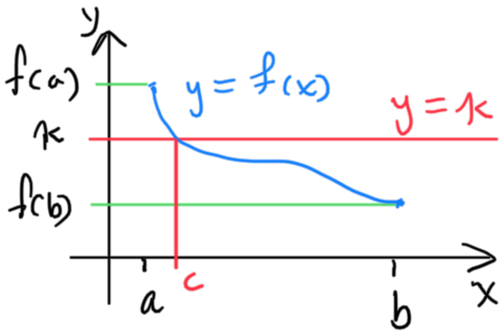
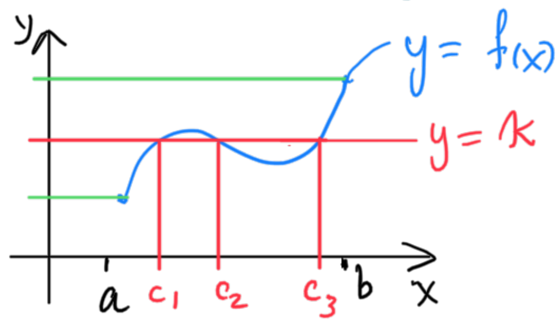
The Mean Value Theorem (M.V.T.)

Recall the intermediate value theorem (I.V.T.):

If f is a cts func. on $[a,b]$ and k is any number s.t. $f(a) < k < f(b)$, then $\exists$ at least one number $c \in (a,b)$ s.t. $f(c) = k$.

(there exists) (continuous) (such that) (belongs)



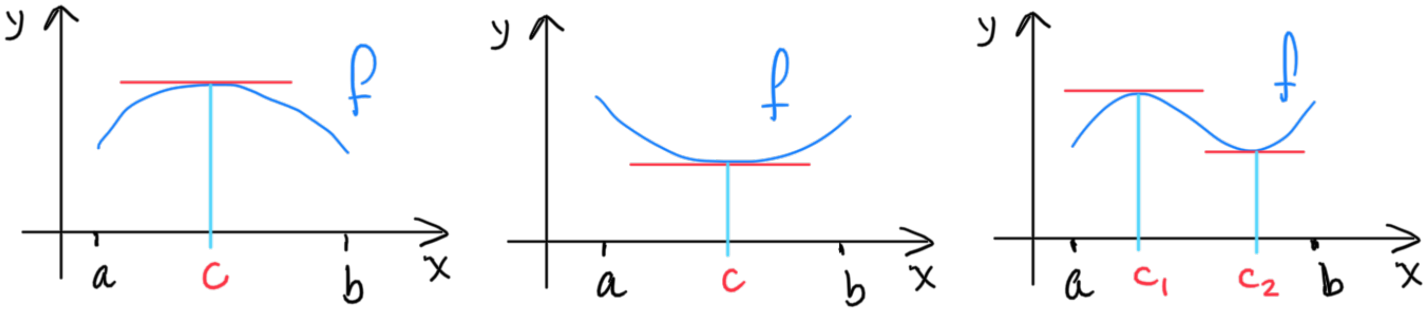


Many results in Mathematics depend on another central fact, which is called the <u>Mean Value Theorem</u> **(M.V.T.)**.

Before arriving at the M.V.T., we first give :

Rolle's theorem.

Let f be cts on $[a,b]$ and diff. on (a,b). If $f(a)=f(b)$, then there is at least one number $c \in (a,b)$ such that $f'(c)=0$.



Ex. Apply Rolle's thm to the func.
$f(x) = x^4 - 2x^2 - 8$ on the interval $[-2,2]$.

Soln.
- f is cts on $[-2,2]$.
- f is diff. on $(-2,2)$.
- $f(2) = 0 = f(-2)$.

By Rolle's thm, $\exists\, c \in (-2,2)$ s.t. $f'(c)=0$.

there exists — belongs — such that

$f'(c) = 4c^3 - 4c = 0$. Could we find such values of c?

$4c(c^2-1) = 0$, then $c = 0, \pm 1 \in (-2,2)$.

Ex: Show that the polynomial $f(x) = 6x^5 + 13x + 1$ has exactly one real root.

Proof. Existence: ← We first show that a root exists.

- f is cts on $[-1, 0]$.
- $f(-1) = -18 < 0$ and $f(0) = 1 > 0$

$\xrightarrow{\text{I.V.T.}}$ $\exists\, a \in (-1, 0)$ s.t. $f(a) = 0$.

That is, $6a^5 + 13a + 1 = 0$.

Uniqueness: ← We then show that the func. has no other root.

We argue by contradiction.

Suppose that there is another root say b.

Without loss of generality, assume that $a < b$.

- f is cts on $[a, b]$.
- f is diff. on (a, b).
- $f(a) = 0 = f(b)$.

$\xrightarrow{\text{M.V.T.}}$ $\exists\, c \in (a, b)$ s.t. $f'(c) = 0$.

Then $30c^4 + 13 = 0$, but $30c^4 + 13 \geq 13$.

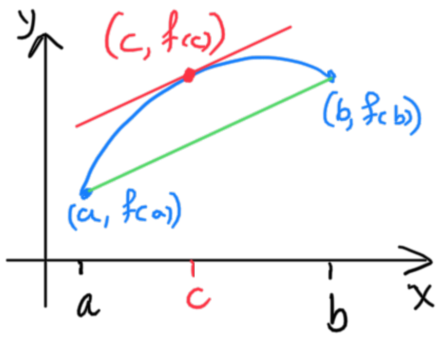
This gives a contradiction!

