# CALCULUS I

- Functions (Lectures 1–7).
- Limits (Lectures 8, 10 and 17).
- Continuity (Lecture 9).
- Differentiation (Lectures 11-16).
- Applications of Differentiation (Lectures 18–21).
- Integrals (Lectures 22–25).

## FUNCTIONS (Lectures 1–7) ← Chapter 1

## 1 Introduction

i. Natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$.

ii. Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.

iii. Rational numbers: $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$.
For example, $2 = \frac{2}{1}$ and $3 = \frac{3}{1}$ are rational numbers, while $\sqrt{2}$ and $\pi$ are irrational numbers.

iv. Real numbers: $\mathbb{R} = (-\infty, \infty)$.

**Definition 1.1.** A function (func.) $f$ is a rule that assigns to each element $x$ in a set $D$ exactly one element, called $f(x)$, called the image of $x$, in a set $R$.

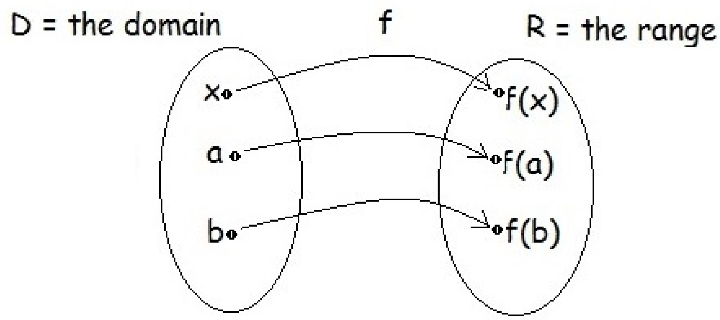



Figure 1: A function $f : D \longrightarrow R$.

**Example 1.1.** *Sketch the graph of the function:*
*(a)* $f(x) = 2x + 1$. *(b)* $g(x) = x^2 + 1$.

**Solution.**

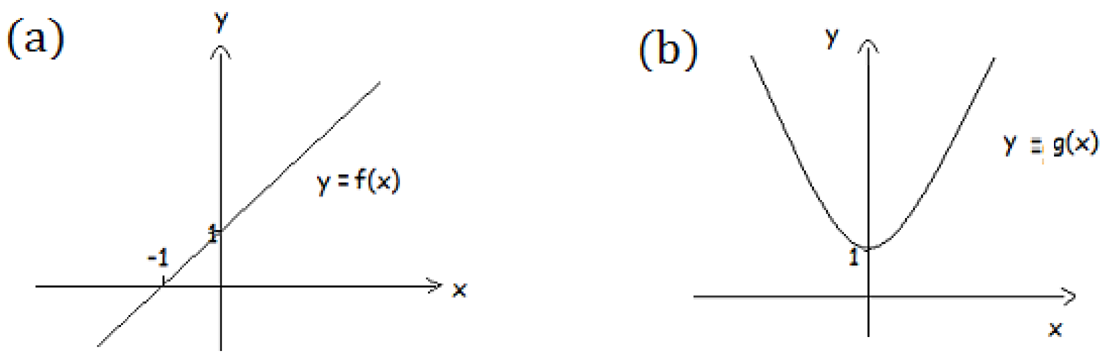



Figure 2: The graphs of $f(x) = 2x + 1$ and $g(x) = x^2 + 1$.

## The vertical line test

**Test 1.1.** *A curve in the $xy-$plane is the graph of a function of $x$ if and only if no vertical line intersects the curve more than once.*

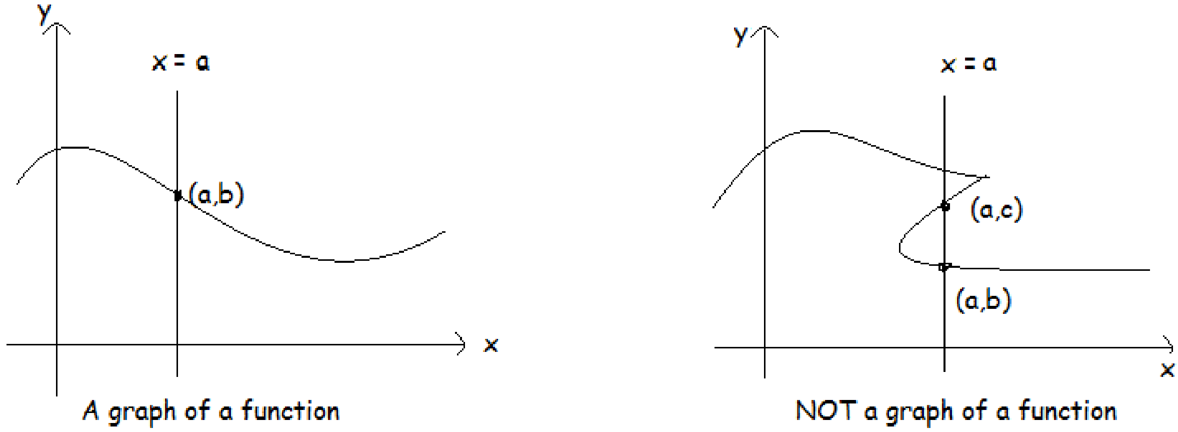



Figure 3: Using the vertical line test to test if a graph is a function.

**Definition 1.2.** Piecewise defined functions are functions defined by different formulas in different parts of their domains.

**Example 1.2.** *Let $f$ be a function defined by*

$$f(x) = \begin{cases} x^2 & \textit{if } x < 0, \\ x + 1 & \textit{if } x \geq 0. \end{cases}$$

*Evaluate $f(-1)$, $f(0)$, and $f(1)$, and then sketch the graph of $f$.*

**Solution.** $f(-1) = (-1)^2 = 1, \quad f(0) = 0{+}1 = 1, \quad f(1) = 1{+}1 = 2.$

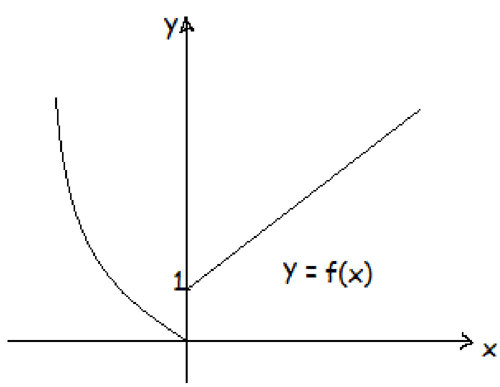



Figure 4: The graph of $f(x)$ given in Example 1.2.

## 2 Shifting, Reflecting, and Stretching Graphs

### Graph shifting

**Fact 2.1.** *Suppose $c > 0$, then to obtain the graph of*

i) $f(x) + c$, *shift the graph of $f(x)$ $c$ units up.*

ii) $f(x) - c$, *shift the graph of $f(x)$ $c$ units down.*

iii) $f(x + c)$, *shift the graph of $f(x)$ $c$ units to the left.*

iv) $f(x - c)$, *shift the graph of $f(x)$ $c$ units to the right.*

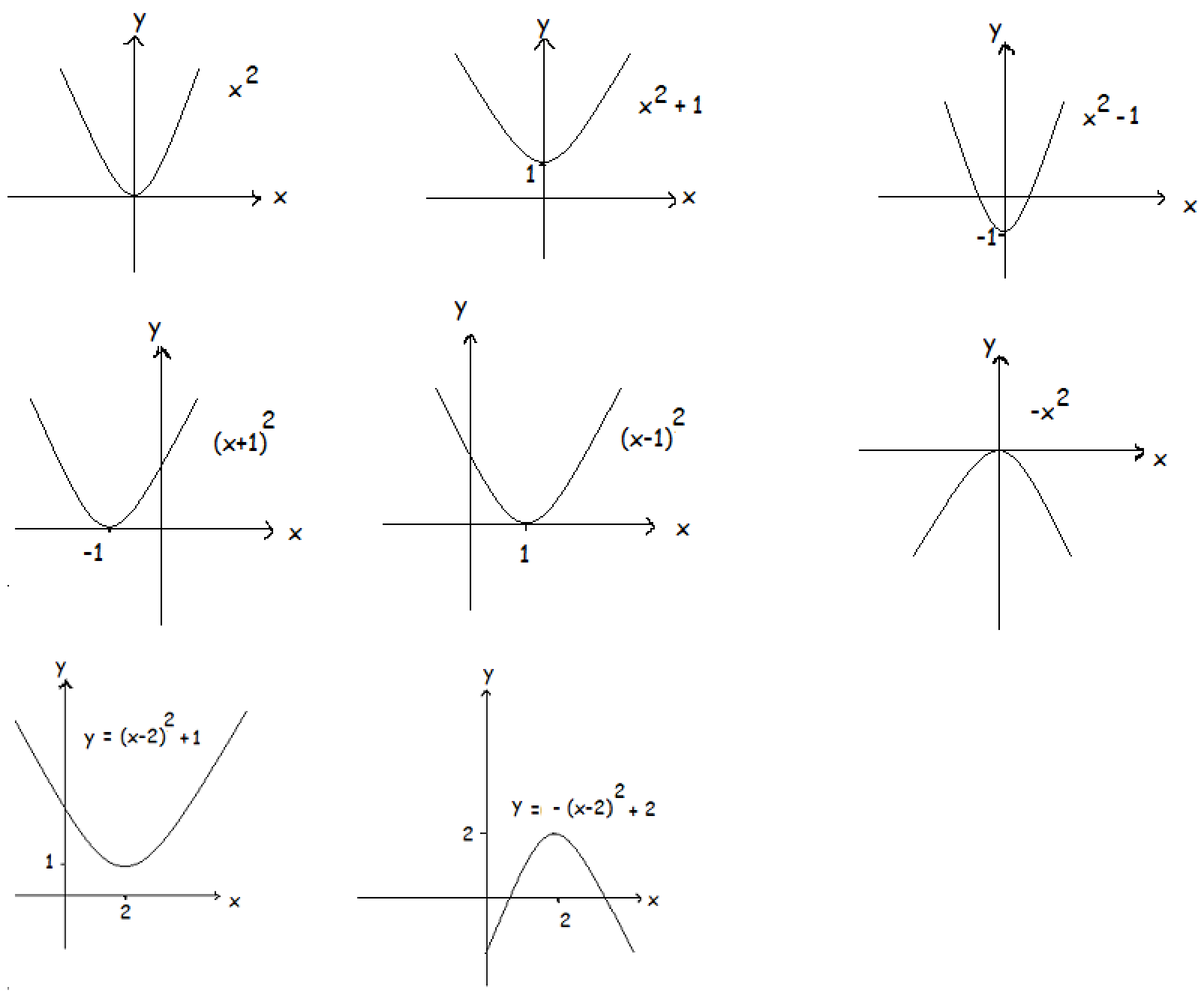



Figure 5: Applying Fact 2.1 to the graph of the function $y = x^2$.

**Example 2.1.** *Explain how the graph of $f(x) = x^2 - 3x + 7$ is obtained from the graph of $g(x) = x^2$.*

**Solution.** Notice that $f(x) = (x - \frac{3}{2})^2 + \frac{19}{4}$. This means that we have two steps:

Step 1: Shift the graph of $g(x)$ $\frac{3}{2}$ units right, then
Step 2: Shift the resulting graph in Step 1 $\frac{19}{4}$ units up.

