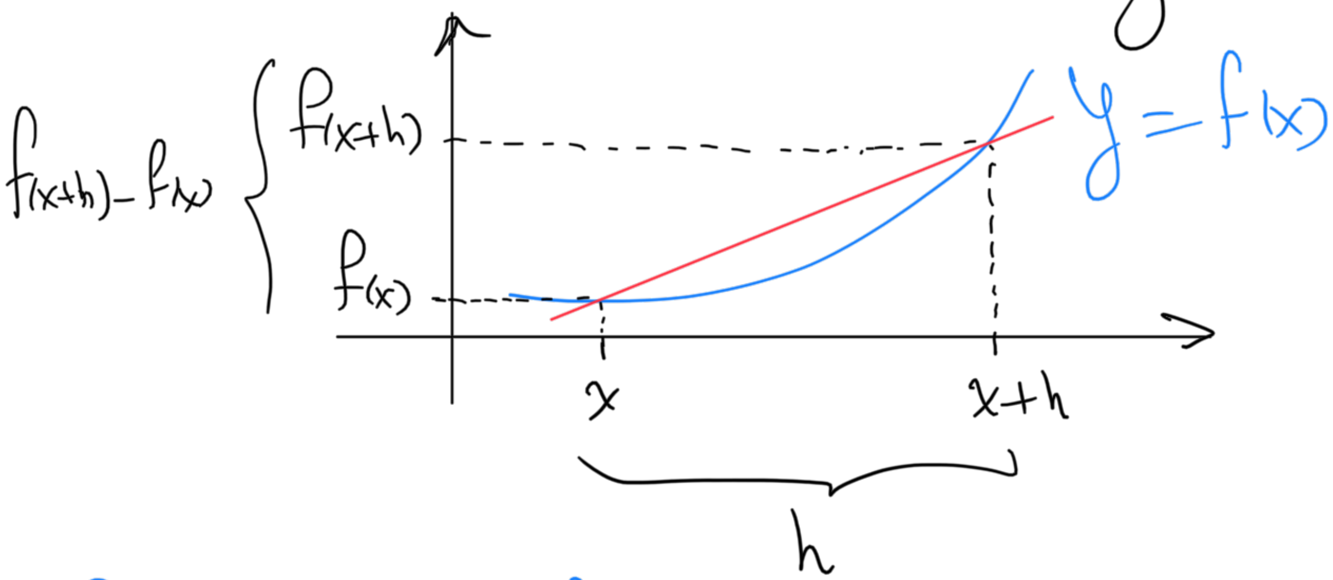
Differentiation.

The Derivative.

Def. A func. f is said to be differentiable (diff.) at x if $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists.

If this limit exists, it is called the derivative of f at x and is denoted by $f'(x)$.



Geometrically, $f'(x)$ is the slope of the graph at the point $(x, f(x))$. The line that passes through the point $(x, f(x))$ with slope $f'(x)$ is called the tangent line to the graph of f at $(x, f(x))$.

Alternative form for the derivative

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}.$$

Ex. Let $f(x) = 3x+1$. Compute $f'(1)$.

Soln $f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0} \frac{3(\cancel{1}+h) + \cancel{1} - (\cancel{3} + \cancel{1})}{h}$$

$$= \lim_{h \to 0} \frac{3\cancel{h}}{\cancel{h}} = 3.$$

Ex. (*) Let $f(x) = 3x^2+1$. Find $f'(x)$.

Soln. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{3(x+h)^2 + 1 - (3x^2+1)}{h}$$

$$= \lim_{h \to 0} \frac{3(\cancel{x^2} + 2xh + h^2) + \cancel{1} - \cancel{3x^2} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{3(2x\cancel{h} + h^{\cancel{2}})}{\cancel{h}}$$

$$f'(x) = \lim_{h \to 0} (6x + h) = 6x.$$

For instance, $f'(1) = f'(x)\big|_{x=1} = 6(1) = 6.$

Ex. Find $f'(x)$ for the function

(1) $f(x) = \sqrt{x}$, $x \geq 0$.

