{9}{16}$,
$m_2 = f(\frac{1}{2}) = \frac{1}{4}$,

$[x_0, x_1] = [\frac{3}{4}, 1]$.
$\Delta x_1 = 1 - \frac{3}{4} = \frac{1}{4}$.
$M_3 = f(1) = 1$.
$m_3 = f(\frac{3}{4}) = \frac{9}{16}$.

$$\mathcal{U}_f(P) = M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3$$
$$= \frac{1}{4} * \frac{1}{2} + \frac{9}{16} * \frac{1}{4} + 1 * \frac{1}{4} \cong 0.39.$$

$$\mathcal{L}_f(P) = m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3$$
$$= 0 * \frac{1}{2} + \frac{1}{4} * \frac{1}{4} + \frac{9}{16} * \frac{1}{4} \cong 0.203.$$

Def. The unique number I that satisfies the inequality $\mathcal{L}_f(P) \le I \le \mathcal{U}_f(P)$ for all partitions P of $[a,b]$ is called the definite integral of f from a to b, denoted by $\int_a^b f(x)\,dx$.

Ex. Prove that $\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$.

Proof. $f(x) = x$, $x \in [a,b]$.

Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a,b]$.

For each $P_i = [x_{i-1}, x_i]$, $\Delta x_i = x_i - x_{i-1}$.

$$M_i = \max_{x \in P_i} f(x) = x_i \quad \text{and} \quad m_i = \min_{x \in P_i} f(x) = x_{i-1}.$$

Then $$\mathcal{U}_f(P) = \sum_{i=1}^n \Delta x_i M_i = \sum_{i=1}^n x_i \Delta x_i,$$

and $$\mathcal{L}_f(P) = \sum_{i=1}^n \Delta x_i m_i = \sum_{i=1}^n x_{i-1} \Delta x_i.$$

Now, $$x_{i-1} \le \frac{x_i + x_{i-1}}{2} \le x_i, \quad \forall i.$$

Then $$x_{i-1} \Delta x_i \le \frac{x_i + x_{i-1}}{2} \Delta x_i \le x_i \Delta x_i, \quad \forall i.$$

Note that $\dfrac{x_i + x_{i-1}}{2} \Delta x_i = \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1})$

$$= \frac{1}{2}\left(x_i^2 - x_{i-1}^2\right).$$

Then $\underbrace{\sum_{i=1}^{n} x_{i-1}\,\Delta x_i}_{\mathcal{L}_f(P)} \le \underbrace{\sum_{i=1}^{n} \frac{1}{2}\left(x_i^2 - x_{i-1}^2\right)}_{=\frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}(b^2 - a^2)} \le \underbrace{\sum_{i=1}^{n} x_i\,\Delta x_i}_{U_f(P)}.$

It follows that

$$\mathcal{L}_f(P) \le \frac{1}{2}(b^2 - a^2) \le \mathcal{L}_f(P), \text{ for all partition } P.$$

Thus, $\displaystyle\int_a^b f(x)\,dx = \frac{1}{2}(b^2 - a^2)$.

Exc. Prove that (1) $\displaystyle\int_a^b k\,dx = k(b-a)$; k: Constant.

(2) $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$.

The definite integral as the limit of Riemann sum.

Defn. Let $\{\overset{a}{\overset{\shortparallel}{x_0}}, x_1, \dots, \overset{b}{\overset{\shortparallel}{x_n}}\}$ be a regular partition of the interval $[a, b]$ with

$$x_k = x_0 + (\Delta x)k = x_0 + \left(\frac{b-a}{n}\right)k, \quad \forall k.$$

For any func. f defined on $[a,b]$, the definite integral of f from a to b is

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x}, \text{ where } x_k^* \in [x_{k-1}, x_k].$$

When the limit exists, we say that f is <u>integrable</u> on $[a,b]$.

This integral gives the <u>area</u> of the region below the graph of f.

<u>Thm.</u> If f is cts on $[a,b]$ (or has only a finite number of jump discontinuities), then f is integrable on $[a,b]$.

<u>Remark</u>: If n is any positive integer, then

1) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ 2) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Ex. Compute $\int_0^2 (x^2-2x)\,dx$ exactly.

Soln. $\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $x_0 = 0$.

Take $x_k^* = x_k = \left(\frac{2}{n}\right)k$.

