The chain rule

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Thm. If g is diff. at x and f is diff. at $g(x)$, then $f\circ g$ is diff. at x and

$$\boxed{(f\circ g)'(x) = \frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x).}$$

Ex. $\frac{d}{dx}(x^3+1)^2 = \frac{d}{dx}(x^6+2x^3+1) = 6x^5+6x^2.$

Using the chain rule, we have

$$\frac{d}{dx}(x^3+1)^2 = 2(x^3+1)\frac{d}{dx}(x^3+1) = 2(x^3+1)(3x^2) = \sim$$

An alternative form of chain rule

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}} \qquad y \rightarrow u \rightarrow x$$

Explanation: let $u = g(x)$, then $\frac{du}{dx} = g'(x)$

and $y = f(g(x)) = f(u)$, so $\frac{dy}{du} = f'(u).$

Then $\frac{dy}{dx} = \frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x) = f'(u)\, g'(x) = \frac{dy}{du}\cdot\frac{du}{dx}.$

Ex. Differentiate $y=(x^3+e^3)^5$.

constant

Soln. $y' = 5(x^3+e^3)^4 \frac{d}{dx}(x^3+e^3)$

$= 5(x^3+e^3)^4(3x^2)$.

$\boxed{\text{Or}}$ let $u = x^3+e^3$, then $y=u^5$ and

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 5u^4(3x^2+0) = 5(x^3+e^3)^4(3x^2).$$

Ex. $\frac{d}{dx}\left(\frac{8}{(x^3+1)^2}\right) = \frac{d}{dx}\left(8(x^3+1)^{-2}\right)$

$= 8(-2)(x^3+1)^{-3}(x^3+1)'$

$= 8(-2)(x^3+1)^{-3}(3x^2)$

$= -48x^2/(x^3+1)^3$.

Ex. $\frac{d}{dx}\left(x+\frac{1}{x}\right)^{-3} = -3\left(x+\frac{1}{x}\right)^{-4}\frac{d}{dx}\left(x+\frac{1}{x}\right)$

$= -3\left(x+\frac{1}{x}\right)^{-4}\left(1-\frac{1}{x^2}\right)$

Remark: $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$. So by the chain rule

$$\boxed{\frac{d}{dx}\sqrt{f(x)} = f'(x)/2\sqrt{f(x)}}.$$

Exc. Find $g'(x)$ if $g(x) = \sqrt{x^2+1}$.

Ex. $$\frac{d}{dx}\left[\left(\sqrt{x^2+4}-3x^2\right)^{3/2}\right] = \frac{3}{2}\left(\sqrt{x^2+4}-3x^2\right)^{\frac{1}{2}}\frac{d}{dx}\left(\sqrt{x^2+4}-3x^2\right)$$

$$= \frac{3}{2}\left(\sqrt{x^2+4}-3x^2\right)^{\frac{1}{2}}\left(\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+4}}-6x\right).$$

Ex. Let $y=\left(2x+(x+1)^3\right)^4$. Find y'.

Soln. $$y' = 4\left(2x+(x+1)^3\right)^3\frac{d}{dx}\left(2x+(x+1)^3\right)$$

$$= 4\left(2x+(x+1)^3\right)^3\left(2+3(x+1)^2\right).$$

Exc. Let $y=4x(x^3+3)^3$. Find y'.

Remark: $$\boxed{\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dt}\cdot\frac{dt}{dx}}$$

$$\begin{tikzcd} y \arrow[r] \arrow[rrr, bend right] & u \arrow[r] & t \arrow[r] & x \end{tikzcd}$$

Ex. Find $\frac{dy}{dx}$ at $x = 9$, given that

$$y = \frac{u+2}{u-1}, \qquad u = (3t-7)^2, \qquad \& \quad t = \sqrt{x}$$

Soln.