## Graph reflecting

**Fact 2.2.** *To obtain the graph of*

*i) $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.*

*ii) $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.*

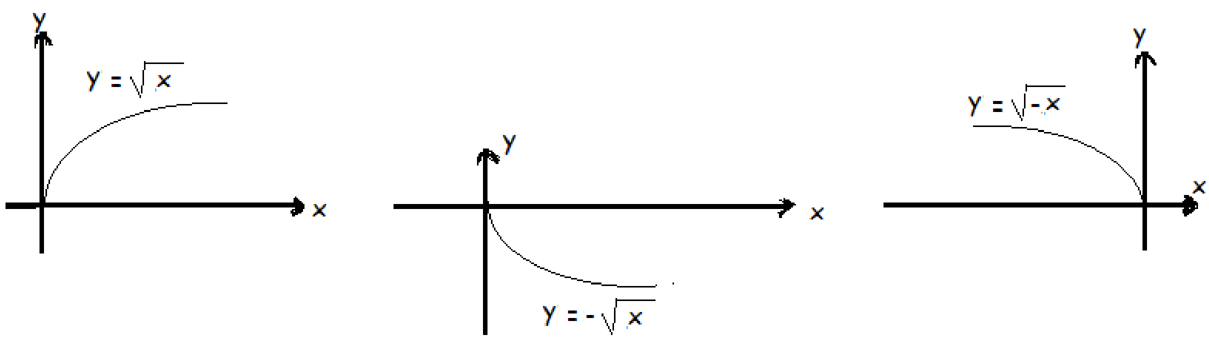



Figure 6: Applying Fact 2.2 to the graph of the function $y = \sqrt{x}$.

**Example 2.2.** *Let $g(x) = x^2 + 3x$. If $g(x)$ is shifted 2 units right and 4 units down, then reflected about the $y-axis$, we obtain a new function, say $h(x)$, which is given by (circle the correct answer):*

$(a)$ $h(x) = x^2 + 7x + 6.$ $(b)$ $h(x) = x^2 + x - 6.$

$(c)$ $h(x) = x^2 - x - 6.$ $(d)$ $h(x) = x^2 - 7x + 6.$

**Solution.** Note that

- shifting 2 units right, yields

$$g_1(x) = g(x-2) = (x-2)^2 + 3(x-2), \quad \text{then}$$

- shifting 4 units down, yields

$$g_2(x) = g_1(x) - 4 = (x-2)^2 + 3(x-2) - 4 = x^2 - x - 6, \quad \text{then}$$

- reflecting about the $y-axis$, yields

$$h(x) = g_2(-x) = (-x)^2 - (-x) - 6 = x^2 + x - 6.$$

Thus, the correct answer is (b).

## Graph stretching and shrinking

**Fact 2.3.** *Let $c > 1$ be a constant. To obtain the graph of*

*i) $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of $c$.*

*ii) $y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of $c$.*

*iii) $y = \frac{1}{c}f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of $c$.*

*iv) $y = f\left(\frac{1}{c}x\right)$, stretch the graph of $y = f(x)$ horizontally by a factor of $c$.*

**The trigonometric function $y = \cos x$ will be introduced and studied in details later**. The graph of this function is shown below. We also show the (vertical and horizontal) shrinking and stretching of the graph of this function.

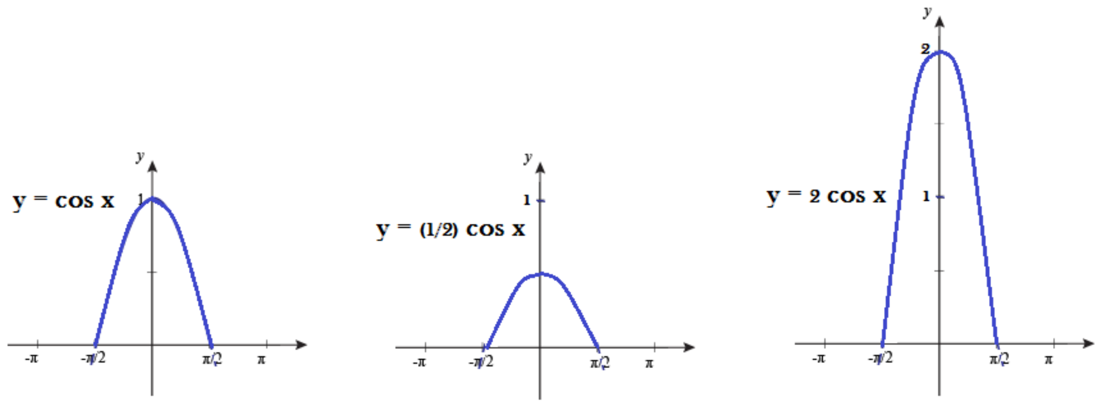



Figure 7: Vertical shrinking and stretching the graph of $y = \cos x$.

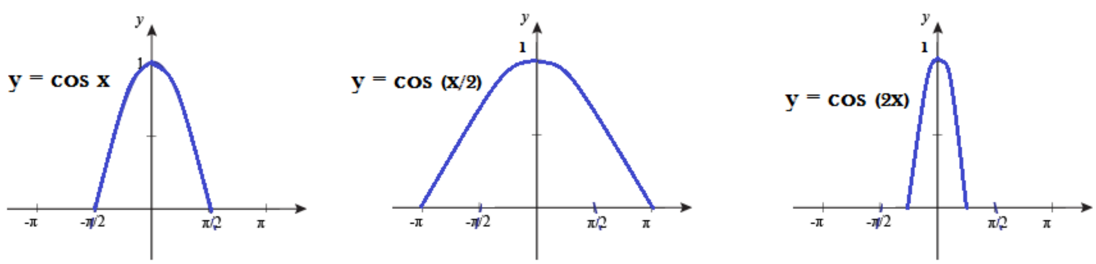
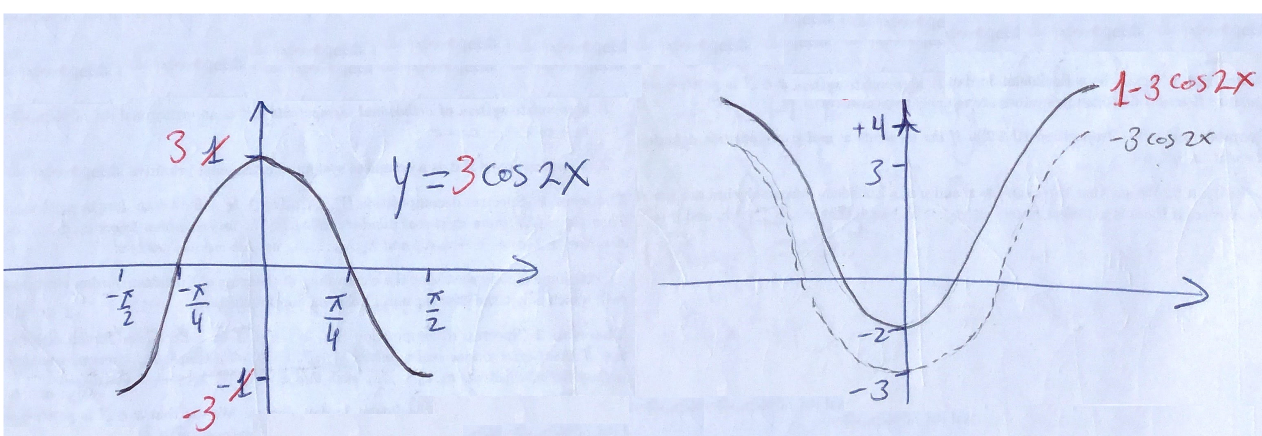



Figure 8: Horizontal stretching and shrinking the graph of $y = \cos x$.

**Example 2.3.** *Use the graph of* $y = \cos x$ *to sketch the graph of* $y = 1 - 3\cos 2x$*, where* $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

**Solution.** Vertical stretching, reflecting, and shifting the graph of $y = \cos x$ yield the graph of $y = 1 - 3\cos 2x$ as follow.



## 3 Functions and domains

Recall that: For a function $f : D \to R$, it is defined that

- $D_f$ (or simply $D$): is the domain of the function $f$.
- $R_f$ (or simply $R$): is the range of the function $f$; $R_f = \left\{f(x) : x \in D_f\right\}$, i.e., the image of D under $f$.

## Function types and domains

(1) **Polynomials.**

**Definition 3.1.** Let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$. A function of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is called a polynomial (poly.) of degree $n$.

**Example 3.1.** *$P(x) = x^5 + 3x + 1$ is a polynomial of degree 5, but the func. $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ is NOT a polynomial as $\frac{1}{2} \notin \mathbb{N}$.*

**Fact 3.1.** *D(Any polynomial)$= \mathbb{R} = (-\infty, \infty)$.*

**Example 3.2.** *$D(x^3 + x^2 + 2) = \mathbb{R}$.*

(2) **Rational functions.**

**Definition 3.2.** Let $P(x)$ and $Q(x)$ be polys where $Q(x)$ is not zero. A function of the form $R(x) = \frac{P(x)}{Q(x)}$ is called a rational function.

**Example 3.3.** *$R_1(x) = \frac{x+1}{7x^3+5}$ and $R_2(x) = \frac{1}{x}$ are rational funcs, but the func. $f(x) = \frac{x}{\sqrt{x}+1}$ is NOT rational (Why?).*

**Fact 3.2.** *D(Any rational) $= D\left(\frac{P(x)}{Q(x)}\right) = \{x \in \mathbb{R} : Q(x) \neq 0\}$.*

**Example 3.4.** $D\left(\frac{x^3}{x^2-1}\right) = \mathbb{R} - \{\pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. *And* $D\left(\frac{x^3}{x^2+1}\right) = \mathbb{R}$.

(3) **Root functions.**

**Definition 3.3.** Let $g(x)$ be a polynomial and $n \in \mathbb{N}$. A function of the form $f(x) = \sqrt[n]{g(x)} = (g(x))^{\frac{1}{n}}$ is called a root function.

**Fact 3.3.** *D(Any root func.) is as follows:*

$$D\left(\sqrt[n]{g(x)}\right) = \begin{cases} \mathbb{R} & , \text{ if } n \text{ is odd,} \\ \{x \in \mathbb{R} : g(x) \geq 0\} & , \text{ if } n \text{ is even.} \end{cases}$$

**Example 3.5.** *According to Fact* 3.3*, we have*

*(1)* $D\left(\sqrt[3]{x}\right) = \mathbb{R}$.

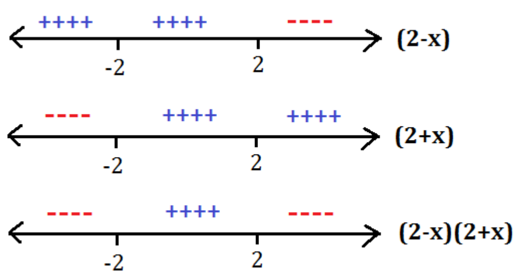
*(2)* $D\left(\sqrt{x}\right) = [0, \infty)$.

*(3)* $D\left(\sqrt{x+1}\right) = [-1, \infty)$.

*(4)* $D\left(\sqrt[4]{x^2-1}\right) = (-\infty, -1] \cup [1, \infty)$
$(x^2 - 1 \geq 0 \implies x^2 \geq 1 \implies x > 1 \text{ or } x < 1)$.

**Example 3.6.** *Let* $f(x) = \frac{1}{\sqrt{4-x^2}}$. *Find* $D_f$.

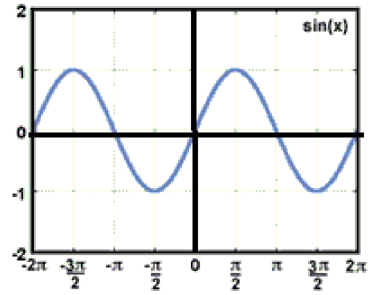
**Solution.** We look for $x$ that satisfies $4 - x^2 > 0$, or equivalently $(2-x)(2+x) > 0$.




