Derivatives of inverse and hyperbolic functions.

Derivatives of inverse functions.

Fact (The derivative of inverse func.)

Assume that f is a 1-1 diff. func. and its inverse func. f^{-1} is also diff., then we can

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$, provided that $f'(f^{-1}(x)) \neq 0$.

$f'(y) \neq 0$.

Proof: Let $y = f^{-1}(x)$, then $f(y) = f(f^{-1}(x)) = x$.
Using implicit differentiation, we have $f'(y)\, y' = 1$.
Hence $(f^{-1})'(x) = y' = \dfrac{1}{f'(y)} = \dfrac{1}{f'(f^{-1}(x))}$. Done!

Fact: If $f(a) = b$, then $\boxed{(f^{-1})'(b) = \dfrac{1}{f'(a)}}$, provided that $f'(a) \neq 0$.

Ex. Let $f(x) = x^5 + 8x^3 + x + 1$ (1-1 func.)
Calculate (A) $(f^{-1})'(1)$ (B) $(f^{-1})'(11)$.

Soln. $f'(x) = 5x^4 + 24x^2 + 1$.

(A) Note that $f(0) = 1$, then

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{1} = 1.$$

(B) Note that $f(1) = 11$, then

$$(f^{-1})'(11) = \frac{1}{f'(1)} = \frac{1}{30}.$$

Derivatives of inverse trigonometric functions

Facts:

(1) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

(2) $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$.

(3) $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$.

(4) $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$.

(5) $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$.

(6) $\frac{d}{dx} \csc^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$.

(7) $\frac{d}{dx} \sin^{-1}(f(x)) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$.

(8) $\frac{d}{dx} \cos^{-1}(f(x)) = \frac{-f'(x)}{\sqrt{1-(f(x))^2}}$.

(9) $\frac{d}{dx} \tan^{-1}(f(x)) = \frac{f'(x)}{1+(f(x))^2}$.

(10) $\frac{d}{dx} \cot^{-1}(f(x)) = \frac{-f'(x)}{1+(f(x))^2}$,

(11) $\frac{d}{dx} \sec^{-1}(f(x)) = \frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$.

(12) $\frac{d}{dx} \csc^{-1}(f(x)) = \frac{-f'(x)}{f(x)\sqrt{(f(x))^2-1}}$.

<u>Proofs:</u> (1) Let $y = \sin^{-1} x$, then $\sin y = x$.

It follows that $\frac{d}{dx}\sin y = \frac{d}{dx}(x)$. So $\cos y \frac{dy}{dx} = 1$.

Then $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$.

(3) Let $y = \tan^{-1} x$, then $\tan y = x$.

It follows that $\sec^2 y \, y' = 1$.

So $y' = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$.

(2, 4–6) <u>Exc.</u>

(7–12) By chain rule.

<u>Examples:</u>

(1) $\frac{d}{dx}\sin^{-1}(x^3) = \frac{3x^2}{\sqrt{1-(x^3)^2}} = \frac{3x^2}{\sqrt{1-x^6}}$.

(2) $\frac{d}{dx}\tan^{-1}(e^{\pi x}) = \frac{\pi e^{\pi x}}{1+e^{2\pi x}}$.

(3) $\frac{d}{dx}\sec^{-1}(\sqrt{\ln x}) =$ <u>Exc.</u>

Ex. Find y' for $y = (\arcsin 2x)^2$.

Soln. $y' = 2(\sin^{-1} 2x) * \dfrac{2}{\sqrt{1-4x^2}}$.

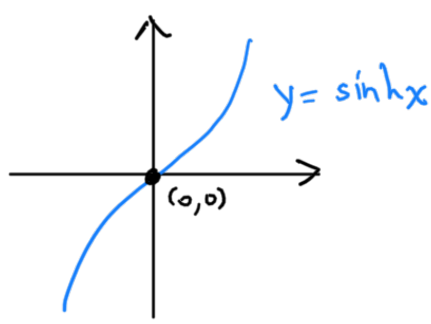
Ex. Find dy for $y = x \tan^{-1}(4x)$.

Soln. $dy = \left(x * \dfrac{4}{1+16x^2} + \tan^{-1} x \right) dx$.

Hyperbolic functions and their derivatives.

Hyperbolic sine

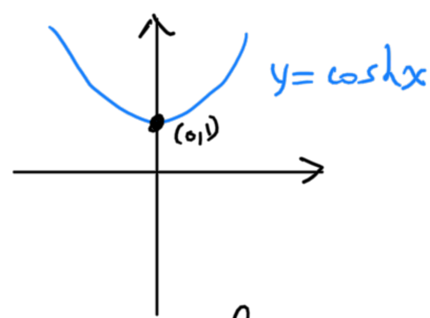
$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$



Odd func.

