Indeterminate forms and l'Hospital's rule.

L'Hospital's rule (L.R.)

(except possibly at a)

Assume that f and g are diff. with $g'(x) \neq 0$ and that $\lim_{x \to c} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ or $\pm\infty$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Remark: L.R. also holds if $x \to c$ is replaced with any of the following;

$x \to c^+$, $x \to c^-$, $x \to \infty$, or $x \to -\infty$.

Ex. Find the limit.

(1) $\lim_{x \to 0} \frac{1 - \cos x}{\sin x}$ $\left(\frac{0}{0}\right)$

$$\overset{\text{L.R.}}{=} \lim_{x \to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(\sin x)} = \lim_{x \to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0.$$

(2) $\lim\limits_{x\to\infty} \dfrac{e^x}{x}\ \left(\frac{\infty}{\infty}\right) \overset{L.R.}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{1} = \infty.$

(3) $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x}\ \left(\frac{\infty}{\infty}\right) \overset{L.R.}{=} \lim\limits_{x\to\infty} \dfrac{2x}{e^x} \overset{L.R.}{=} \lim\limits_{x\to\infty} \dfrac{2}{e^x} = 0.$

(4) $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\csc x}\ \left(\frac{\infty}{\infty}\right) \overset{L.R.}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\csc x \cot x}$

$$= \lim_{x\to 0^+}\left(-\frac{\sin x}{x}\tan x\right)$$

$$= -\left(\lim_{x\to 0^+}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\tan x\right)$$

$$= (-1)(0) = \text{Zero}.$$

(5) $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^r} \overset{L.R.}{=} \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{r x^{r-1}} = \lim\limits_{x\to\infty} \dfrac{1}{r x^r} = 0.$

(r is any positive real number)

(6) $\lim\limits_{x\to\infty} \dfrac{x^r}{e^x} \overset{L.R.}{=} \lim\limits_{x\to\infty} \dfrac{r x^{r-1}}{e^x}$

$$\overset{L.R.}{=} \lim_{x\to\infty} \frac{r(r-1)x^{r-2}}{e^x}$$

$$\overset{L.R.}{=}$$

$$\vdots$$

$$\overset{L.R.}{=} \lim_{x\to\infty} \frac{r!}{e^x} = 0.$$

Note: The L.R. can be used with five additional forms

Indeterminate forms	Determinate forms
$0/0$	$\infty + \infty = \infty$
$\pm\infty/\pm\infty$	$-\infty - \infty = -\infty$
$\infty - \infty$	$\infty \cdot \infty = \infty$
$0 \cdot \infty$	$0^{\infty} = 0$
0^0	$0^{-\infty} = \infty$
1^{∞}	
∞^0	
Use the L.R.	Do not use the L.R.

Ex. Find the limit.

(1) $\lim_{x \to 0^+} \sqrt{x} \ln x$ $(0 \cdot \infty)$

$$= \lim_{x \to 0^+} \frac{\ln x}{1/\sqrt{x}} \overset{L.R.}{=} \lim_{x \to 0^+} \frac{1/x}{-\frac{1}{2} x^{-\frac{3}{2}}} = \lim_{x \to 0^+} -2\sqrt{x} = 0.$$

(2) $\lim_{x \to 0} \left(\frac{1}{\ln(x+1)} - \frac{1}{x} \right)$ $(\infty - \infty)$

$$= \lim_{x \to 0} \frac{x - \ln(x+1)}{x \ln(x+1)}$$

$$\overset{L.R.}{=} \lim_{x\to 0} \frac{1-\frac{1}{x+1}}{\left(\frac{1}{x+1}\right)x + \ln(x+1)} \left(\frac{x+1}{x+1}\right)$$

$$= \lim_{x\to 0} \frac{x+\cancel{1}-\cancel{1}}{x+(x+1)\ln(x+1)}$$

$$= \lim_{x\to 0} \frac{1}{1+\ln(x+1)+\cancel{(x+1)}\frac{1}{\cancel{(x+1)}}} = \frac{1}{2}.$$

(3) $\lim_{x\to\infty}(e^x - x)$ $(\infty - \infty)$

$$= \lim_{x\to\infty} x\left(\frac{e^x}{x}-1\right)$$

$$= \left(\lim_{x\to\infty} x\right)\left[\left(\lim_{x\to\infty}\frac{e^x}{x}\right)-1\right]$$

$$\overset{L.R.}{=} \left(\lim_{x\to\infty} x\right)\left[\left(\lim_{x\to\infty}\frac{e^x}{1}\right)-1\right]$$

$$= \infty.$$

Remark: If $\lim_{x\to c} [f(x)]^{g(x)}$ has one of the indeterminate forms 0^0, ∞^0, or 1^∞, then by letting $y = [f(x)]^{g(x)}$, we have for $f(x) > 0$ that $\ln y = \ln [f(x)]^{g(x)} = g(x) \ln [f(x)]$.

Then $\lim_{x\to c} \ln y = \lim_{x\to c} g(x) \ln (f(x))$ will have the indeterminate form $0 \cdot \infty$ which can deal with as above example (item (1)).

Ex- Find the limit.

(1) $\lim_{x\to 1^+} x^{\frac{1}{x-1}}$ (1^∞)

Soln- Let $y = x^{\frac{1}{x-1}}$, then $\ln y = \ln x^{\frac{1}{x-1}} = \frac{\ln x}{x-1}$.

It follows that $\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{\ln x}{x-1}$

$$\stackrel{L.R.}{=} \lim_{x\to 1^+} \frac{\frac{1}{x}}{1} = 1.$$

$$\therefore \lim_{x\to 1^+} y = \lim_{x\to 1^+} e^{\ln y} = e^{1}.$$

(2) $\lim_{x\to 0^+} x^x$ (0^0)

Soln. Let $y = x^x$, then $\ln y = x\ln x$.

It follows that $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x\ln x$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

$$\overset{L.R.}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} (-x) = 0.$$

$$\therefore \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{0} = 1.$$

(3) $\lim_{x\to 0^+} (\sin x)^x = 1.$ Exc.

(4) $\lim_{x \to \infty} (x+1)^{2/x}$ (∞^0)

Sln. Let $y = (x+1)^{2/x}$, then $\ln y = \frac{2}{x} \ln(x+1)$.

Then $\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{2\ln(x+1)}{x}$

$$\overset{L.R.}{=} \lim_{x \to \infty} \frac{\frac{2}{x+1}}{1} = 0.$$

$\therefore \lim_{x \to \infty} y = \lim_{x \to \infty} e^{\ln y} = e^0 = 1.$

Exc. Find the limit.

(1) $\lim_{x \to \infty} \frac{e^{2/x} - 1}{1/x}$. FINAL ANS. 2.

(2) $\lim_{x \to \infty} \frac{2^x}{x^2}$. FINAL ANS. ∞

(3) $\lim_{x \to \infty} (3^x + 4^x)^{\frac{1}{x}}$. FINAL ANS. 4

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References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

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د. بهاء محمود الزالق
The University of Jordan
Dr. Baha Alzalg
baha2math@gmail.com

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