This implies that $-2 < x < 2$. Therefore, $D_f = (-2, 2)$.

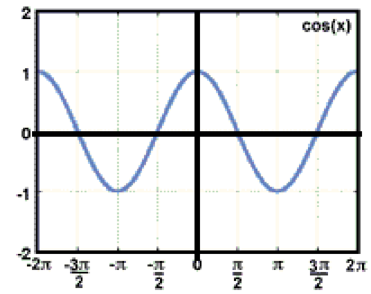
## (4) **Trigonometric funcs.**

(a) The sine function:
$y = \sin x$,
$D = \mathbb{R}$,
$R = [-1, 1]$.



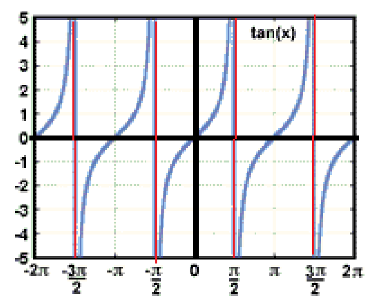

The graph of $y = \sin x$.

(b) The cosine function:
$y = \cos x$,
$D = \mathbb{R}$,
$R = [-1, 1]$.



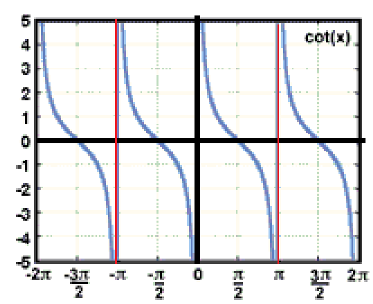

The graph of $y = \cos x$.

(c) The tangent function:
$y = \tan x = \frac{\sin x}{\cos x}$,
$D = \{x \in \mathbb{R} : \cos x \neq 0\}$
$= \left\{x \in \mathbb{R} : x \neq \frac{\pi}{2} + n\pi; n \in \mathbb{Z}\right\}$,
$R = \mathbb{R}$.



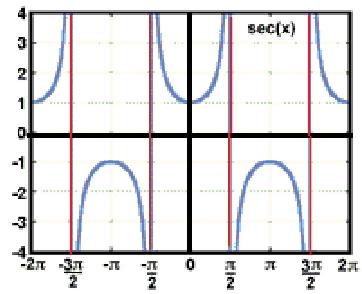

The graph of $y = \tan x$.

(d) The cotangent func.:
$y = \cot x = \frac{\cos x}{\sin x}$,
$D = \{x \in \mathbb{R} : \sin x \neq 0\}$
$= \{x \in \mathbb{R} : x \neq n\pi; n \in \mathbb{Z}\}$,
$R = \mathbb{R}$.



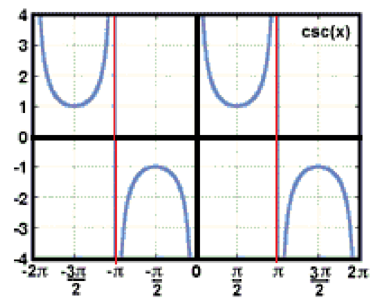

The graph of $y = \cot x$.

(e) The secant function:
$y = \sec x = \frac{1}{\cos x}$,
$D = \{x \in \mathbb{R} : \cos x \neq 0\}$
$= \left\{x \in \mathbb{R} : x \neq \frac{\pi}{2} + n\pi; n \in \mathbb{Z}\right\}$,
$R = (-\infty, -1] \cup [1, \infty)$.




The graph of $y = \sec x$.

(f) The cosecant function:
$y = \csc x = \frac{1}{\sin x}$,
$D = \{x \in \mathbb{R} : \sin x \neq 0\}$
$= \{x \in \mathbb{R} : x \neq n\pi; n \in \mathbb{Z}\}$,
$R = (-\infty, -1] \cup [1, \infty)$.




The graph of $y = \csc x$.

**Example 3.7.** *Find the domain and the range of the following functions:*

*1)* $f(x) = 2\cos x$. *2)* $g(x) = 1 - 3\sin(\frac{x}{2})$. *3)* $h(x) = \sqrt{1-x} - 1$.

**Solution.**

1) $D_f = \mathbb{R}$.

Note that $-1 \leq \cos x \leq 1$, hence $-2 \leq 2\cos x \leq 2$.

Therefore $R_f = [-2, 2]$.

2) $D_g = \mathbb{R}$.

Note that $-1 \leq \sin(\frac{x}{2}) \leq 1$, so $-3 \leq -3\sin(\frac{x}{2}) \leq 3$,

and hence $-2 \leq 1 - 3\sin(\frac{x}{2}) \leq 4$. Therefore $R_g = [-2, 4]$.

3) Note that $1 - x \geq 0 \implies 1 \geq x \implies x \in (-\infty, 1]$.

Thus, $D_h = (-\infty, 1]$.

Note also that $\sqrt{1-x} \geq 0 \implies h(x) = \sqrt{1-x} - 1 \geq -1$.

Therefore, $R_h = [-1, \infty)$.

**Example 3.8.** *Let* $f(x) = \sqrt{2x+4}$*, where* $x \in [0, 6]$*. Find* $R_f$*.*

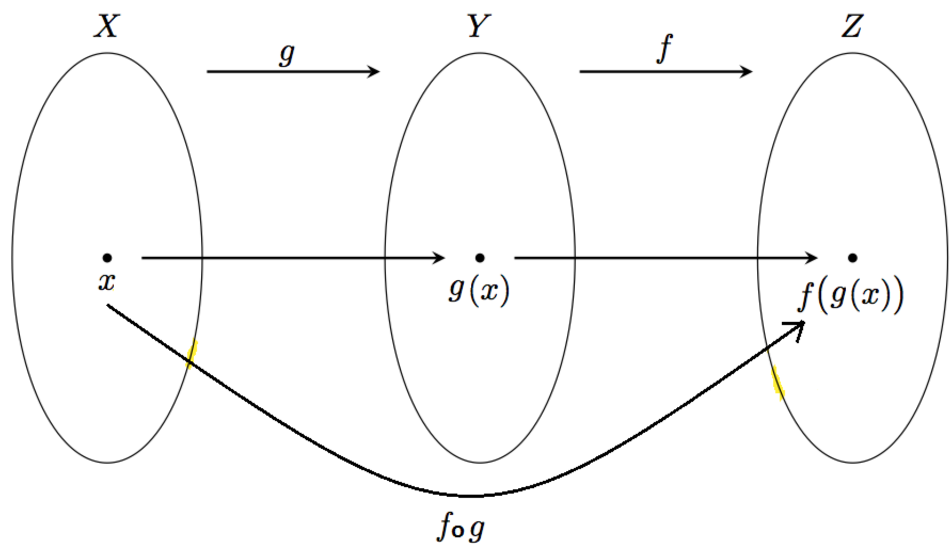
**Solution.** Note that

$$\begin{aligned} x \in [0,6] &\implies 0 \leq x \leq 6 \\ &\implies 0 \leq 2x \leq 12 \\ &\implies 4 \leq 2x + 4 \leq 16 \\ &\implies 2 \leq f(x) = \sqrt{2x+4} \leq 4. \end{aligned}$$

Therefore, $R_f = [2, 4]$.

# 4 Composition; Odd and even functions

## Composition of functions




**Definition 4.1.** Let $f$ and $g$ be two functions. The composition of $f$ and $g$, denoted by $f \circ g$, is the function defined on the set $\left\{x \in D_g : g(x) \in D_f\right\}$ by $(f \circ g)(x) = f(g(x))$.

**Example 4.1.** *Let $g(x) = x^2$ and $f(x) = x + 3$.*
*Find $(f \circ g)(x)$ and $(g \circ f)(x)$.*

**Solution.** First, $(f \circ g)(x) = f(g(x)) = g(x) + 3 = x^2 + 3$.
Second , $(g \circ f)(x) = g(f(x)) = (f(x))^2 = (x + 3)^2$.

**Example 4.2.** *Let $f$ and $g$ be defined as in the above example and let $h(x) = \cos\sqrt{x}$. Find $(f \circ g \circ h)(x)$ and $(g \circ h \circ f)(x)$.*

**Solution.** First,

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\cos\sqrt{x})) = f(\cos^2\sqrt{x}) = (\cos^2\sqrt{x}) + 3.$$

Second,

$$(g \circ h \circ f)(x) = g(h(f(x))) = g(h(x+3)) = g(\cos\sqrt{x+3}) = \cos^2\sqrt{x+3}.$$

**Example 4.3.** *Find $f \circ g \circ h$ where $f(x) = \dfrac{x}{x+1}$, $g(x) = x^{10}$ and $h(x) = x + 3$.*

**Solution.**

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+3)) = f((x+3)^{10}) = \frac{(x+3)^{10}}{(x+3)^{10}+1}.$$

**Example 4.4.** *Find functions $f$, $g$ and $h$ such that*

$$(f \circ g \circ h)(x) = \cos^2(x+9).$$

**Solution.** One can take $h(x) = x + 9$, $g(x) = \cos x$ and $f(x) = x^2$.

**Example 4.5.** *Find the value of $x$ so that $(f \circ g)(x) = (g \circ f)(x)$, where $f(x) = x^2$ and $g(x) = x + 1$.*

**Solution.** Note that, $(f \circ g)(x) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$, and that $(g \circ f)(x) = g(x^2) = x^2 + 1$.
Therefore, we have $(f \circ g)(x) = (g \circ f)(x) \implies (x^2+1)+2x = x^2+1 \implies 2x = 0 \implies x = 0$.

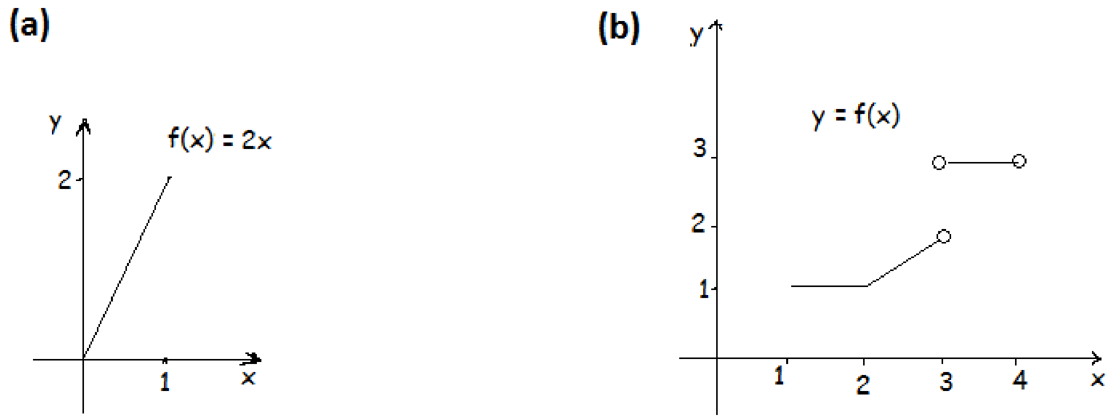
**Example 4.6.** *If $g(x) = 3x+1$ and $f$ is unknown, but has the domain $D_f = [4, 7)$, find the domain $D_{f \circ g}$.*

**Solution.** Note that $(f \circ g)(x) = f(g(x)) = f(3x + 1)$. So, to find the domain of $f \circ g$, we wanna $4 \leq 3x + 1 < 7$, but

$$4 \leq 3x + 1 < 7 \implies 3 \leq 3x < 6 \implies 1 \leq x < 2.$$

This means that $D_{f \circ g} = [1, 2)$.