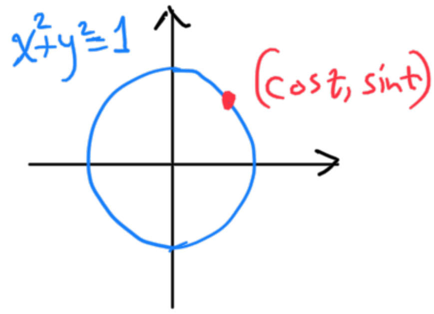
Hyperbolic cosine

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

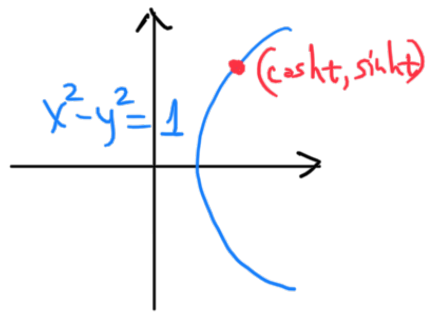


Even func.

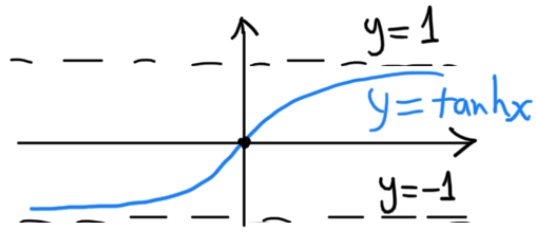
Notes: (1) The point $(\cos t, \sin t)$ lies on the circle $x^2+y^2=1$.



(2) The point $(\cosh t, \sinh t)$ lies on the hyperbola $x^2-y^2=1$.



Other hyperbolic functions.



- $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.
- $\coth x = \dfrac{\cosh x}{\sinh x}$.
- $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$.
- $\operatorname{csch} x = \dfrac{1}{\sinh x}$.

Hyperbolic identities

- $\sinh(-x) = -\sinh x$.
- $\cosh(-x) = \cosh x$.
- $\cosh^2 x - \sinh^2 x = 1$.
- $1 - \tanh^2 x = \operatorname{sech}^2 x$.
- $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

Facts: (1) $\frac{d}{dx}\sinh x = \cosh x$. (2) $\frac{d}{dx}\cosh x = \sinh x$.

(3) $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$. (4) $\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$.

(5) $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x\tanh x$. (6) $\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x\coth x$.

Proof. (3)
$$\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$$
$$= \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x}$$
$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$
$$= \frac{1}{\cosh^2 x}$$
$$= \operatorname{sech}^2 x.$$ Done

Ex. (1) $\frac{d}{dx}\coth\sqrt{x} = \frac{-1}{2\sqrt{x}}\operatorname{csch}^2\sqrt{x}$.

(2) $\frac{d}{dx}\operatorname{csch}(\ln\sqrt{x}) = \frac{-1}{2x}\operatorname{csch}(\ln\sqrt{x})\cot(\ln\sqrt{x})$.

Inverse hyperbolic functions and their derivatives

Facts: (1) $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$, $x \in \mathbb{R}$.

(2) $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$, $x \geq 1$.

(3) $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$, $-1 < x < 1$.

Proof: (1) Let $y = \sinh^{-1} x$, then $\sinh y = x$.

It follows that $\frac{1}{2}(e^y - e^{-y}) = x$.

Then $e^y - e^{-y} = 2x$, or $e^y - 2x - e^{-y} = 0$.

Multiplying by e^y, we have $e^{2y} - 2x e^y - 1 = 0$.

Then $e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$.

Note that $e^y > 0$, but $x - \sqrt{x^2+1} < 0$.

Then, $e^y = x + \sqrt{x^2+1}$.

Thus $y = \ln e^y = \ln\left(x + \sqrt{x^2+1}\right) = \sinh^{-1} x$.

(2) and (3) Exc.

Facts: (1) $\frac{d}{dx} \sinh^{-1} x = 1/\sqrt{1+x^2}$.

(2) $\frac{d}{dx} \cosh^{-1} x = 1/\sqrt{x^2-1}$.

(3) $\frac{d}{dx} \tanh^{-1} x = 1/(1-x^2)$.

Proof: (1) Let $y = \sinh^{-1} x$, then $\sinh y = x$.
By implicit diff., we have $\cosh y \frac{dy}{dx} = 1$.
Then $\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$.

Alternative proof of (1):

$$\frac{d}{dx} \sinh^{-1} x = \frac{d}{dx} \ln\left(x + \sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \frac{d}{dx}\left(x+\sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\cancel{\sqrt{x^2+1} + x}\ 1}{\cancel{\left(x+\sqrt{x^2+1}\right)}\sqrt{x^2+1}}$$

$$= 1/\sqrt{x^2+1}.$$

(2) and (3) Exc.

Ex. $\frac{d}{dx} \tanh^{-1}(\sin\sqrt{x}) = \frac{1}{1-(\sin\sqrt{x})^2} \frac{d}{dx} \sin\sqrt{x}$

$$= \frac{1}{1-\sin^2\sqrt{x}} (\cos\sqrt{x}) \frac{1}{2\sqrt{x}}$$

$$= \frac{\cancel{\cos\sqrt{x}}}{\cos^{\cancel{2}}\sqrt{x}} \frac{1}{2\sqrt{x}}$$

$$= \frac{\sec\sqrt{x}}{2\sqrt{x}}.$$

Searching keywords:

- Derivatives of inverse functions
- Derivatives of hyperbolic functions and inverse hyperbolic functions
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

د. بهاء محمود الزالق
The University of Jordan
Dr. Baha Alzalg
baha2math@gmail.com