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} (x_k^2 - 2x_k)\Delta x$$

$$= \sum_{k=1}^{n} \left[\left(\frac{2k}{n}\right)^2 - 2\left(\frac{2k}{n}\right)\right]\left(\frac{2}{n}\right)$$

$$= \sum_{k=1}^{n} \left(\frac{4k^2}{n^2} - \frac{4k}{n}\right)\left(\frac{2}{n}\right)$$

$$= \frac{8}{n^3}\sum_{k=1}^{n} k^2 - \frac{8}{n^2}\sum_{k=1}^{n} k$$

$$= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{8}{n^3}\right)\frac{n(n+1)}{2}$$

$$\xrightarrow{\text{as } n\to\infty} \frac{8}{6} - 4 = -\frac{4}{3}.$$

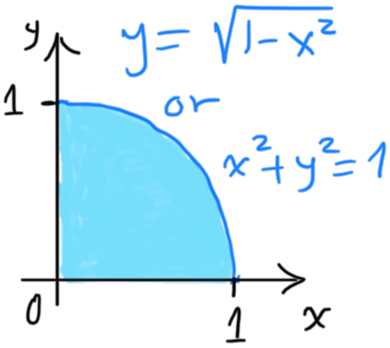
$\therefore \int_0^2 (x^2-2x)\,dx = -\frac{4}{3}$.

Exc. Show that $\int_2^3 (x^2-2x)\,dx = \frac{4}{3}$.

Ex. Evalute the following integrals by interpreting each in terms of areas.

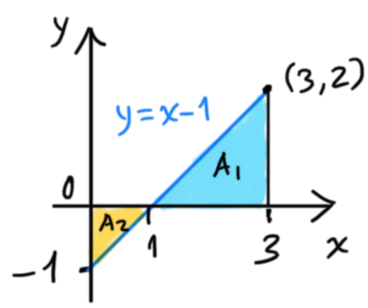
(A) $\int_0^1 \sqrt{1-x^2}\, dx$ (B) $\int_0^3 (x-1)\, dx$.

Soln. (A) This integral is the area under the curve $y=\sqrt{1-x}$ from 0 to 1, which is the quarter-circle with radius 1. Therefore



$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}.$$

(B) This integral is the difference of the areas of two triangles:



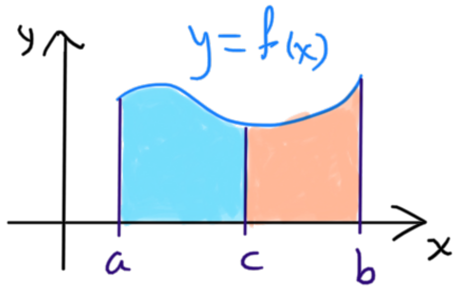
$$\int_0^3 (x-1)\, dx = A_1 - A_2 = \frac{1}{2}(2\cdot 2) - \frac{1}{2}(1\cdot 1)$$
$$= 1.5.$$

Properties of the definite integral.

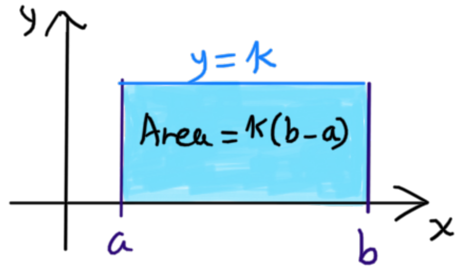
1. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$.

2. $\int_a^a f(x)\, dx = 0$.

3. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



4. $\displaystyle\int_a^b k\,dx = k(b-a)$; k is any constant.



5. $\displaystyle\int_a^b \left[\alpha f(x) \pm \beta g(x)\right] dx = \alpha \int_a^b f(x)\,dx \pm \beta \int_a^b g(x)\,dx.$

α and β are any constants.

Ex. If it is known that $\displaystyle\int_0^{10} f(x)\,dx = 17$ and $\displaystyle\int_0^{8} f(x)\,dx = 12$, find $\displaystyle\int_8^{10} f(x)\,dx$.

Soln. $\displaystyle\underbrace{\int_0^{10} f(x)\,dx}_{17} = \underbrace{\int_0^{8} f(x)\,dx}_{12} + \underbrace{\int_8^{10} f(x)\,dx}_{?}$. Thus $\displaystyle\int_8^{10} = 17-12 = 5.$

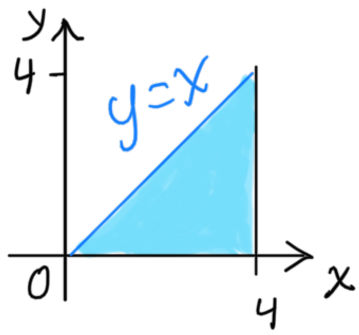
Ex. $\displaystyle\int_0^4 \left(x + 2\sqrt{16-x^2}\right) dx.$

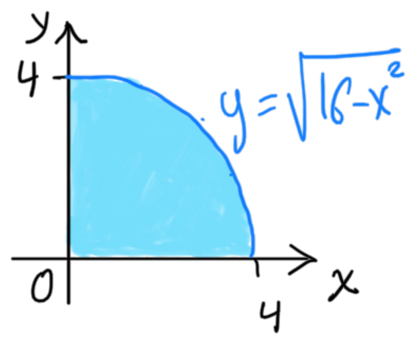
Soln. $\displaystyle\int_0^4 x\,dx = \frac{1}{2}(4 * 4) = 8.$