(2) $f(x) = 1/x$, $x \neq 0$.

Soln. (1) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(\cancel{x} + \overset{1}{\cancel{h}}) - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}.$$

(2) Exc.

Ex. Decide whether or not the following func. is differentiable at the given point.

(1) $f(x) = \begin{cases} 4 & \text{if } x<2 \\ 2x & \text{if } x \geq 2 \end{cases}$ at $x=2$.

Soln. $\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{2(2+h)-4}{h}$

$= \lim_{h \to 0^+} \frac{\cancel{4} + 2\cancel{h} - \cancel{4}}{\cancel{h}} = 2.$

$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{4-4}{0} = 0.$

$\therefore f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$ DNE

Thus, f is not differentiable at $x=2$.

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(2) $f(x) = \begin{cases} x^2 & , x \leq 1 \\ 2x-1 & , x>1 \end{cases}$ at $x=1$ Exc.

Thm. If $f(x)$ is differentiable at $x=a$, then $f(x)$ is continuous at $x=a$.

Remark: The converse of the above thm is not true in general.
For example the func.

$$f(x)=\begin{cases} 4 & \text{if } x<2, \\ 2x & \text{if } x\geq 2. \end{cases}$$

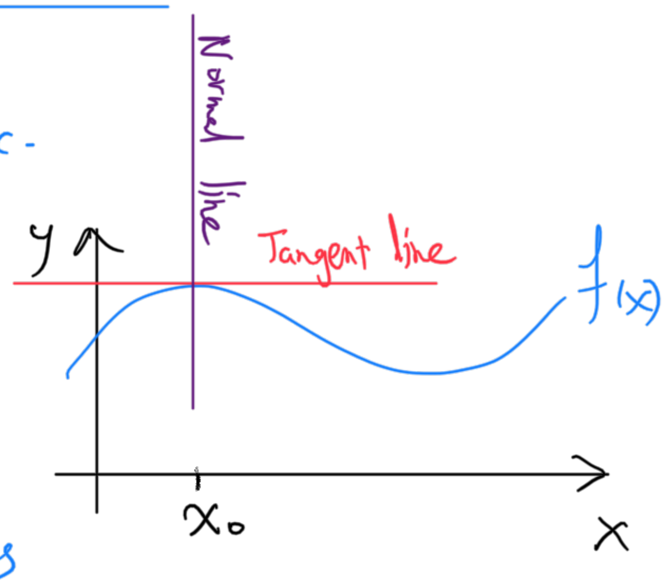
(given in the above Ex. item (1))

is continuous at $x=2$, but we have seen from the above example that $f(x)$ is not diff. at $x=2$.

Remark: If $f(x)$ is not continuous at $x=a$, then f is not diff. at $x=a$.

Tangent lines and normal lines

Def. Let f be a diff. func. at $x = x_0$. Then



(1) The tangent line at the point $(x_0, f(x_0))$ has slope $f'(x_0)$ and has the equation

$$\boxed{y - f(x_0) = f'(x_0)(x - x_0)}.$$

(2) The line through $(x_0, f(x_0))$ that is perpendicular to the tangent line is called the normal line and has slope $-1/f'(x_0)$ and has the equation

$$\boxed{y - f(x_0) = \frac{-1}{f'(x_0)}(x - x_0)}.$$

Ex. Find equations for the tangent and normal lines to the graph of $f(x) = 3x^2 + 1$ at the point $(1, f(1)) = (1, 4)$.

Soln. From Example (*), we have

$$f'(x) = 6x \text{ , then } f'(1) = 6 .$$

- The equation of the tangent line is

$$y - f(1) = f'(1)(x-1) ,$$

then $y - 4 = 6(x-1)$.

Thus $y = 6x - 2$.

- The equation of the normal line is

$$y - f(1) = \frac{-1}{f'(1)}(x-1) ,$$

then $y - 4 = \frac{-1}{6}(x-1)$.

Thus, $y = \frac{1}{6}(25 - x)$.