$$\frac{dy}{du} = \frac{-3}{(u-1)^{-2}} \qquad \frac{du}{dt} = 6(3t-7) \qquad \frac{dt}{dx} = \frac{1}{2\sqrt{x}}$$

$$\left.\frac{dy}{du}\right|_{u=4} = \frac{-1}{3} \qquad \left.\frac{du}{dt}\right|_{t=3} = 12 \qquad \left.\frac{dt}{dx}\right|_{x=9} = \frac{1}{6}.$$

$$x = 9 \Longrightarrow t = \sqrt{9} = 3 \Longrightarrow u = (3(3)-7)^2 = 4$$

Then $\frac{dy}{dx} = \left.\frac{dy}{du}\right|_{u=4} \left.\frac{du}{dt}\right|_{t=3} \left.\frac{dt}{dx}\right|_{x=9} = \frac{-1}{3} * 12 * \frac{1}{6} = \frac{-2}{3}.$

Ex. Find $f'(x)$ if $\frac{d}{dx}\left(f(2x)\right) = x^2$.

Soln. $2f'(2x) = x^2$. Let $u = 2x$, then

$$2f'(u) = (u/2)^2 = u^2/4.$$

It follows that $f'(u) = u^2/8$ or $f'(x) = x^2/8$.

Derivatives of trigonometric functions.

Recall that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim\limits_{x\to 0} \frac{1-\cos x}{x} = 0$.

Thm. 1) $\frac{d}{dx}\sin x = \cos x$ 2) $\frac{d}{dx}\cos x = -\sin x$

3) $\frac{d}{dx}\tan x = \sec^2 x$ 4) $\frac{d}{dx}\cot x = -\csc^2 x$

5) $\frac{d}{dx}\sec x = \sec x \tan x$ 6) $\frac{d}{dx}\csc x = -\csc x \cot x$.

Proof: Recall that $\cos(A+B) = \cos A \cos B - \sin A \sin B$

2)
$$\frac{d}{dx}\cos x = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$
$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$
$$= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h}$$
$$= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$
$$= \cos x\,(0) - \sin x\,(1)$$
$$= -\sin x.$$

3) $\frac{d}{dx} \tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{\cos x\,(\cos x) - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Examples.

1) $\frac{d}{dx} \sin(3x) = \cos(3x)\frac{d}{dx}(3x) = 3\cos(3x).$

2) $\frac{d}{dx} \sin(x^3) = \cos(x^3)\frac{d}{dx}(x^3) = 3x^2\cos(x^3).$

3) $\frac{d}{dx} \sin^3 x = 3\sin^2 x \frac{d}{dx}\sin x = 3\sin^2 x \cos x.$

Ex. $\frac{d}{dx}\left(\frac{1-\sec x}{\tan x}\right) = \frac{d}{dx}(\cot x - \csc x)$

$$= -\csc^2 x + \csc x \cot x.$$

Ex. $\frac{d}{dx} \sin^2(x^3) = 2 \sin(x^3) \frac{d}{dx} \sin(x^3)$

$= 2 \sin(x^3) \cos(x^3) \frac{d}{dx}(x^3)$

$= 2 (\sin x^3)(\cos x^3)(3x^2)$

$= 3x^2 \sin(2x^3).$

Ex. Let $y = \tan x \sin^2(x^3)$. Find y'.

Soln. $y' = \left(\frac{d}{dx} \tan x\right) \sin^2 x^3 + \tan x \left(\frac{d}{dx} \sin^2(x^3)\right)$

$= \sec^2 x \sin^2 x^3 + \tan x \left(3x^2 \sin(2x^3)\right).$

Ex. Find the equation of the tangent line to $y = 3\tan x - 2\csc x$ at $x = \pi/3$.

Soln. $y' = 3\sec^2 x - 2(-\csc x \cot x)$

$= 3\sec^2 x + 2 \csc x \cot x$

Note that $\tan\frac{\pi}{3} = \sqrt{3}$, $\csc\frac{\pi}{3} = \frac{2}{\sqrt{3}}$, $\sec\frac{\pi}{3} = 2$ & $\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$.