$\displaystyle\int_0^4 \sqrt{16-x^2}\,dx = \frac{1}{4} * \pi (4)^2 = 4\pi.$

$\displaystyle = \int_0^4 x\,dx + 2\int_0^4 \sqrt{16-x^2}\,dx$

$= 8 + 2(4\pi) = 8(1+\pi).$



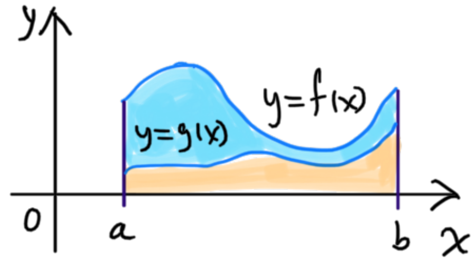


Comparison properties of the integral.

1. If $f(x) \geq 0$, $\forall x \in [a,b]$, then $\int_a^b f(x)\,dx \geq 0$.

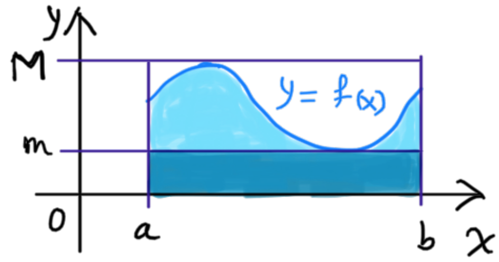
2. If $f(x) \geq g(x)$, $\forall x \in [a,b]$

   then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.



3. If $m \leq f(x) \leq M$, $\forall x \in [a,b]$,

   then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.



Ex. Estimate the given integral.

(1) $\int_1^3 \sqrt{x^2+1}\,dx$.

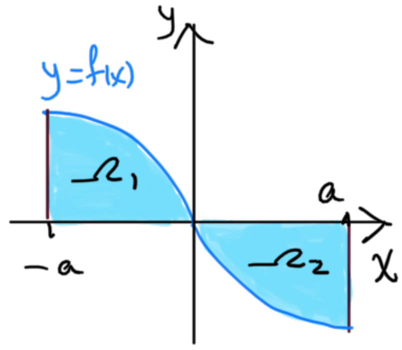
Soln. $\sqrt{2} \leq \sqrt{x^2+1} \leq \sqrt{10}$, $\forall x \in [1,3]$. Then

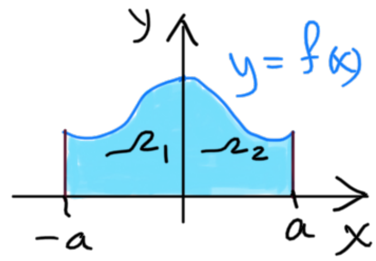
$$2\sqrt{2} \leq \int_1^3 \sqrt{x^2+1}\,dx \leq 2\sqrt{10}.$$

(2) $\int_0^1 e^{-x^2}\,dx$. Exc.

Symmetry

$f(-x) = -f(x)$ $\qquad$ $f(-x) = f(x)$





Fact: Let f be a cts func- on $[-a, a]$, then

1) If f is odd on $[-a, a]$, then
$$\int_{-a}^{a} f(x)\,dx = 0 = \text{Area}(\Omega_1) - \text{Area}(\Omega_2).$$

2) If f is even on $[-a, a]$, then
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx = \text{Area}(\Omega_1) + \text{Area}(\Omega_2).$$

Remark

E . E is E.
E . O is O.
O . O is E.
E + E is E.
O + O is O.
O/E and E/O are O.
E/E and O/O are E.

- O means odd function.
- E means even function.

Ex: Find $\int_{-\pi}^{\pi} (\sin x + x\cos x)^3\,dx$.

Soln.
$$\underbrace{\sin x}_{\text{odd}} + \underbrace{\underbrace{x}_{\text{odd}}\ \underbrace{\cos x}_{\text{even}}}_{\text{odd}}$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{odd}}$$

$$(\sin x + x\cos x)^3 = \underbrace{\underbrace{\underbrace{(\sin x + x\cos x)}_{\text{odd}}\underbrace{(\sin x + x\cos x)}_{\text{odd}}}_{\text{even}}\underbrace{(\sin x + x\cos x)}_{\text{odd}}}_{\text{odd.}}$$

$$\therefore \int_{-\pi}^{\pi} (\sin x + x\cos x)^3\,dx = \text{Zero}.$$

Exc. Prove that $\displaystyle\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\, dx = 0$.

Fact: $\displaystyle\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

Proof. Note that $-|f(x)| \leq f(x) \leq |f(x)|$, $\forall x \in [a,b]$.

Then $\displaystyle -\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$,

Thus, $\displaystyle\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$. Done!

Searching keywords:

- The definite integral, Riemann sum التكامل المحدود، مجموع ريمان
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
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References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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