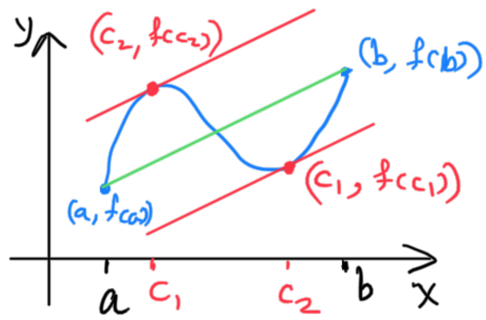
Thus, f has exactly one real root. Done!

The mean value theorem (M.V.T.)

If f is cts on $[a,b]$ and diff. on (a,b), then there is at least one number $c \in (a,b)$ s.t.

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{or} \quad f(b)-f(a) = f'(c)(b-a)}$$





Ex. Apply the M.V.T. to the given func. on the indicated interval.

(1) $f(x) = 3\sqrt{x} - 4x, \quad x \in [1, 4]$.

Soln. • f is cts on $[1,4]$.

• f is diff. on $(1,4)$.

$\xrightarrow{\text{M.V.T.}}$ $\exists\, c \in (1,4)$ s.t. $f'(c) = \dfrac{f(4)-f(1)}{4-1}$.

Could we find the value(s) of c?

Now, $f'(c) = \frac{3}{2\sqrt{c}} - 4 = \frac{-10+1}{3} = \frac{-9}{3} = -3$.

Then $\frac{3}{2\sqrt{c}} = -3+4 = 1$, hence $3 = 2\sqrt{c}$.

This implies that $\sqrt{c} = \frac{3}{2}$, so $c = \frac{9}{4} \in (1,4)$.

(2) $f(x) = \begin{cases} 2+x^3 & , -1 \leq x \leq 1; \\ 3x & , 1 \leq x \leq 2. \end{cases}$

<u>Soln.</u> $f'(x) = \begin{cases} 3x^2 & , -1 < x < 1; \\ 3 & , 1 \leq x < 2. \end{cases}$

- f is cts on $[-1,2]$.
- f is diff. on $(-1,2)$.

$\xrightarrow{\text{M.V.T.}}$ $\exists\, c \in (-1,2)$ s.t. $f'(c) = \frac{f(2)-f(-1)}{2-(-1)}$.

Could we find the value(s) of c?

Now, $f'(c) = \frac{6-1}{3} = \frac{5}{3}$. So, $3c^2 = \frac{5}{3}$.

Then $c^2 = \frac{5}{9}$, hence $c = \pm\frac{\sqrt{5}}{3} \in (-1,2)$.

Applications of the M.V.T.

for all

Ex. Prove that $|\sin a - \sin b| \le |a-b|$, $\forall a,b \in \mathbb{R}$.

Proof. Let $f(x) = \sin x$, $x \in [a,b]$.

- f is cts on $[a,b]$.
- f is diff. on (a,b). In fact, $f'(x) = \cos x$.

$\boxed{\text{M.V.T.}} \Rightarrow \exists\, c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Then $\cos c = \dfrac{\sin b - \sin a}{b-a}$

or $\sin b - \sin a = \cos c\,(b-a)$.

Using the fact that $|\cos c| \le 1$, we have

$|\sin b - \sin a| = |\cos c|\,|b-a| \le |b-a|$. Done

Exc. Prove that $|\sqrt{a} - \sqrt{b}| \le \dfrac{1}{\sqrt{a}}(b-a)$, for all $b > a > 0$.

We can use the M.V.T. to show that:

Thm. If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on (a,b).

Corollary. If $f'(x) = g'(x)$ for all $x \in (a,b)$, then $f(x) = g(x) + C$ where C is a constant.

Proof: Let $h(x) = f(x) - g(x)$, $x \in [a,b]$.

Then $h'(x) = f'(x) - g'(x) = 0$, $\forall x \in (a,b)$.

By the above thm, $h(x) = C$, (C : constant).

Thus, $f(x) - g(x) = C$ or $f(x) = g(x) + C$.

Ex. Show that $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$.

Proof. Let $f(x) = \tan^{-1}x + \cot^{-1}x$.

Then $f'(x) = \frac{1}{1+x^2} + \frac{-1}{1+x^2} = 0$, for all x.

Using the above corollary, $f(x) = C$, for all x.

Note that we can find the value of the constant C by noting that

$$C = f(1) = \tan^{-1}(1) + \cot^{-1}(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Thus, $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$.

A generalization of the M.V.T. :-

The Cauchy Mean Value Theorem

Suppose that f and g are cts on $[a,b]$ and diff. on (a,b). If $g(x) \neq 0$, $\forall x \in (a,b)$, then $\exists$ a number $c \in (a,b)$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} \quad \text{or} \quad \frac{f'(c)}{\int_a^b f'(x)\,dx} = \frac{g'(c)}{\int_a^b g'(x)\,dx}.$$

Searching keywords:

- Use Rolle’s theorem, use the mean value theorem.
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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