Then $y\left(\frac{\pi}{3}\right) = 3\sqrt{3} - \frac{4}{\sqrt{3}}$,

and $y'\left(\frac{\pi}{3}\right) = 3(2)^2 + 2\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = 12 + \frac{4}{3} = \frac{40}{3}$.

The tangent line at $x = \frac{\pi}{3}$ has the eq.

$$y - y\left(\frac{\pi}{3}\right) = y'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right).$$

Thus $y - \left(3\sqrt{3} - \frac{4}{\sqrt{3}}\right) = \frac{40}{3}\left(x - \frac{\pi}{3}\right)$.

Ex. (A) Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$.

(B) For what values of x does the graph of f have a horizontal tangent?

Soln. (A) $f'(x) = \dfrac{\sec x(\tan x - 1)}{(1 + \tan x)^2}$. Exc.

(B) This occurs when $f'(x) = 0$.

Since $\sec x$ is never 0, we see that $f'(x) = 0$ when $\tan x = 1$, i.e., when $x = n\pi + \pi/4$ (n is an integer).

Ex. Let $f(x) = \sin(\cos(\tan x))$. Find $f'(x)$.

Soln. $f'(x) = \cos(\cos(\tan x)) \frac{d}{dx} \cos(\tan x)$

$= \cos(\cos(\tan x)) \left[-\sin(\tan x)\right] \frac{d}{dx}(\tan x)$

$= -\cos(\cos(\tan x)) \sin(\tan x) \sec^2 x$

Ex. Find y' if $y = \sin^2\left(\cos\sqrt{\sin 5x}\right)$.

Soln. $y' = 2\sin\left(\cos\sqrt{\sin 5x}\right)\left(-\sin\sqrt{\sin 5x}\right)$

$* \frac{1}{2}(\sin 5x)^{-1/2} * \cos 5x * 5.$

Ex. Let $f(x) = \sin 3x$. Find $f^{(39)}(\pi)$.

Soln. $f'(x) = 3\cos 3x$

$f''(x) = -3^2 \sin 3x$

$f'''(x) = -3^3 \cos 3x$

$f^{(4)}(x) = 3^4 \sin 3x$

$f^{(5)}(x) = 3^5 \cos 3x$

$\Rightarrow$ $f^{(37)}(x) = 3^{37} \cos 3x$

$f^{(38)}(x) = -3^{38} \sin 3x$

$f^{(39)}(x) = -3^{39} \cos 3x.$

The successive derivatives occur in a cycle of length 4

1	5	9	*	*	–	–	–	*	37
2	6	10	*	*	–	–	–	*	38
3	7	11	15	19	23	27	31	35	39
4	8	12	*	*	–	–	–	*	40

So $f^{(39)}(\pi) = -3^{39}\cos(3\pi) = 3^{39}$.

Derivatives of exponential and logarithmic functions

Facts: 1) $\frac{d}{dx} e^x = e^x$. 2) $\frac{d}{dx} \ln x = \frac{1}{x}$.

3) $\frac{d}{dx} e^{f(x)} = f'(x)\, e^{f(x)}$. 4) $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$.

Ex. At what point on the curve $y = e^x$ is the tangent line parellel to the line $y = 2x$?

Soln. Because $y' = e^x$, the slope of the tangent line at $x = a$ is e^a.

This tangent line is parallel to the line $y = 2x$ if they have the same slope, i.e. if $e^a = 2$.

Then $a = \ln e^a = \ln 2$.

Thus, the point is $(a, y(a)) = (\ln 2, 2)$.

Ex. $f(t) = \tan(e^t) + e^{\tan t}$.

$$f'(t) = e^t \sec^2 e^t + \sec^2 t \, e^{\tan t}.$$

Ex. $\frac{d}{dx} \ln \sin x = \frac{\cos x}{\sin x} = \cot x$.