**Example 4.7.** *For each part ((a) and (b)), use the graph of $f$ to find the domain and the range of $f$.*




**Solution.**

(a) $D_f = [0, 1]$ and $R_f = [0, 2]$.

(b) $D_f = [1, 3) \cup (3, 4)$ and $R_f = [1, 2) \cup \{3\}$.

## Odd and even functions

**Definition 4.2.** A function $f$ is said to be an:

(i) odd function iff $f(-x) = -f(x)$ for all $x \in D_f$.

(ii) even function iff $f(-x) = f(x)$ for all $x \in D_f$.

**Example 4.8.** *It is known that* $(-x)^2 = x^2$ *and* $\cos(-x) = \cos x$, *while* $(-x)^3 = -x^3$ *and* $\sin(-x) = -\sin x$. *This means that:*

- *The functions* $y_1 = x^2$ *and* $y_2 = \cos x$ *are even functions.*
- *The function* $y_3 = x^3$ *and* $y_4 = \sin x$ *are odd functions.*

**Remark 4.1.** *The graph of any odd function is symmetric about the origin, while the graph of any even function is symmetric about the* $y$*-axis.*

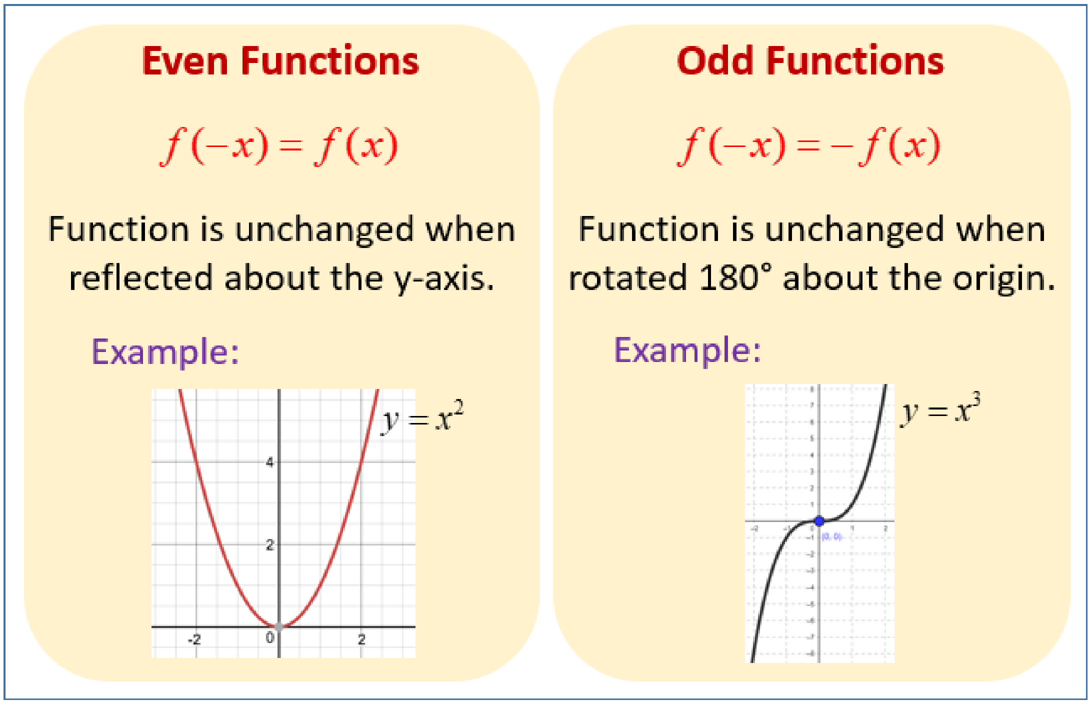



Figure 9: If a function is even, its graph is symmetric about the $y$-axis. If it is odd, its graph is symmetric about the origin. This picture was taken from: https://www.onlinemathlearning.com/even-and-odd-functions.html.

**Example 4.9.** *Which of the following functions is even?*

*(a)* $f(x) = \frac{\sin x}{|x|}$.

*(b)* $g(x) = \frac{\cos x}{x^3+x}$.

*(c)* $h(x) = 1 + \cos x$.

**Solution.** Note that

(a) $f(-x) = \frac{\sin -x}{|-x|} = \frac{-\sin x}{|x|} = -\frac{\sin x}{|x|} = -f(x)$. So, $f$ is odd.

(b) $g(-x) = \frac{\cos(-x)}{(-x)^3+(-x)} = \frac{\cos x}{-x^3-x} = -\frac{\cos x}{x^3+x} = -g(x)$. So, $g$ is odd.

(c) $h(-x) = 1 + \cos -x = 1 + \cos x = h(x)$. So, $h$ is even.

Given this, the correct answer is (c).

**Example 4.10.** *Determine whether the function*

$$f(x) = 2x^3 - 3x^2 - 4x + 4$$

*is even, odd, or neither.*

**Solution.** Note that

$$f(-x) = 2(-x)^3 - 3(-x)^2 - 4(-x) + 4 = -2x^3 - 3x^2 + 4x + 4.$$

This means that $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. Thus, $f$ is neither even nor odd.

**Some properties:**

Let $E$ denote any even function and $O$ any odd function. Then

| | | |
|---|---|---|
| (1) $E \pm E = E$. | (2) $O + O = O$. | (3) $E \cdot E = E$. |
| (4) $O \cdot O = E$. | (5) $O \cdot E = O$. | (6) $E/E = E$. |
| (7) $O/O = E$. | (8) $O/E = O$. | (9) $E/O = O$. |
| (10) $O \circ O = O$. | (11) $E \circ O = E$. | (12) $\left\{ \begin{array}{c} O \\ E \end{array} \right\} \circ E = E$. |

Question: What about $O - O$?

# 5 Logarithmic and exponential functions

## The logarithmic function

**Definition 5.1.** For any positive number $B \neq 1$, the logarithm with base $B$ is defined by $c = \log_B A$ iff $B^C = A$.

For example, we have

$$\begin{array}{lcl} 10^1 = 10 & \Longrightarrow & \log_{10} 10 = 1. \\ 10^0 = 1 & \Longrightarrow & \log_{10} 1 = 0. \\ 10^3 = 1000 & \Longrightarrow & \log_{10} 1000 = 3. \\ 10^{-3} = 0.001 & \Longrightarrow & \log_{10} 0.001 = -3. \end{array}$$

**Fact 5.1** (Properties of $\log_{10} \cdot$)**.** *We have:*

*(1)* $\log_{10}(AB) = \log_{10} A + \log_{10} B.$

*(2)* $\log_{10}(A/B) = \log_{10} A - \log_{10} B.$

*(3)* $\log_{10} A^B = B \log_{10} A.$

**Definition 5.2.** For any positive number $B \neq 1$, the logarithm function with base $B$ is defined by

$$\underbrace{y = \log_B x}_{\text{Logarithm func.}} \quad \text{iff} \quad \underbrace{x = B^y}_{\substack{\text{Exponential func.} \\ \text{(to be studied later)}}} .$$

**Two important special cases:**

- $\log_{10} x$ is abbreviated by $\log x$.
- $\log_e x$ is abbreviated by $\ln x$ (natural logarithmic function). Recall that $e$ is the Neperian number; $e \approx 2.71828182845904$.

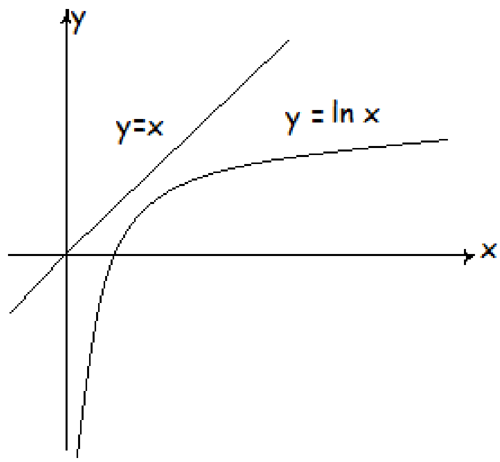



Figure 10: The graph of $y = \ln x$.

It is clear that $D(\ln x) = (0, \infty)$ and that $R(\ln x) = \mathbb{R}$.

In particular, $y = \ln x$ is $\begin{cases} \text{negative} & \text{when } 0 < x < 1, \\ \text{zero} & \text{when } x = 1, \\ \text{positive} & \text{when } x > 1. \end{cases}$

**Example 5.1.** *Find the domain of the following functions*

(1) $f(x) = \ln(1-x)$. (2) $g(x) = \ln(x^2-4)$. (3) $h(x) = \dfrac{\sqrt{x+2}}{\log x}$.

**Solution.**

(1) Note that $1 - x > 0 \implies x < 1$. Thus, $D_f = (-\infty, 1)$.

(2) Note that $x^2 - 4 > 0 \implies 4 < x^2 \implies x > 2$ or $x < -2$.
Thus, $D_g = (-\infty, -2) \cup (2, \infty)$.

(3) Note that $x + 2 \geq 0 \implies x \geq -2$,
and that $\log x \neq 0 \implies x \neq 1$.
Also, $\log x$ is defined only when $x > 0$.
Thus, $D_h = (0, \infty) - \{1\}$.

**Fact 5.2** (Coming soon)**.** *Let $A, B \in \mathbb{R}$ with $B > 0$. Then*

$$\ln e^A = A \quad \textit{and} \quad e^{\ln B} = B.$$

**Example 5.2.** *If $D_f = \mathbb{R} - \{3\}$ and $g(x) = \ln x$, find $D_{(f \circ g)}$.*

**Solution.** Note that $(f \circ g)(x) = f(\ln x)$.
Because $D_f = \mathbb{R} - \{3\}$, we look for $x \in (0, \infty)$ so that $\ln x \neq 3$.
Thus, $D_{(f \circ g)} = (0, \infty) - \left\{e^3\right\}$.

**Some properties of $\ln x$:**

**Fact 5.3.** *For any positive numbers x and y, we have*

*(1)* $\ln(xy) = \ln x + \ln y$.

*(2)* $\ln(\frac{1}{y}) = -\ln y$.

*(3)* $\ln(\frac{x}{y}) = \ln x - \ln y$.

*(4)* $\ln(x^r) = r \ln x$, *for any rational number r.*

**Example 5.3.** *Use the properties of logarithms to expand* $\ln\left(\dfrac{x^3y^4}{z^5}\right)$.

**Solution.**

$$\begin{aligned}
\ln\left(\frac{x^3y^4}{z^5}\right) &= \ln(x^3y^4) - \ln(z^5) \\
&= \ln(x^3) + \ln(y^4) - \ln(z^5) \\
&= 3\ln(x) + 4\ln(y) - 5\ln(z).
\end{aligned}$$

**Example 5.4.** *Given that* $\ln 3 = 1.1$, $\ln 4 = 1.39$ *and* $\ln 5 = 1.61$. *Estimate:* (1) $\ln 0.2$. (2) $\ln 2.4$.

**Solution.**

(1) $\ln(0.2) = \ln(\frac{2}{10}) = \ln(\frac{1}{5}) = -\ln(5) = -1.61$.

(2) $\ln(2.4) = \ln(\frac{24}{10}) = \ln(\frac{12}{5}) = \ln\left(\frac{(3)(4)}{5}\right) = \ln(3) + \ln(4) - \ln(5)$
$= 1.1 + 1.39 - 1.61 = 0.88$.