Some differentiation formulas

1) $f(x) = K \Rightarrow f'(x) = 0$. ($K$: any constant)

2) $f(x) = x \Rightarrow f'(x) = 1$.

3) $f(x) = x^r \Rightarrow f'(x) = r x^{r-1}$ (r: any real number)

Examples:

1) If $f(x) = x^{1010}$, then $f'(x) = 1010\, x^{1009}$.

2) If $f(x) = x^{\pi}$, then $f'(x) = \pi\, x^{\pi - 1}$.

3) If $f(x) = \frac{1}{x} = x^{-1}$, then $f'(x) = -x^{-2} = -1/x^2$.

4) If $f(x) = \sqrt{x} = x^{1/2}$, then $f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

5) If $f(x) = \sqrt[3]{x^2} = x^{3/2}$, then $f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$.

Thm. If f and g are diff. at x, then the functions $f \pm g$, kf (k: a constant), $f \cdot g$, $1/f$, and f/g are diff. at x. Moreover,

1) $(f \pm g)'(x) = f'(x) \pm g'(x)$.

2) $(kf)'(x) = k\, f'(x)$.

3) $(f \cdot g)'(x) = f(x)\, g'(x) + f'(x)\, g(x)$.

4) $\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f(x))^2}$, provided $f(x) \neq 0$.

5) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{(g(x))^2}$, provided $g(x) \neq 0$.

Proof. We prove item (3).

$$(f\cdot g)'(x) = \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + g(x) \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= f(x) \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} + g(x) \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= f(x)\,g'(x) + g(x)\,f'(x).$$ Done

as $h \to 0$, $x+h \to x$. Since f is diff., f is cts

thm $\Rightarrow f(x+h) \to f(x)$.

Notation. Given $y = f(x)$, then

$$f'(x) = y' = \underbrace{\frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x)}_{\text{Leibniz notation}}.$$

The expression $\frac{d}{dx}$ is called a <u>differential operator</u>.

Ex. Find the derivative.

1) $y = 2x^6 + 3\sqrt{x}$.

Soln. $\frac{d}{dx}\left(2x^6 + 3\sqrt{x}\right) = 2\frac{d}{dx}\left(x^6\right) + 3\frac{d}{dx}\left(\sqrt{x}\right)$

$= 12x^5 + \frac{3}{2\sqrt{x}}$.

2) $y = \frac{4x^2 - 3x + 2\sqrt{x}}{x} = \frac{4x^2}{x} - \frac{3x}{x} + \frac{2\sqrt{x}}{x}$

$= 4x - 3 + 2x^{-1/2}$

Then $y' = 4 - 0 + 2\left(-\frac{1}{2}\right)x^{-3/2} = 4 - x^{-3/2}$.

3) $f(x) = \sqrt{x}\left(x^2 + x\right) = x^{5/2} + x^{3/2}$

Soln. $\frac{d}{dx}f(x) = \frac{d}{dx}\left(x^{5/2} + x^{3/2}\right)$

$= \frac{5}{2}x^{3/2} + \frac{3}{2}x^{1/2} = \frac{\sqrt{x}}{2}\left(5x+3\right)$.

4) $y = \left(x^4 - 3x^2 + 3x\right)\left(x^3 - 2x + 3\right)$.

Soln. $y' = \left(4x^3 - 6x + 3\right)\left(x^3 - 2x + 3\right)$
$+ \left(x^4 - 3x^2 + 2x\right)\left(3x^2 - 2\right)$

5) $g(x) = \dfrac{x^2-2}{x^2+1}$.

$\underline{\text{Soln}}$ $g'(x) = \dfrac{(x^2-2)'(x^2+1) - (x^2-2)(x^2+1)'}{(x^2+1)^2}$

$$= \frac{2x(x^2+1) - (x^2-2)(2x)}{(x^2+1)^2} = \frac{6x}{(x^2+1)^2}.$$

6) $y = (x^2-1)\dfrac{(x^3+3x^2)}{(x^2+2)}$ Exc.