Ex. $\frac{d}{dx} \ln \sqrt{x} = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}} = \frac{1}{2x}$.

or $\frac{d}{dx} \ln x^{\frac{1}{2}} = \frac{1}{2} \frac{d}{dx} \ln x = \frac{1}{2} \cdot \frac{1}{x}$.

Ex. $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left(\ln(x+1) - \frac{1}{2}\ln(x-2)\right)$

$$= \frac{1}{x+1} - \left(\frac{1}{2}\right)\frac{1}{x-2}.$$

Facts: 1) $\frac{d}{dx} a^x = a^x \ln a$. 2) $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$.

3) $\frac{d}{dx} a^{f(x)} = f'(x) a^{f(x)} \ln a$. 4) $\frac{d}{dx} \log_a f(x) = \frac{f'(x)}{f(x) \ln a}$.

here $a \neq 1$.

Ex. $\frac{d}{dx} 2^{x^2} = 2^{x^2} \ln 2 \, (2x) = 2^{x^2+1} (\ln 2) x$.

Ex. $\frac{d}{dx} (\ln x)^4 = 4 (\ln x)^3 \frac{d}{dx} \ln x = \frac{4}{x} (\ln x)^3$.

Ex. $\frac{d}{dx} \ln (\ln x) = \frac{\frac{d}{dx} \ln x}{\ln x} = \frac{1}{x \ln x}$.

Ex. $\frac{d}{dx} e^{e^x} = e^{e^x} e^x = e^{(e^x + x)}$.

Ex. $\frac{d}{dx} (x e^{-x}) = x e^{-x} (-1) + e^{-x} (-1) = e^{-x} (1-x)$.

Ex. Let $f(x) = x^x$ for $x > 0$. Find $f'(x)$.

Soln. $\ln f(x) = \ln x^x = x \ln x$, then

$$\frac{f'(x)}{f(x)} = \frac{d}{dx} \ln f(x) = \frac{d}{dx} (x \ln x) = \ln x + 1.$$

Then $f'(x) = f(x) (\ln x + 1) = x^x (\ln x + 1)$.

Exc. Find $\frac{d}{dx} x^{\sqrt{x}}$, for $x > 0$.

Ex. Find $\frac{dy}{dx}$ if $y = (\ln x)^{\tan x}$.

Soln. Let $\ln y = \tan x \ln(\ln x)$.

Then $\frac{1}{y}\frac{dy}{dx} = \tan x * \frac{\frac{1}{x}}{\ln x} + \ln(\ln x) * \sec^2 x$

Thus $\frac{dy}{dx} = (\ln x)^{\tan x}\left[\frac{\tan x}{x \ln x} + \sec^2 x \ln(\ln x)\right]$.

Ex. Find $f'(x)$ for $f(x) = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Soln. $\ln f(x) = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)$

$$= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2).$$

Then $\frac{f'(x)}{f(x)} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x}{x^2+1} - 5\cdot\frac{3}{3x+2}$.

Thus, $f'(x) = f(x)\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$

$$= \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right].$$

Ex. Prove that $\lim_{x \to 0} \left(1+\frac{1}{x}\right)^{\frac{1}{x}} = e$.

Proof: Let $f(x) = \ln x$, then

$f'(x) = \frac{1}{x}$. So $f'(1) = 1$.

On the other hand, using definition, we have

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1+x) - \underbrace{\ln(1)}_{0}}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1+x)}{x}$$

$$= \lim_{x \to 0} \ln(1+x)^{1/x}.$$

Thus, $\lim_{x \to 0} \left(1+\frac{1}{x}\right)^{1/x} = 1$. It follows that

$$\lim_{x \to 0} (1+x)^{1/x} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}}$$

$$= e^{\lim_{x \to 0} \ln(1+x)^{1/x}}$$

$$= e^{1}. \text{ Done!}$$

Searching keywords:

- Chain rule قاعدة السلسلة
- Derivatives of trigonometric functions مشتقات الاقترانات المثلثية
- Derivatives of exponential and logarithmic functions مشتقات الاقترانات الأسية واللوغاريتمية
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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