**Example 5.5.** *Solve the following equation for $x$:*

$$3\ln x - \ln(x^2 + 2x) = 0.$$

**Solution.** Note that

$$0 = 3\ln x - \ln(x^2 + 2x) = \ln(x^3) - \ln(x^2 + 2x) = \ln\left(\frac{x^3}{x^2 + 2x}\right)$$

implies that $x^3/(x^2 + 2x) = 1$. Now,

$$\begin{aligned} \tfrac{x^3}{x^2+2x} = 1 \quad &\Longrightarrow \quad x^3 = x^2 + 2x \\ &\Longrightarrow \quad x^3 - x^2 - 2x = 0 \\ &\Longrightarrow \quad x(x^2 - x - 2) = 0 \\ &\Longrightarrow \quad x(x-2)(x+1) = 0 \\ &\Longrightarrow \quad x = 0, x = 2, x = -1. \end{aligned}$$

Because $D(\ln x) = (0, \infty)$, the values $x = -1$ and $x = 0$ are impossible, and hence the value $x = 2$ is the only solution of the equation.

**Example 5.6.** *Is the function* $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ *odd or even?*

**Solution.** Note that

$$
\begin{aligned}
f(-x) &= \ln\left(-x + \sqrt{(-x)^2 + 1}\right) \\
&= \ln\left(-x + \sqrt{x^2 + 1}\right) \\
&= \ln\left(\left(-x + \sqrt{x^2 + 1}\right)\left(\frac{x + \sqrt{x^2 + 1}}{x + \sqrt{x^2 + 1}}\right)\right) \\
&= \ln\left(\frac{x^2 + 1 - x^2}{x + \sqrt{x^2 + 1}}\right) \\
&= \ln\left(\frac{1}{x + \sqrt{x^2 + 1}}\right) \\
&= -\ln\left(x + \sqrt{x^2 + 1}\right) \\
&= -f(x).
\end{aligned}
$$

Thus, $f(x)$ is an odd function.

**Fact 5.4.** $\log_B x = \dfrac{\ln x}{\ln B}$, *for* $x > 0$. *Here,* $B$ *is a positive number.*

**Example 5.7.** *Find the value of* $z = (\log_2 3)(\log_3 4)(\log_4 5)\ldots(\log_{15} 16)$.

**Solution.**

$$
\begin{aligned}
z &= (\log_2 3)(\log_3 4)(\log_4 5)\ldots(\log_{15} 16) \\
&= \left(\frac{\ln 3}{\ln 2}\right)\left(\frac{\ln 4}{\ln 3}\right)\left(\frac{\ln 5}{\ln 4}\right)\cdots\left(\frac{\ln 16}{\ln 15}\right) \\
&= \left(\frac{\ln 16}{\ln 2}\right) \\
&= 4\left(\frac{\ln 2}{\ln 2}\right) \\
&= 4.
\end{aligned}
$$

**A special case of Fact 5.4:** $\log_P(P^t) = \dfrac{\ln P^t}{\ln P} = t$**.**

For example, $\log_2 32 = \log_2 2^5 = 5$.

# The exponential function

Recall that $e$ is the Neperian number; $e \approx 2.71828182845904$.

**Definition 5.3.** The function $f(x) = e^x, x \in \mathbb{R}$, is called the natural exponential function.

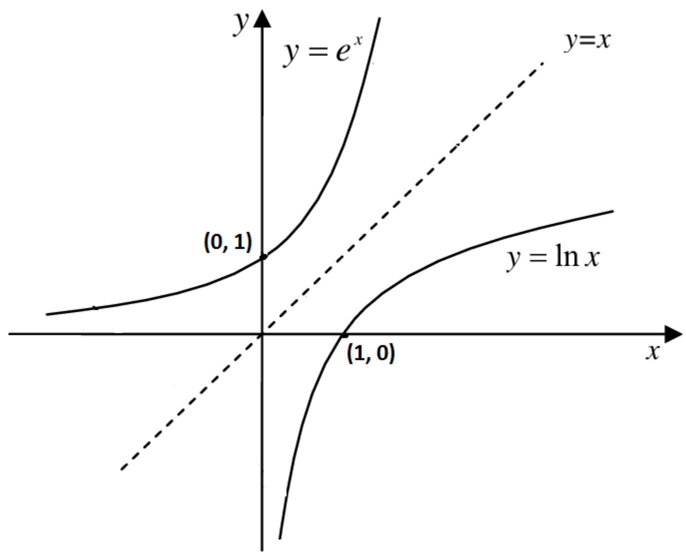



Figure 11: The graphs of $y = e^x$ and $y = \ln x$. Note that $e^0 = 1$ and $\ln 1 = 0$.

Note that $D(e^x) = \mathbb{R} = R(\ln x)$ and $R(e^x) = (0, \infty) = D(\ln x)$.

**Some properties of $e^x$:**

**Fact 5.5.** *(1)* $e^x > 0$ *for all* $x \in \mathbb{R}$.

*(2)* $e^{x+y} = e^x e^y$.

*(3)* $e^{-x} = \dfrac{1}{e^x}$.

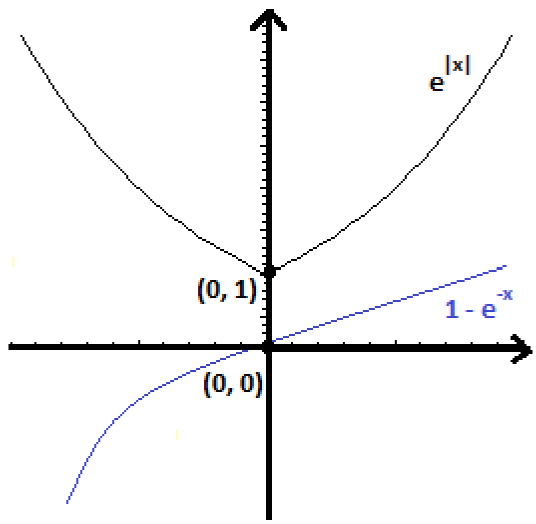
*(4)* $e^{x-y} = \dfrac{e^x}{e^y}$.

*(5)* $\ln e^x = x$, *for all* $x \in \mathbb{R}$.

*(6)* $e^{\ln x} = x$, *for all* $x > 0$.

**Example 5.8.** *Sketch the graph of* $y_1 = 1 - e^{-x}$ *and* $y_2 = e^{|x|}$.

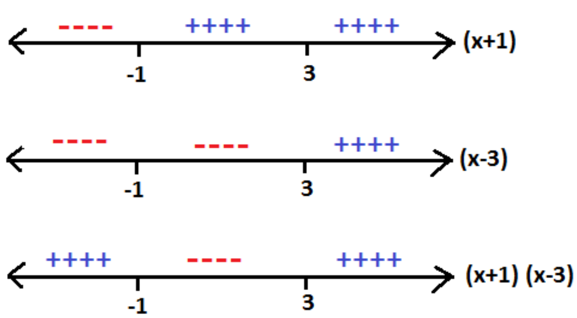
**Solution.** By reflecting and shifting the graph of $y = e^x$, we get




**Example 5.9.** *Solve the inequality* $\ln|x^2 - 2x - 2| \leq 0$.

**Solution.** Note that

$$\begin{aligned} \ln|x^2 - 2x - 2| \leq 0 &\implies e^{\ln|x^2-2x-2|} \leq e^0 \\ &\implies x^2 - 2x - 2 \leq 1 \\ &\implies x^2 - 2x - 3 \leq 0 \\ &\implies (x-3)(x+1) \leq 0. \end{aligned}$$




The solution set is $\{x \in \mathbb{R} : -1 \leq x \leq 3\} = [-1, 3]$.

**Example 5.10.** *Solve $e^{2x} - 3e^x = 4$.*

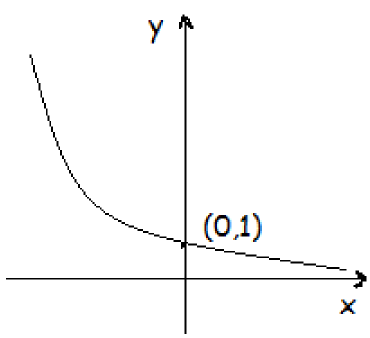
**Solution.** Note that

$$\begin{aligned} e^{2x} - 3e^x = 4 &\implies y^2 - 3y = 4 \qquad \text{(by setting } \; y = e^x\text{)} \\ &\implies y^2 - 3y - 4 = 0 \\ &\implies (y-4)(y+1) = 0 \\ &\implies e^x = y = 4 \;\text{ and hence }\; \ln x = 4, \\ &\qquad \text{or } \; e^x = y = -1 \;\; \text{(impossible).} \end{aligned}$$

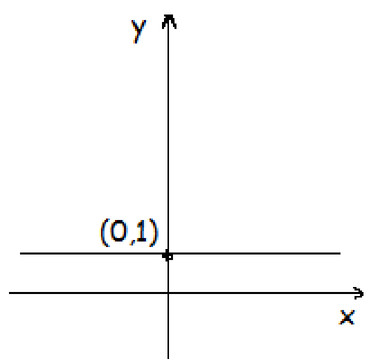
**Example 5.11.** *Find $R(3e^x - 1)$.*

**Solution.** Note that $e^x > 0 \implies 3e^x > 0 \implies 3e^x - 1 > -1$. Therefore $R(3e^x - 1) = (-1, \infty)$.

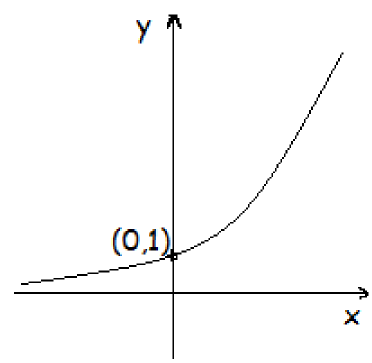
**Definition 5.4.** Let $p$ be a positive constant. The func. $f(x) = p^x$ is called the exponential function with base $p$. Here, we call $p$ the base and $x$ the exponent.




y = p^x, 0< p<1




y = 1^x




y = p^x, p>1

Note that $D(p^x) = \mathbb{R}$ and $R(p^x) = (0, \infty)$.

**Some properties of $p^x$:**

**Fact 5.6.** *(1)* $p^{x+y} = p^x p^y$ *and* $p^{x-y} = \dfrac{p^x}{p^y}$.

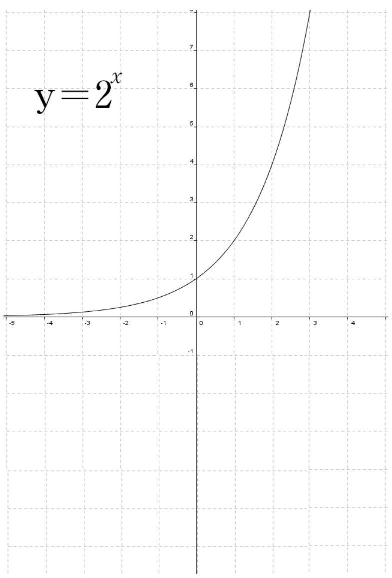
*(2)* $(p^x)^y = p^{xy}$ *and* $(pq)^x = p^x q^x$.