$\underline{\text{Fact}}$: $(f \cdot g \cdot h)' = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$.

$\underline{\text{Ex.}}$ Let $y = x^{3/2}(x^2-2)(x^3-x+1)$. Find y'.

$\underline{\text{Soln.}}$ $\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}}(x^2-2)(x^3-x+1)$

$$+ x^{\frac{3}{2}}(2x)(x^3-x+1)$$

$$+ x^{3/2}(x^2-2)(3x^2-1).$$

$\underline{\text{Ex.}}$ Assume that f and g are diff. with $f(1) = -2$, $f'(1) = 3$, $g(0) = 3$, and $g'(0) = -1$.

(A) Find an equation of the tangent line to the graph of the func. $h(x) = x^2 f(x)$ at $x = 1$.

(B) Find an equation of the tangent line to the graph of $q(x) = x^2/g(x)$ at $x = 0$.

Soln.

(A) $h(1) = (1)^2 f(1) = -2$.

$h'(x) = x^2 f'(x) + 2x f(x)$, then

$h'(1) = (1)^2 (3) + (2)(-2) = -1$.

The equation of the tangent line is

$$y - h(1) = h'(1)(x-1),$$

then $y - (-2) = (-1)(x-1)$.

Thus, $y = -x - 1$.

(B) Exc.

Fact: If $f(x) = f_1(x) f_2(x) \cdots f_n(x)$

then $f'(x) = f(x)\left[\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}\right]$.

Ex. Let $g(x) = \dfrac{x(x-2)(x-3)}{(x-1)^2}$. Find $g'(x)$.

Soln. $g(x) = x(x-2)(x-3)(x-1)^{-2}$. Then

$$g'(x) = g(x)\left[\frac{1}{x} + \frac{1}{x-2} + \frac{1}{x-3} + \frac{-2(x-1)^{-3}}{(x-1)^{-2}}\right]$$

$$= \frac{x(x-2)(x-3)}{(x-1)^2}\left[\frac{1}{x} + \frac{1}{x-2} + \frac{1}{x-3} + \frac{2}{x-1}\right].$$

Higher-order derivatives.

Let $y = f(x) = 3x^4 - 2x^2 + 1$.

Order	Prime notation	Leibniz notation	Example
1	$y' = f'(x)$	$\frac{df}{dx}$	$12x^3 - 4x$
2	$y'' = f''(x)$	$\frac{d^2f}{dx^2}$	$36x^2 - 4$
3	$y''' = f'''(x)$	$\frac{d^3x}{df^3}$	$72x$
4	$y^{(4)} = f^{(4)}(x)$	$\frac{d^4f}{dx^4}$	72
5	$y^{(5)} = f^{(5)}(x)$	$\frac{d^5f}{dx^5}$	0

Note that $f^{(n)}(x) = \frac{d^n}{dx^n} f(x) = 0$, for $n \geq 5$.

Ex. $y = \frac{1}{x} = x^{-1}$,

$y' = -x^{-2}$,

$y'' = 2x^{-3}$,

$y''' = -6x^{-4}$,

$y^{(4)} = 24x^{-5}$,

$\vdots$

$y^{(n)} = (-1)^n n! \, x^{n-1}$.

Recall that

$n! = n(n-1)(n-2)\cdots 3.2.1$

$4! = 4.3.2.1 = 24$

$3! = 3.2.1 = 6$

$2! = 2.1 = 2$

$1! = 1$

$0! = 1$

Ex. $\frac{d}{dx}\left[x^2 \frac{d^2}{dx^2}(1+x^2)\right] = \frac{d}{dx}(2x^2) = 4x.$

$= \frac{d}{dx}(2x)$

$= 2.$

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- Differentiation, derivative الاشتقاق، المشتقة
- Tangent line, normal line معادلة المماس ومعادلة العمودي على المماس
- The University of Jordan الجامعة الأردنية
- Calculus I 1 تفاضل وتكامل
- Baha Alzalg بهاء الزالق

References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For any comments or concerns, please use my email to contact me.

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