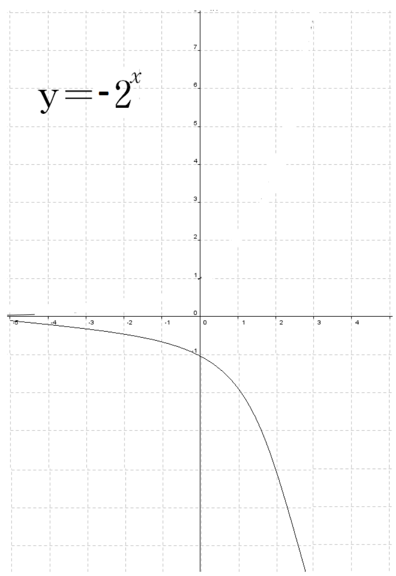
*(3)* $\log_p p^x = x$*, for all* $x \in \mathbb{R}$.

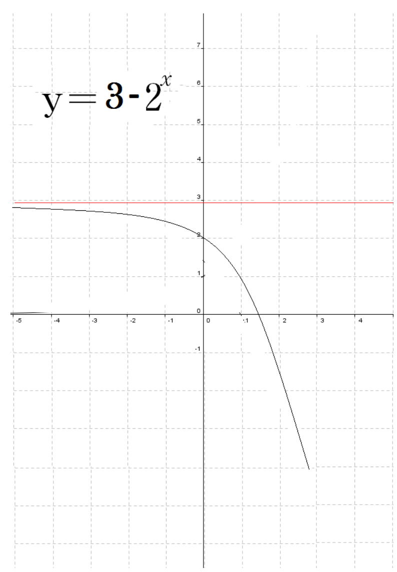
*(4)* $p^{\log_p x} = x$*, for all* $x > 0$.

**Example 5.12.** *Use the graph of* $y = 2^x$ *to sketch the graph of* $y = 3 - 2^x$.

**Solution.**



# 6 One-to-one and inverse functions

## One-to-one functions

A one-to-one function (1-1 func.) is the one that maps distinct elements of its domain to distinct elements of its range.

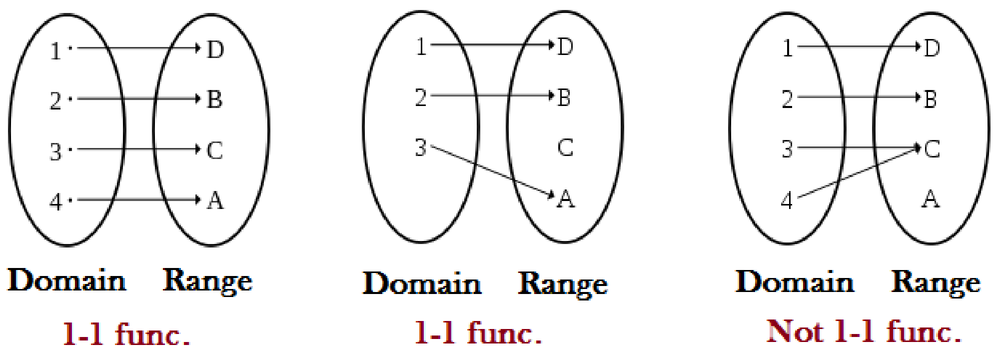



Figure 12: These pictures were taken from: wikipedia.org.

## The horizontal line test

**Test 6.1.** *A function is 1-1 iff every horizontal line intersects its graph in at most one point.*

For example, the function $f(x) = x^2$ is not 1-1, while $f(x) = x^3$ is a 1-1 function.

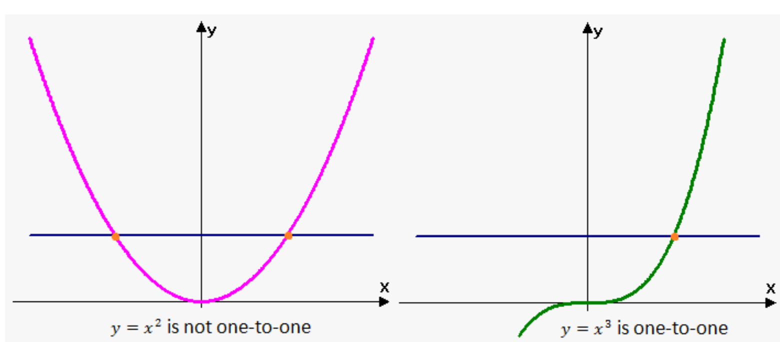



Figure 13: These pictures were taken from: https://nohemiportfiolio2012-2013.weebly.com/polynomials-and-rational-functions.html.

**Definition 6.1.** A function $f$ is said to be 1-1 iff the equality $f(a) = f(b)$ implies that $a = b$, for all $a, b \in D_f$.

**Example 6.1.** *Determine which of the following functions is 1-1.*

*(1)* $f(x) = x^3 - 8$.

*(2)* $g(x) = \dfrac{x}{x-8}$.

*(3)* $h(x) = x^4 + 3$.

**Solution.**

(1) Let $a, b \in D_f$ and assume that $f(a) = f(b)$, then

$$f(a) = f(b) \implies a^3 - 8 = b^3 - 8 \implies a^3 = b^3 \implies a = b.$$

Thus, $f$ is 1-1.

(2) Let $a, b \in D_g$ and assume that $g(a) = g(b)$, then

$$\frac{a}{a-8} = \frac{b}{b-8} \implies ab - 8a = ab - 8b \implies 8a = 8b \implies a = b.$$

Thus, $g$ is 1-1.

(3) Let $a, b \in D_h$ and assume that $h(a) = h(b)$, then

$$h(a) = h(b) \implies a^4 + 3 = b^4 + 3 \implies a^4 = b^4 \implies a = \pm b!!$$

So, w are not able to prove that $a = b$ given $h(a) = h(b)$.

Note that $1, -1 \in D_h$ and that $h(1) = 4 = h(-1)$, while $1 \neq -1$. So, $h$ is not 1-1.

**Fact 6.1.** *If $f$ is an increasing or decreasing function on its domain, then it is 1-1.*

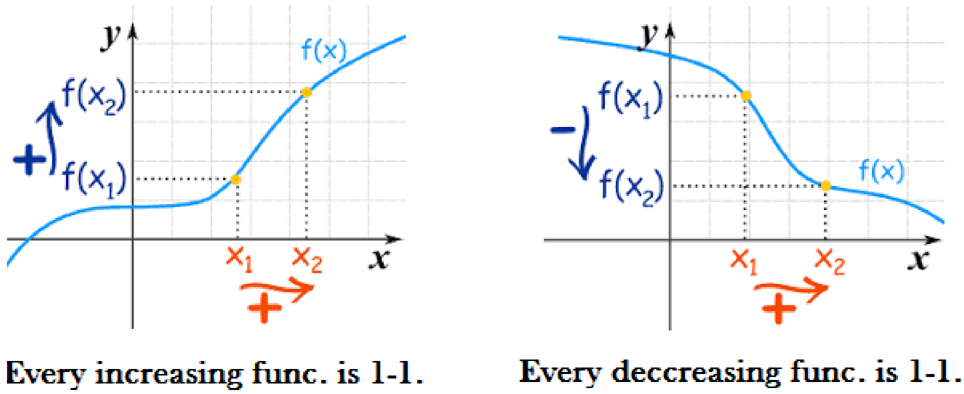



Figure 14: These pictures were taken from: https://www.mathsisfun.com/sets/functions-increasing.html.

**Example 6.2.** *One of the following functions is not 1-1:*

*(a)* $f(x) = \sin x, x \in [0, \pi]$.

*(b)* $f(x) = |x + 1|, x \geq 0$.

*(c)* $f(x) = x^2 + 1, x \geq 0$.

*(d)* $f(x) = \cos x, x \in [-\pi, 0]$.

**Solution.** The function $f(x) = \sin x, x \in [0, \pi]$ is not 1-1 because $f(\frac{\pi}{3}) = f(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$ while $\frac{\pi}{3} \neq \frac{2\pi}{3}$. So the correct answer is (a).

## Inverse functions

**Definition 6.2.** Let $f$ be a 1-1 function. The inverse of $f$, denoted by $f^{-1}$, is the unique function defined on the range of $f$ that satisfies the equation $f(f^{-1}(x)) = x$, for all $x \in R_f$.

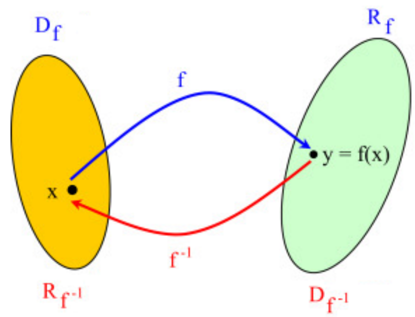



Figure 15: This picture was taken from: https://www.sciencedirect.com/topics/mathematics/inverse-function.

**Example 6.3.** *If* $f(x) = 2x^3 + 3x + 1$ *and* $f^{-1}(x) = 1$*. Find* $x$*.*

**Solution.** $f^{-1}(x) = 1 \implies x = f(f^{-1}(x)) = f(1) = 2 + 3 + 1 = 6.$

**Fact 6.2.**

$$f(f^{-1}(x)) = x, \textit{ for all } x \in R_f, \quad \textit{and} \quad f^{-1}(f(x)) = x, \textit{ for all } x \in D_f.$$

**Question: How to find $f^{-1}(x)$ given $f(x)$?**

**Answer: We apply the following three steps consecutively.**

(1) Let $y = f^{-1}(x)$.

(2) $f(y) = f(f^{-1}(x)) = x$.

(3) Solve for $y$.

**Example 6.4.** *Find a rule for* $f^{-1}$ *for the following 1-1 functions:*

*(a)* $f(x) = x^3$.

*(b)* $f(x) = 5x + 1$.

*(c)* $f(x) = \sqrt{x^3 + 5}$.

*(d)* $f(x) = \dfrac{3x}{x-4}$.

*(e)* $f(x) = (1 - x^3)^{\frac{1}{5}} + 2$.

**Solution.**

$$\begin{aligned}
\text{(a) Letting } y = f^{-1}(x) &\implies f(y) = f(f^{-1}(x)) = x \\
&\implies y^3 = x \\
&\implies y = x^{\frac{1}{3}} \\
&\implies f^{-1}(x) = x^{\frac{1}{3}}.
\end{aligned}$$

$$\begin{aligned}
\text{(b) Letting } y = f^{-1}(x) &\implies f(y) = f(f^{-1}(x)) = x \\
&\implies 5y + 1 = x \\
&\implies y = \tfrac{x-1}{5} = f^{-1}(x).
\end{aligned}$$

$$\begin{aligned}
\text{(c) Letting } y = f^{-1}(x) &\implies f(y) = f(f^{-1}(x)) = x \\
&\implies \sqrt{y^3 + 5} = x \\
&\implies y^3 + 5 = x^2 \\
&\implies y^3 = x^2 - 5 \\
&\implies y = (x^2 - 5)^{\frac{1}{3}} = f^{-1}(x).
\end{aligned}$$

$$\begin{aligned}
\text{(d) Letting } y = f^{-1}(x) &\implies f(y) = f(f^{-1}(x)) = x \\
&\implies \tfrac{3y}{y-4} = x \\
&\implies 3y = xy - 4x \\
&\implies 4x = xy - 3y \\
&\implies y = \tfrac{4x}{x-3} = f^{-1}(x).
\end{aligned}$$

(e) Exc. (Final Ans. is: $f^{-1}(x) = [1 - (x-2)^5]^{\frac{1}{3}}$).

**Fact 6.3.** *The graph of $f^{-1}$ can be obtained by reflecting the graph of $f$ about the line $y = x$.*

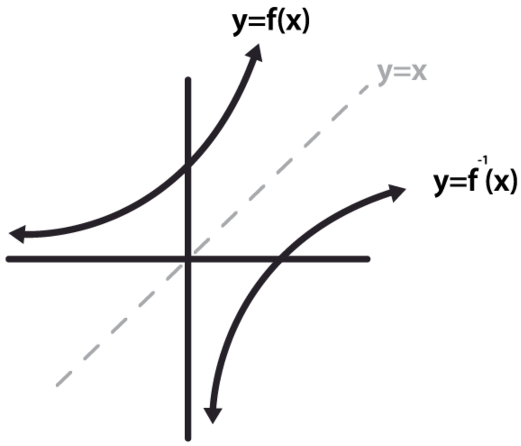
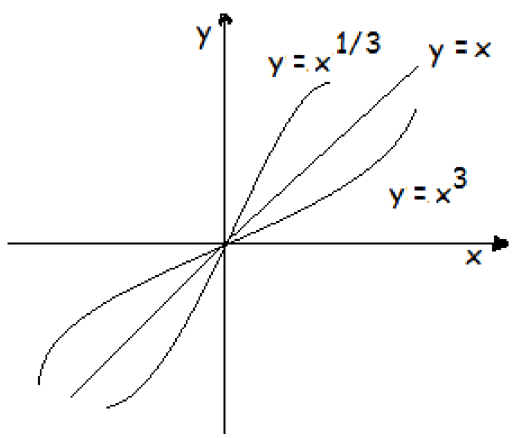



Figure 16: This picture was taken from: https://calcworkshop.com/trig-equations/inverse-functions/.

For example, take $f(x) = x^3$, we have shown that $f^{-1}(x) = x^{\frac{1}{3}}$.




Note that $f^{-1}(x)$ does NOT mean $\dfrac{1}{f(x)}$.
(Consider, for example, $f(x) = x^3$).
In fact, $\dfrac{1}{f(x)} = (f(x))^{-1}$.

**Fact 6.4.**

$$D_{f^{-1}} = R_f \ \ \textit{and} \ \ R_{f^{-1}} = D_f.$$

**Example 6.5.** *Let* $f(x) = \dfrac{x}{x-3}$. *Find* (1) $D_f$. (2) $R_f$.

**Solution.**

(1) Clearly, $D_f = \mathbb{R} - \{3\}$.

(2) We know that $R_f = D_{f^{-1}}$. So, we first find $f^{-1}(x)$.

$$\begin{aligned}
\text{Letting } y = f^{-1}(x) &\implies f(y) = f(f^{-1}(x)) = x \\
&\implies \tfrac{y}{y-3} = x \\
&\implies xy - 3x = y \\
&\implies xy - y = 3x. \\
&\implies y = \tfrac{3x}{x-1} = f^{-1}(x).
\end{aligned}$$

Thus, $R_f = D_{f^{-1}} = \mathbb{R} - \{1\}$.

**Example 6.6.** *Let* $f(x) = \frac{1}{\sqrt{4-x^2}}$. *Find* (1) $D_f$. (2) $R_f$.

**Solution.**

(1) $x \in D_f \iff 4 - x^2 > 0 \iff 4 > x^2 \iff -2 < x < 2$. Thus, $D_f = (-2, 2)$.

(2) Note that $f(x)$ is not a 1-1 function, so we cannot use the inverse to find its range. However, we know that $4 - x^2$ is increasing on $(-2, 0]$ and decreasing on $[0, 2)$. So, the function $f(x)$ is decreasing on $(-2, 0]$ and increasing on $[0, 2)$. Therefore, $f(0) = \frac{1}{2}$ is the absolute minimum and $f \to \infty$ as $x \to \pm 2$. Thus, $R_f = [0, \infty)$.

**Example 6.7.** *Let* $f(x) = 2x^2 + 3x + 1, x \leq \frac{-3}{4}$. *Then,* $f^{-1}(0)$ *equals*

*(a)* 1.

*(b)* 0.

*(c)* −1.

*(d)* $\{-\frac{1}{2}, -1\}$.

**Solution.** Let $y = f^{-1}(0)$, then $f(y) = 0$. That is,

$$f(y) = 2y^2 + 3y + 1 = (2y + 1)(y + 1) = 0.$$

It follows that $y = -1, \frac{-1}{2}$.
Thus, $f^{-1}(0) = -1$ only, because $\frac{-1}{2} > \frac{-3}{4}$.
So, the correct answer is (c).

**Fact 6.5.** $\ln x = y \iff e^y = x$.
*More generally,* $\log_B x = y \iff B^y = x$.

**Example 6.8.** *Let* $h(x) = e^{\cos x}, x \in (0, \pi)$. *Find* $h^{-1}(x)$.

**Solution.**

$$\begin{aligned} \text{Letting } y = h^{-1}(x) &\implies h(y) = h(h^{-1}(x)) = x \\ &\implies e^{\cos y} = x \\ &\implies \cos y = \ln x \\ &\implies y = \cos^{-1}(\ln x) = h^{-1}(x). \end{aligned}$$

# 7 Inverse trigonometric functions

Recall that:

(i) $f$ is 1-1 $\iff$ f has an inverse.

(ii) $f^{-1}(f(x)) = x,$ for all $x \in D_f$.

(iii) $f(f^{-1}(x)) = x,$ for all $x \in R_f$.

## Inverse sine

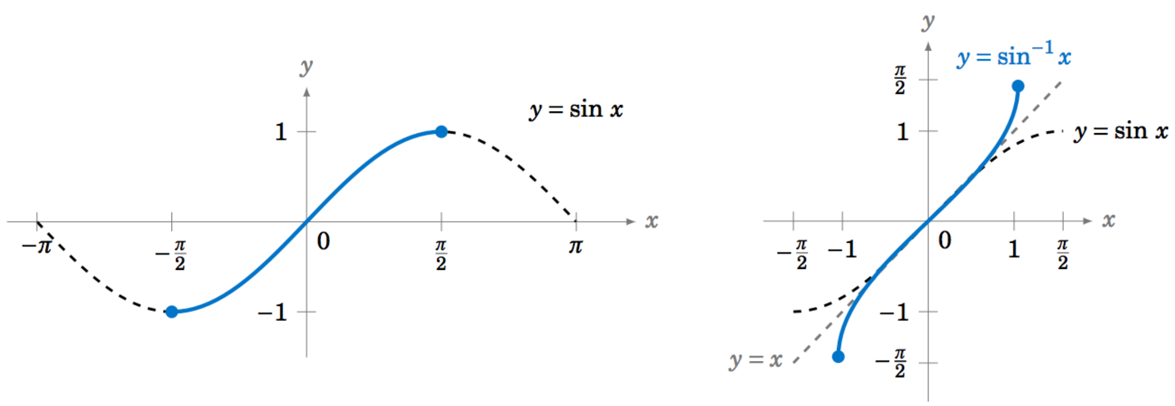



Figure 17: These pictures were taken from: https : //math.libretexts.org/Courses/Mount$_R$oyal$_U$niversity/MATH$_1$200%3A$_C$alculus$_f$or$_S$cientists$_I$

The function $y = \sin x$ is 1-1 on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
Then, $\sin x$ has an inverse, say $\sin^{-1}(x)$, on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and

$$\begin{aligned} \sin(\sin^{-1}(x)) &= x \quad \text{for all} \quad x \in [-1, 1], \\ \sin^{-1}(\sin(x)) &= x \quad \text{for all} \quad x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]. \end{aligned}$$

Recall that $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$.

**Example 7.1.** *Calculate if defined*

*(a)* $\sin(\sin^{-1}(\frac{1}{4}))$.

*(b)* $\sin(\sin^{-1}(4))$.

*(c)* $\sin^{-1}(\sin(\frac{\pi}{4}))$.

*(d)* $\sin^{-1}(\sin(\frac{9\pi}{4}))$.

*(e)* $\sin^{-1}(\sin(\frac{7\pi}{4}))$.

*(f)* $\csc(\sin^{-1}(\frac{1}{3}))$.

**Solution.**

(a) $\sin(\sin^{-1}(\frac{1}{4})) = \frac{1}{4}$.

(b) $\sin(\sin^{-1}(4)) \neq 4$; it is not defined because $4 \notin [-1, 1]$.

(c) $\sin^{-1}(\sin(\frac{\pi}{4})) = \frac{\pi}{4}$.

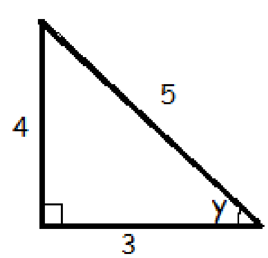
(d) $\sin^{-1}(\sin(\frac{9\pi}{4})) = \sin^{-1}(\sin(\frac{9\pi}{4} - 2\pi)) = \sin^{-1}(\sin(\frac{\pi}{4})) = \frac{\pi}{4}$.

(e) $\sin^{-1}(\sin(\frac{7\pi}{4})) = \sin^{-1}(\sin(\frac{7\pi}{4} - 2\pi)) = \sin^{-1}(\sin(\frac{-\pi}{4})) = -\frac{\pi}{4}$.

(f) $\csc(\sin^{-1}(\frac{1}{3})) = \dfrac{1}{\sin(\sin^{-1}(\frac{1}{3}))} = \dfrac{1}{1/3} = 3$.

**Example 7.2.** *Find* $\tan(2\sin^{-1}(\frac{4}{5}))$.

**Solution.** Let $y = \sin^{-1}(\frac{4}{5})$, then $\sin y = \frac{4}{5}$.



By referring to the right-angle triangle, it follows that

$$\begin{aligned}
\tan(2\sin^{-1}(\tfrac{4}{5})) &= \tan(2y) \\
&= \frac{\sin 2y}{\cos 2y} \\
&= \frac{2\sin y\cos y}{\cos^2(y) - \sin^2(y)} \\
&= \frac{2(4/5)(3/5)}{(3/5)^2 - (4/5)^2} \\
&= -24/7.
\end{aligned}$$

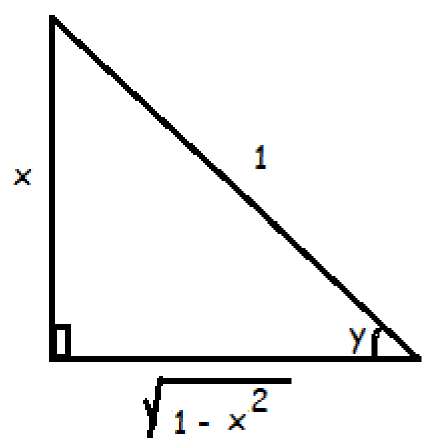
**Fact 7.1.** $y = \sin^{-1} x \iff x = \sin y$ *and* $-\pi/2 \le y \le \pi/2$.

**Example 7.3.** *Show that*

*(a)* $\cos(\sin^{-1} x) = \sqrt{1-x^2}$.

*(b)* $\cot(\sin^{-1} x) = \frac{\sqrt{1-x^2}}{x}$.

**Solution.** Let $y = \sin^{-1} x$, then $\sin y = x = \frac{x}{1}$. Then, by referring to the right-angle triangle: we have




(a) $\cos(\sin^{-1} x) = \cos y = \sqrt{1-x^2}$.

(b) $\cot(\sin^{-1} x) = \cot y = \frac{\sqrt{1-x^2}}{x}$.

**Example 7.4.** *Find the range of the function* $f(x) = 4\sin^{-1}(x) + \pi$.

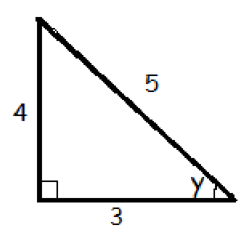
**Solution.** Note that

$$\begin{aligned} -\tfrac{\pi}{2} \leq \sin^{-1} x \leq \tfrac{\pi}{2} &\implies -2\pi \leq 4\sin^{-1} x \leq 2\pi \\ &\implies -\pi \leq \underbrace{4\sin^{-1} x + \pi}_{f(x)} \leq 3\pi. \end{aligned}$$

Thus, $R_f = [-\pi, 3\pi]$.

**Example 7.5.** *If* $\sin x = \frac{4}{5}$ *and* $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, *find* $\cos x$.

**Solution.** By referring to the right-angle triangle, we have $\cos x = \pm\frac{3}{5}$.




Since $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we get $\cos x = -\frac{3}{5}$.

# Inverse cosine

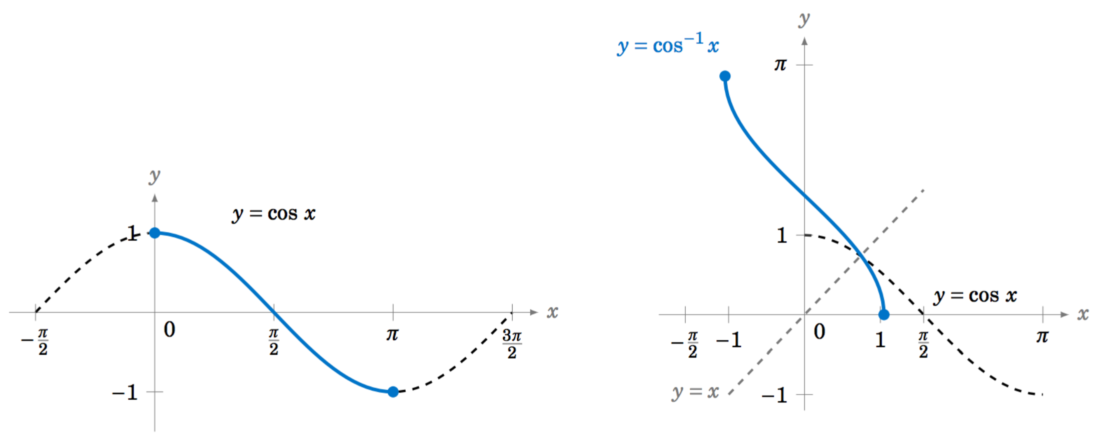



Figure 18: These pictures were taken from: https : //math.libretexts.org/Courses/Mount$_{R}$oyal$_{U}$niversity/MATH$_{1}$200%3A$_{C}$alculus$_{f}$or$_{S}$cientists$_{I}$

The function $y = \cos x$ is 1-1 on $[0, \pi]$.
Then, $\cos x$ has an inverse, say $\cos^{-1}(x)$, on $[0, \pi]$, and

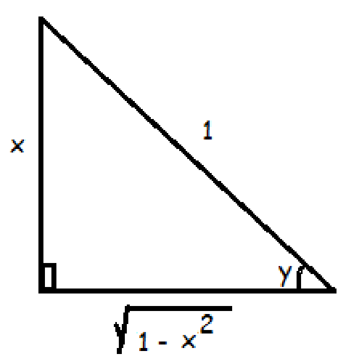
$$\cos(\cos^{-1} x) = x \quad \text{for all} \quad x \in [-1, 1],$$
$$\cos^{-1}(\cos x) = x \quad \text{for all} \quad x \in [0, \pi].$$

**Fact 7.2.** $y = \cos^{-1} x \iff x = \cos y$ *and* $0 \leq y \leq \pi$.

Note that $D_{\cos^{-1} x} = [-1, 1]$ and $R_{\cos^{-1} x} = [0, \pi]$.

**Example 7.6.** *Simplify the expression* $\tan(\cos^{-1}(x))$.

**Solution.** Let $y = \cos^{-1} x$, then $\cos y = x$. By referring to the




right-angle triangle, we obtain $\tan(\cos^{-1}(x)) = \tan y = \frac{\sqrt{1-x^2}}{x}$.

**Example 7.7.** *Let* $f(x) = 3\cos^{-1}(2x+1) + \pi$*. Find* $D_f$ *and* $R_f$*.*

**Solution.** First

$$\begin{aligned} x \in D_f &\iff -1 \le 2x + 1 \le 1 \\ &\iff -2 \le 2x \le 0 \\ &\iff -1 \le x \le 0. \end{aligned}$$

Therefore, $D_f = [-1, 0]$.

Second,

$$\begin{aligned} R_{\cos^{-1}(2x+1)} = [0, \pi] &\implies 0 \le \cos^{-1}(2x+1) \le \pi \\ &\implies 0 \le 3\cos^{-1}(2x+1) \le 3\pi \\ &\implies \pi \le 3\cos^{-1}(2x+1) + \pi \le 4\pi \\ &\implies \pi \le f(x) \le 4\pi. \end{aligned}$$

Therefore, $R_f = [\pi, 4\pi]$.

# Inverse tangent function

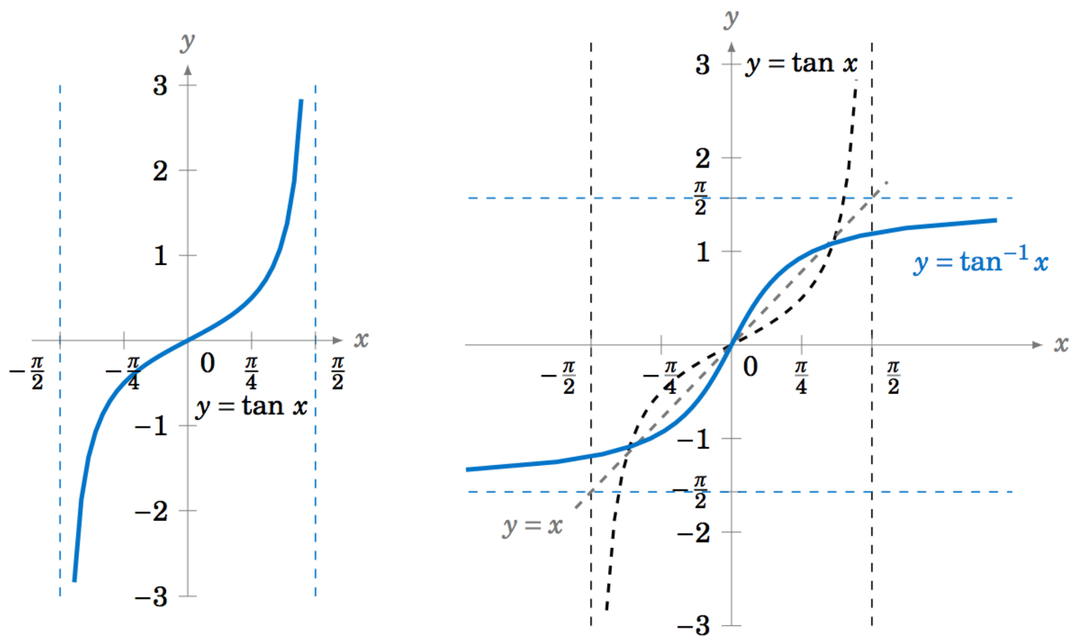



Figure 19: These pictures were taken from: *https : //math.libretexts.org/Courses/Mount$_R$oyal$_U$niversity/MATH$_1$200%3A$_C$alculus$_f$or$_S$cientists$_I$*

The function $y = \tan x$ is 1-1 on $(-\frac{\pi}{2}, \frac{\pi}{2})$.
Then, $\tan x$ has an inverse, say $\tan^{-1}(x)$, on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and

$$\begin{aligned} \tan(\tan^{-1} x) &= x \quad \text{for all} \quad x \in \mathbb{R}, \\ \tan^{-1}(\tan x) &= x \quad \text{for all} \quad x \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}). \end{aligned}$$

**Fact 7.3.** $y = \tan^{-1} x \iff x = \tan y$ *and* $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

**Example 7.8.** *Simplify the expression* $\cos(\tan^{-1} x)$.

**Solution.** Let $y = \tan^{-1} x$, then $\tan y = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
It follows that $\cos(\tan^{-1} x) = \cos(y) = \frac{1}{\sec x} = \frac{1}{\sqrt{1+\tan^2 y}} = \frac{1}{\sqrt{1+x^2}}$.

# Inverse cot, inverse sec, and inverse csc

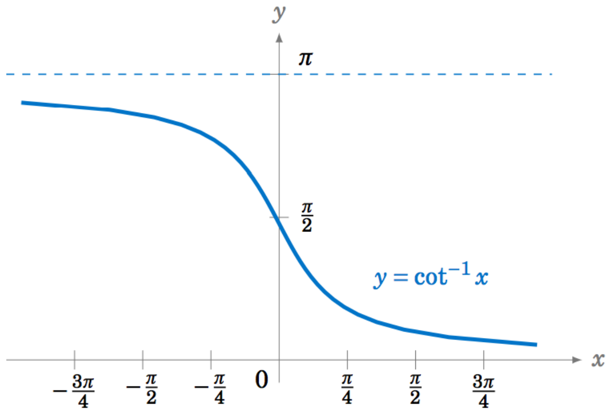



Figure 20: This picture was taken from: *https : //math.libretexts.org/Courses/Mount$_R$oyal$_U$niversity/MATH$_1$200%3A$_C$alculus$_f$or$_S$cientists$_I$*

**Fact 7.4.** $y = \cot^{-1} x$ *with* $x \in \mathbb{R} \iff x = \cot y$ *and* $y \in (0, \pi)$.

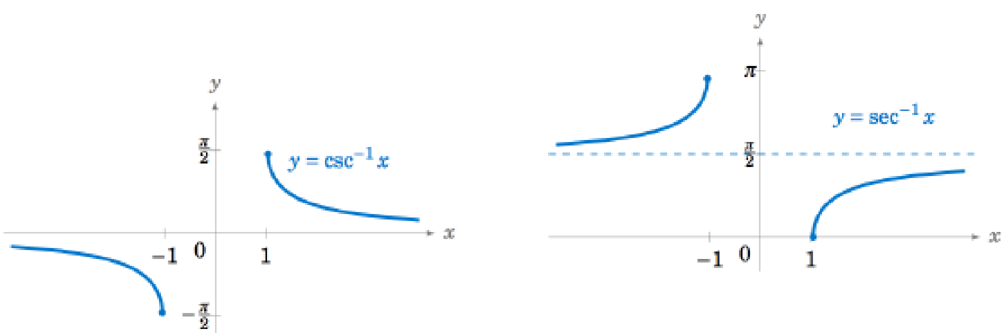



Figure 21: These pictures were taken from: *https : //math.libretexts.org/Courses/Mount$_R$oyal$_U$niversity/MATH$_1$200%3A$_C$alculus$_f$or$_S$cientists$_I$*

**Fact 7.5.** *(a)* $y = \sec^{-1} x$ *with* $|x| \geq 1 \iff x = \sec y$ *and*
$y \in [0, \pi/2) \cup [\pi, 3\pi/2)$.

*(b)* $y = \csc^{-1} x$ *with* $|x| \geq 1 \iff x = \csc y$ *and*
$y \in (0, \pi/2] \cup (\pi, 3\pi/2]$.

Searching keywords:

Functions.

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References: See the course website

http://sites.ju.edu.jo/sites/Alzalg/Pages/101.aspx

For comments or concerns, please contact me by email.


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